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ON THE CONSTRUCTION OF PHYSICAL STATESIN THE DUAL RESONANCE MODEL

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A B S T R A C T

We are presenting a method of direct calculation of the states which satisfy the gauge conditions in the case of the intercept $\alpha_0 = 1$. This is done by applying a non-unitary one-to-one mapping on the space of the states created by the spatial components of the oscillators. We discuss the validity of this result.

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1. INTRODUCTION

In the case of the trajectory intercept equal to one ($\alpha_0 = 1$) in the dual resonance model, Virasoro ¹⁾ has pointed out the existence of an infinite number of Ward-like identities which imply that a large number of intermediate states actually do not contribute to the amplitude (spurious states). A great amount of work ^{2),3),4)} has recently been devoted to the analysis of the level structure in this case and, in particular, to try to prove that the Virasoro algebra is sufficient to eliminate all the ghosts (negative norm states) from the physical spectrum. This last point has been explicitly checked in Ref. 4) for the first eight daughter trajectories by means of numerical computations of the norm of the spurious states. Here, on the contrary, we shall try to construct directly the physical states $\{|\psi\rangle; L_n|\psi\rangle = 0, \forall n=1,2,\dots,\infty\}$. Our method is an extension of the one used in Ref. 3) to give in a closed form the solutions of, separately anyone of the equations $L_n|\psi\rangle = 0$: we want a solution of, simultaneously, all the equations $L_n|\psi\rangle = 0$.

We shall first give in Section 2, some general mathematical results and, in particular, Theorem 2 which will be used later. In Section 3, we shall exhibit our procedure and show that, in a neighbourhood of the mass-shell, the space of the solutions is obtained by applying a non-unitary one-to-one linear mapping on the space of the states created by the spatial components of the oscillators. Finally, in the last section, we shall discuss our results and make some concluding remarks concerning, in particular, the norms of the states.

2. GENERAL MATHEMATICAL RESULTS

Let $\mathcal{F}$ be a Fock space and $|\psi\rangle$ any vector of this space. $|\psi\rangle$ is of degree n if $|\psi\rangle = \sum_{\alpha} c_{\alpha} |\psi^{\alpha}\rangle$, $|\psi^{\alpha}\rangle = \prod_i (b^{(i)})^{\alpha_i} |0\rangle$ where $\alpha = \sum_i i \alpha_i$ and $n = \sup \alpha$. Let L be an operator acting on $\mathcal{F}$ and such that it lowers the degree of any vector of $\mathcal{F}$. Now, suppose there exists an operator C such that $[L, C] = 1$; we have the following theorem.

Theorem 1

Let X be an operator of finite degree in $\mathcal{F}$ (i.e., it transforms any vector of finite degree into a vector of finite degree) ⁵⁾, then

$$\tilde{X} = \lim_{n \rightarrow +\infty} X^{(n)} \quad \text{commutes with } L \text{ on } \mathcal{F}$$

where $X^{(n)} = X^{(n-1)} - \frac{C}{n} [L, X^{(n-1)}]$ and $X^{(0)} = X$ (A)

Moreover,

$$[L, X^{(n)}] = (-1)^n \frac{C^n}{n!} \underbrace{[L [L \dots [L, X] \dots]]}_{n+1} = (-1)^n \frac{C^n}{n!} (ad L)^{n+1} X \quad (B)$$

Proof

One has to check this formula on any $|\psi^\alpha\rangle$. For α fixed, there exists N_α such that $L^{N_\alpha}|\psi^\alpha\rangle = 0$ and M_α such that $L^{M_\alpha}X|\psi^\alpha\rangle = 0$ (X is of finite degree). Let us call $R = \sup(N_\alpha, M_\alpha)$, then, according to (B) $\tilde{X} = X^{(R)}$, since on $|\psi^\alpha\rangle$, $[\bar{L}, X^{(R)}] = 0$. Therefore, we only have to prove (B) and we do it by induction. Suppose that (B) holds up to $n-1$. Then,

$$\begin{aligned} [L, X^{(n)}] &= [L, X^{(n-1)}] - \frac{1}{n} [L, X^{(n-1)}] - \frac{C}{n} [L, X^{(n-1)}] \\ &= \frac{n-1}{n} (-1)^{n-1} \frac{C^{n-1}}{(n-1)!} (adL)^n X - \frac{C}{n} (-1)^{n-1} adL \left(\frac{C^{n-1}}{(n-1)!} (adL)^n X \right) \\ &= \frac{n-1}{n!} (-1)^{n-1} C^{n-1} (adL)^n X - \frac{n-1}{n!} (-1)^{n-1} C^{n-1} (adL)^n X \\ &\quad - \frac{C^n}{n!} (-1)^n (adL)^{n+1} X = (-1)^n \frac{C^n}{n!} (adL)^{n+1} X \end{aligned}$$

where we have used the distributivity of adL .

