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K^0 DECAYS, UNITARITY AND TCP

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A B S T R A C T

The unitarity and TCP properties of weak decay amplitudes are discussed in an S matrix approach. The K_S decay amplitudes are expressed in terms of the K_L amplitudes.

The recent discussions ^{1),2)} on the $K \rightarrow \mu^+ \mu^-$ puzzle involve unitarity and TCP invariance in an essential way. Since the corresponding operators are in general not directly applicable to decaying states, but refer to operations on scattering states, we use in this paper an S matrix approach ³⁾. We will obtain the restrictions on the decay amplitudes due to unitarity and TCP invariance by considering the unitary and TCP invariant scattering of the decay products of unstable particles at an energy close to the resonance masses. Such a treatment is independent of perturbation theory and does not involve TCP properties of unstable states.

The assumptions we will use in consecutive steps and which are relevant to the case of the K^0 decays are the following :

- i) in the close neighbourhood of the two K resonances, the energy dependence of the unitary S matrix is entirely given by two complex poles (S and L pole);
- ii) the resonance scattering proceeds coherently via distinct spin zero superpositions of scattering states;
- iii) TCP invariance holds for the scattering process.

Under these conditions, it is found that the K_L decay amplitudes determine uniquely the K_S decay amplitudes and vice versa. The connection involves the non-resonating part of the S matrix (energy independent in the close neighbourhood of the resonances) which is dominated by strong or electromagnetic interactions.

I. CONSEQUENCES OF UNITARITY

According to assumption i), the s wave part of the S matrix for the scattering process in the close neighbourhood of the K resonances can be decomposed into three terms :

$$S = S_0 + i \frac{\tilde{T}_S}{M_S - E} + i \frac{\tilde{T}_L}{M_L - E} \quad (1)$$

$$M_{S,L} = E_{S,L} - \frac{i}{2} \Gamma_{S,L} \quad E_{S,L}, \Gamma_{S,L} : \text{real constants}$$

The essential point is the energy independence of the operators S_0 , $\tilde{T}_S$, $\tilde{T}_L$ in the tiny energy region of interest. The unitarity requirement

$$S \cdot S'^{\dagger} = S'^{\dagger} \cdot S = 1 \quad (2)$$

now enforces the unitarity of S_0 , the S matrix of the non-resonant background

$$S_0 \cdot S_0^{\dagger} = S_0^{\dagger} \cdot S_0 = 1 \quad (3)$$

The consequences of the unitarity relation (2) for the operators $\tilde{T}_{S,L}$ occurring in (1) can best be expressed in terms of the quantities $T_{S,L}$ defined by

$$\tilde{T}_{S,L} = S_0^{1/2} T_{S,L} S_0^{1/2} \quad (4)$$

One obtains from (1) and (2) with

$$\begin{aligned} \sigma &= i / (E_L - E_S + i \frac{\Gamma_S + \Gamma_L}{2}) \\ T_S &= \frac{1}{\Gamma_S} T_S T_S^{\dagger} + \sigma T_S T_L^{\dagger} = \frac{1}{\Gamma_S} T_S^{\dagger} T_S + \sigma T_L^{\dagger} T_S \\ T_L &= \frac{1}{\Gamma_L} T_L T_L^{\dagger} + \sigma^* T_L T_S^{\dagger} = \frac{1}{\Gamma_L} T_L^{\dagger} T_L + \sigma^* T_S^{\dagger} T_L \end{aligned} \quad (5)$$

For the operators T_S , T_L we will now use the special form

$$T_S = |S\rangle\langle S'|, \quad T_L = |L\rangle\langle L'| \quad (6)$$

where the states $|S\rangle$, $|S'\rangle$, $|L\rangle$, $|L'\rangle$ (not to be confused with the K_S , K_L states in the usual treatment) are superpositions of spin zero scattering states at an energy corresponding to the average K mass. More precisely written, Eq. (6) should contain an energy integral

$$T_S = \int dE |S(E)\rangle\langle S'(E)|$$

in which energy dependent superpositions of scattering states occur. We are, however, only interested in matrix elements taken at energy values in the close neighbourhood of the resonance masses. The energy dependence of $S(E)$ and $S'(E)$ can then be neglected according to assumption i). In the formulae which follow, the energy momentum δ function is factored out such that the matrix elements do not contain it any more. The ansatz (6) specifies the

assumption ii) given above. It rules out an incoherent admixture of states. An incoherent admixture would in fact not correspond to the physical picture of formation and decay via two single unstable particles. In our formulation only observable amplitudes occur, we avoid speaking of unitarity and TCP properties of the unstable states K_S, K_L or K_1, K_2 ^{*)}. The unitarity relation (5) provides for a connection between the primed and unprimed states. We find immediately

$$\begin{aligned} |S'\rangle &= \frac{1}{\Gamma_S} \langle S'S' | S \rangle + \sigma^* \langle L'S' | L \rangle \\ |L'\rangle &= \frac{1}{\Gamma_L} \langle L'L' | L \rangle + \sigma \langle S'L' | S \rangle \\ |S\rangle &= \frac{1}{\Gamma_S} \langle S | S \rangle |S'\rangle + \sigma \langle L | S \rangle |L'\rangle \\ |L\rangle &= \frac{1}{\Gamma_L} \langle L | L \rangle |L'\rangle + \sigma^* \langle S | L \rangle |S'\rangle \end{aligned} \quad (7)$$

Consistency between the first two with the last two equations in (7) requires

$$\begin{aligned} \langle S'S' \rangle &= \frac{\Gamma_S}{\Gamma_L} \frac{\langle L | L \rangle}{D}, \quad \langle L'L' \rangle = \frac{\Gamma_L}{\Gamma_S} \frac{\langle S | S \rangle}{D} \\ \langle L'S' \rangle &= -\frac{\sigma}{\sigma^*} \frac{\langle L | S \rangle}{D}, \quad D'D = 1 \\ D &= \frac{1}{\Gamma_S \Gamma_L} \langle S | S \rangle \langle L | L \rangle - |\sigma|^2 |\langle S | L \rangle|^2 \end{aligned} \quad (8)$$

Thus, as a consequence of unitarity, the states $|S'\rangle, |L'\rangle$ can be expressed in terms of the states $|S\rangle$ and $|L\rangle$. By taking the norm of the states $|S'\rangle, |L'\rangle$ equal to the norm of the states $|S\rangle, |L\rangle$, respectively, we have ^{**)}

$$\begin{aligned} \frac{\langle S | S \rangle}{\Gamma_S} &= \frac{\langle L | L \rangle}{\Gamma_L} = \sqrt{1 + |\lambda|^2} \\ \lambda &= \sigma \langle L | S \rangle = -\sigma^* \langle L' | S' \rangle \end{aligned} \quad (9)$$

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*) Nevertheless, the unstable mesons π and μ are used as initial and final states in the scattering process; we are only interested in the K problem near the K mass.

**) To take the norms equal is necessary for the interpretation and for the definition of the decay amplitudes [see Eq. (11)]. One may also consider it as an assumption to be included in ii).

and

$$\begin{aligned} |S'\rangle &= \sqrt{1+|\lambda|^2} |S\rangle - \lambda |L\rangle \\ |L'\rangle &= \sqrt{1+|\lambda|^2} |L\rangle - \lambda^* |S\rangle \end{aligned} \quad (10)$$

In order to illustrate the unitarity relations (9), let us introduce the amplitudes of the K_S and K_L particles for the decay into a scattering state $|j\rangle$. According to (1) and (4), they are given by

$$A_j^S = \langle j | S_0^{1/2} | S \rangle, \quad A_j^L = \langle j | S_0^{1/2} | L \rangle \quad (11)$$

We now have from (9), for example,

$$\Gamma_S = \frac{1}{\sqrt{1+|\lambda|^2}} \langle S | S \rangle = \frac{1}{\sqrt{1+|\lambda|^2}} \sum_j |A_j^S|^2 \quad (12)$$

Equations (9) also contain the Bell-Steinberger rule ⁴⁾

$$\begin{aligned} \langle L | S \rangle &= \sum_j A_j^{L*} A_j^S = \frac{\lambda}{\sigma} \\ &= \lambda \left(\frac{\Gamma_L + \Gamma_S}{2} - i(E_L - E_S) \right) \end{aligned} \quad (13)$$

A change of phase of the states $|S\rangle$, $|L\rangle$ introduces via (10) a corresponding phase change of $|S'\rangle$, $|L'\rangle$ leaving the operators T_S , T_L unaltered. We therefore may choose the parameter λ real :

$$\lambda = \sigma \langle L | S \rangle = \lambda^* \quad (14)$$

As long as TCP invariance is not assumed, the relations (9) remain the only strict requirements for the states $|S\rangle$ and $|L\rangle$. A possible breakdown of TCP invariance in connection with the $K-2\mu$ puzzle is discussed in Refs. 5) and 6).

II. CONSEQUENCES OF TCP INVARIANCE

To proceed, we now invoke TCP invariance for the scattering process [assumption iii].

It requires

$$\langle f | S | i \rangle = \langle \hat{i} | S | \hat{f} \rangle \quad (15)$$

for any two states i and f . We denote here the TCP conjugate of a state - which corresponds to the result of the TCP operation on a free state - by the symbol $\hat{}$. Using Eqs. (1), (4) and (6) and remembering the assumed energy independence of the operators S_0, T_S, T_L we find

$$\langle f | S_0 | i \rangle = \langle \hat{i} | S_0 | \hat{f} \rangle \quad (16)$$

and

$$\begin{aligned} & \langle f | S_0^{1/2} | S(L) \rangle \langle S'(L') | S_0^{1/2} | i \rangle \\ &= \langle \hat{i} | S_0^{1/2} | S(L) \rangle \langle S'(L') | S_0^{1/2} | \hat{f} \rangle \\ &= \langle \hat{S}(\hat{L}) | S_0^{1/2} | i \rangle \langle f | S_0^{1/2} | \hat{S}'(\hat{L}') \rangle \end{aligned} \quad (17)$$

Thus, the conclusion from TCP invariance is [besides Eq. (16)] the proportionality of the primed states with the TCP conjugate states of $|S\rangle$ and $|L\rangle$

$$|S'\rangle = e^{i\theta} |\hat{S}\rangle, \quad |L'\rangle = -e^{i\theta} |\hat{L}\rangle \quad (18)$$

In Eq. (18) the constraint (14) on (9) has been used. θ is a real parameter. We could take $\theta=0$ without changing the operators T_S, T_L . However, because of the conventional phase choices used in the description of K decays, we will keep this parameter.

To arrive now at the combined consequences of unitarity and TCP invariance we decompose the states $|S\rangle, |L\rangle$ into an orthonormal basis

$$\begin{aligned} |S\rangle &= \sum_i \alpha_i^S |e_i\rangle + \sum_i \beta_i^S |o_i\rangle \\ |L\rangle &= \sum_i \alpha_i^L |e_i\rangle + \sum_i \beta_i^L |o_i\rangle \end{aligned} \quad (19)$$

where the basis vectors have the property

$$\begin{aligned}
 CP|e_i\rangle &= |e_i\rangle & CP|o_i\rangle &= -|o_i\rangle \\
 |\hat{e}_i\rangle &= |e_i\rangle & |\hat{o}_i\rangle &= -|o_i\rangle \\
 \langle e_j|e_i\rangle &= \langle o_j|o_i\rangle = \delta_{ji} & , \quad \langle o_j|e_i\rangle &= 0
 \end{aligned} \tag{20}$$

In this way a classification of the scattering states with regard to their CP properties is performed. The coefficients in (19) are the decay amplitudes of the K_S and K_L mesons without the effect of the operator $S_0^{\frac{1}{2}}$. According to (10) and (18) these reduced decay amplitudes fulfil the relations

$$\begin{aligned}
 \frac{\alpha_i^L}{\alpha_i^S} &= \frac{1}{\lambda} \left(\sqrt{1+\lambda^2} - e^{i\theta} \frac{\alpha_i^{*S}}{\alpha_i^S} \right) = \frac{\lambda}{\sqrt{1+\lambda^2} + e^{i\theta} \frac{\alpha_i^{*L}}{\alpha_i^L}} \\
 \frac{\beta_i^S}{\beta_i^L} &= \frac{1}{\lambda} \left(\sqrt{1+\lambda^2} - e^{i\theta} \frac{\beta_i^{*L}}{\beta_i^L} \right) = \frac{\lambda}{\sqrt{1+\lambda^2} + e^{i\theta} \frac{\beta_i^{*S}}{\beta_i^S}}
 \end{aligned} \tag{21}$$

The formulae (16) and (21) contain the results of the TCP invariance requirement. The most remarkable point about Eq. (21) is the fact that the decay amplitudes of the K_S particle are obtainable from the decay amplitudes of the K_L particle.

Equation (21) justifies, in a more general way, the usual treatment in which one concludes from

$$\begin{aligned}
 |K_S\rangle &= \frac{1}{\sqrt{1+|\varepsilon|^2}} (|K_1\rangle + \varepsilon |K_2\rangle) \\
 |K_L\rangle &= \frac{1}{\sqrt{1+|\varepsilon|^2}} (|K_2\rangle + \varepsilon |K_1\rangle)
 \end{aligned} \tag{22}$$

and

$$\begin{aligned}
 \alpha_i^S &= \langle e_i | H_W | K_S \rangle = a_i + i\varepsilon b_i \\
 \alpha_i^L &= \langle e_i | H_W | K_L \rangle = i b_i + \varepsilon a_i \quad \text{etc.}
 \end{aligned}$$

that the quantities a_i, b_i are real, except for the effect of the final state interaction ⁷⁾. The matrix $S_0^{\frac{1}{2}}$, defined as the non-resonating (energy independent) part of the S matrix in Eq. (1) takes care of this latter effect [Eqs. (4), (11)]. A comparison of (21) with (22) gives λ and θ in terms of the conventional complex parameter ε occurring in (22) :

$$\lambda = \frac{\varepsilon + \varepsilon^*}{\sqrt{(1 - \varepsilon^2)(1 - \varepsilon^{*2})}} \quad , \quad e^{i\theta} = \left(\frac{1 - \varepsilon^2}{1 - \varepsilon^{*2}} \right)^{1/2} \quad (23)$$

Although λ and θ are very small quantities, it is clear from (21) that one cannot always approximate θ by zero and $\sqrt{1 + \lambda^2}$ by 1. The smallness of λ provides, however, for an interesting dependence of the relative phases of the reduced K_S and K_L decay amplitudes.

We sketch the dependence (21) of the phase of α_S on the phase of α_L in the Figure. In the region $-\pi/2 < \varphi(\alpha_L) < \pi/2$ the phase of α_S is for all practical purposes zero. In this region α_L/α_S is of order λ . At $\varphi(\alpha_L) \sim \pi/2$ $\varphi(\alpha_L)$ stays at this value, while $\varphi(\alpha_S)$ rises. In the latter region α_L/α_S is of order $1/\lambda$ describing a "large" CP violation. The region of phase values for which α_L/α_S is of order 1 is very small. It is situated around the edges of the curve with an extension of order λ *).

As stated before, the quantities α_i, β_i which we have just discussed are unfortunately not directly measurable in the decay processes but connected to the observed amplitudes via the matrix S_0 . Eliminating the α 's and β 's by using (11), (21), (23) we arrive at the desired relation between the amplitudes A_j^S and A_j^L

$$\begin{aligned} A_j^S &= \frac{1 - \varepsilon^2}{\varepsilon + \varepsilon^*} \left(A_j^L + \sum_k \langle j | S_0 | k \rangle \hat{k} \bar{A}_k^L \right) + \varepsilon A_j^L \\ A_j^L &= \frac{1 - \varepsilon^2}{\varepsilon + \varepsilon^*} \left(A_j^S - \sum_k \langle j | S_0 | k \rangle \hat{k} \bar{A}_k^S \right) + \varepsilon A_j^S \end{aligned} \quad (24)$$

The normalization of each amplitude is, as in previous formulae, such that the absolute square gives the partial transition rate. In the sum in (24) TCP conjugate states appear giving a sign change for CP odd contributions. According to our derivation, Eq. (24) is quite generally valid as long as TCP

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*) Thus, from a purely statistical point of view one would expect a very small probability for a decay channel to have $\alpha_S/\alpha_L \approx 1$. The channels would prefer $\alpha_L/\alpha_S \approx \lambda$ or possibly $\alpha_L/\alpha_S \approx 1/\lambda$ if coupled weakly enough to be compatible with (9).

invariance holds. The operator S_0 may still have CP violating parts. The question whether or not such CP violating terms show up already at the order of 10^{-3} is a very important one.

III. THE $K-2\mu$ and $K-2\gamma$ AMPLITUDES

The relation between K_S and K_L amplitudes given in Eq. (24) is of particular relevance for the $K-2\mu$ and $K-2\gamma$ decays. We will not give a thorough discussion on these decays here, a few remarks might be appropriate, however.

Since the $K-2\mu$ and $K-2\gamma$ amplitudes are very small ⁸⁾, the unitarity restrictions (9) are unessential for them and Eq. (24) is the only relation to be observed. Thus, contrary to the limiting case where CP is conserved, there is no lower bound for the $K_L \rightarrow \mu\mu$ amplitude itself. Instead, a very small amplitude will lead to a relatively large $K_S \rightarrow 2\mu$ transition probability ¹⁾. We will assume in the following CP conservation for S_0 and use in the sum in (24) only the $\mu\mu \pm$, $\gamma\gamma \pm$, $2\pi+$ and $3\pi-$ contributions which we will abbreviate by $\mu \pm$, $\gamma \pm$, $\pi \pm$. The $\pm$ sign refers to CP=+1 and CP=-1 channels, respectively.

By neglecting in addition obviously small terms, we obtain in this approximation, with $S_0 = 1 + 2i t_0$

$$\begin{aligned}
 A^S(\mu-) &\approx \frac{2}{\varepsilon + \varepsilon^*} \left(i \Im A^L(\mu-) - i \langle \mu- | t_0 | \gamma- \rangle \bar{A}^L(\gamma-) - i \langle \mu- | t_0 | \pi- \rangle \bar{A}^L(\pi-) \right. \\
 &\quad \left. + \varepsilon A^L(\mu-) \right), \\
 A^S(\mu+) &\approx \frac{2}{\varepsilon + \varepsilon^*} \left(\Re A^L(\mu+) + i \langle \mu+ | t_0 | \gamma+ \rangle \bar{A}^L(\gamma+) + i \langle \mu+ | t_0 | \pi+ \rangle \bar{A}^L(\pi+) \right. \\
 &\quad \left. + \varepsilon A^L(\mu+) \right), \quad (25) \\
 A^S(\gamma-) &\approx \frac{2}{\varepsilon + \varepsilon^*} \left(i \Im A^L(\gamma-) - i \langle \gamma- | t_0 | \mu- \rangle \bar{A}^L(\mu-) - i \langle \gamma- | t_0 | \pi- \rangle \bar{A}^L(\pi-) \right. \\
 &\quad \left. + \varepsilon A^L(\gamma-) \right), \\
 A^S(\gamma+) &\approx \frac{2}{\varepsilon + \varepsilon^*} \left(\Re A^L(\gamma+) + i \langle \gamma+ | t_0 | \mu+ \rangle \bar{A}^L(\mu+) + i \langle \gamma+ | t_0 | \pi+ \rangle \bar{A}^L(\pi+) \right. \\
 &\quad \left. + \varepsilon A^L(\gamma+) \right).
 \end{aligned}$$

Similar formulae are obtained by expressing the K_L amplitudes in terms of the K_S amplitudes. The first two equations give immediately the Christ-Lee triangle inequality ¹⁾ if the $\pi^\pm$ contribution in the L channel is neglected. The other less certain rules discussed by Christ and Lee are obtained by neglecting the complete π^+ contribution in (25), as well as in the formulae containing the S amplitudes on the right-hand side, i.e., in the S channel. If we keep, however, the π contributions other inequalities can be read off from the equations. For example, from the third formula in (25) follows the inequality

$$|\langle \gamma | t_0 | \pi^- \rangle| (\Gamma_{3\pi}^L)^{1/2} \lesssim \text{Re } \varepsilon (\Gamma_{2\gamma}^S)^{1/2} + (\Gamma_{2\gamma}^L)^{1/2} \quad (26)$$

One also finds

$$|\langle \gamma | t_0 | \pi^+ \rangle| (\Gamma_{2\pi}^S)^{1/2} \lesssim \text{Re } \varepsilon (\Gamma_{2\gamma}^L)^{1/2} + (\Gamma_{2\gamma}^S)^{1/2} + |\langle \gamma | t_0 | \mu^+ \rangle| (\Gamma_{\mu\mu}^S)^{1/2} \quad (27)$$

These equations are helpful for the estimate of the $\langle \mu^\pm | t_0 | \pi^\pm \rangle$ matrix element occurring in (25).

The author takes pleasure in thanking Professors J.S. Bell, M.K. Gaillard, J. Prentki and G.C. Wick for stimulating discussions.

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