Let us now define

$$P_{L,C} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (adL)^n$$

which has a meaning as an operator on the Fock space. This operator $P_{L,C}$ has the following properties.

Lemma 1 X and Y being two operators in the Fock space, one has

- a) $(adL)P_{L,C}(X) = 0$
- b) $P_{L,C}(P_{L,C}(X)) = P_{L,C}(X)$ (projection property)
- c) $[\bar{L}, \tilde{X}] = 0 \iff$ there exists Y such that $X = P_{L,C}(Y)$
- c') Let $|\bar{\psi}\rangle$ be a vector in the Fock space
 $L|\bar{\psi}\rangle = 0 \iff$ there exists an operator φ such that $|\bar{\psi}\rangle = P_{L,C}(\varphi)|0\rangle$
- d) If $[\bar{C}, \tilde{X}] = 0$ then $P_{L,C}(XY) = P_{L,C}(X)P_{L,C}(Y)$ (factorization property)
- e) $P_{L,C}(X) = 0 \iff$ there exists Y such that $X = CY$
- f) For any X $P_{L,C}(CX) = 0$

Proof

Let us prove a). Indeed, we have

$$\begin{aligned} (\text{ad } L) P_{L,C}(X) &= (\text{ad } L) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (\text{ad } L)^n X \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} n C^{n-1} (\text{ad } L)^n X + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (\text{ad } L)^{n+1} X \\ &= -P_{L,C}([L, X]) + P_{L,C}([L, X]) = 0 \end{aligned}$$

From a), it follows b)

$$P_{L,C}(P_{L,C}(X)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (\text{ad } L)^n P_{L,C}(X) = P_{L,C}(X)$$

Let us prove c). If X is such that $(\text{ad } L)X=0$, then $P_{L,C}(X)=X$, which proves the existence of Y (which can be taken equal to X). The converse is a consequence of a). c') is the vector version of c). Indeed, there exists an operator ψ such that $|\bar{\psi}\rangle = \psi|0\rangle$ and $L|\bar{\psi}\rangle = 0$ is equivalent to $(\text{ad } L)\psi|0\rangle$, then the proof follows like in c). Let us now prove d).

$$\begin{aligned} P_{L,C}(XY) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (\text{ad } L)^n (XY) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n \sum_{k=0}^n \binom{n}{k} [(\text{ad } L)^k X] [(\text{ad } L)^{n-k} Y] \end{aligned}$$

Suppose that $(\text{ad } C)[(\text{ad } L)^r X] = 0$. Then, we can commute C^{n-r} with $(\text{ad } L)^r X$ and we get

$$P_{L,C}(XY) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k}{k!} C^k [(\text{ad } L)^k X] \frac{(-1)^{n-k}}{(n-k)!} C^{n-k} [(\text{ad } L)^{n-k} Y] = P_{L,C}(X) P_{L,C}(Y)$$

Therefore, it remains to prove that $(\text{ad } C)[(\text{ad } L)^r X] = 0$. For $r=0$ it is true by assumption; for $r=1$ it is a consequence of Jacobi's identity and the fact that $[L, C] = 1$. Suppose it holds up to $r-1$ ($r \geq 1$) and let us prove it for r . We have

$$(\text{ad } C)[(\text{ad } L)^r X] = \text{ad } C [\text{ad } L [(\text{ad } L)^{r-1} X]]$$

and the last expression is again zero by using Jacobi's identity. The proof of e) is obvious in one direction by using f). We have just to check that $P_{L,C}(X)=0$ leads to $X=CY$. From $P_{L,C}(X)=0$ we get

$$X = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C^n (\text{ad } L)^n X = CY$$

where

$$Y = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C^{n-1} (ad L)^n X$$

Finally, by using d) the proof of f) reduces to $P_{L,C}(C) = 0$. But

$$P_{L,C}(C) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^n (ad L)^n C \quad (ad L)^n C = \begin{cases} C & n=0 \\ 1 & n=1 \\ 0 & n>1 \end{cases}$$

Therefore, $P_{L,C}(C) = C - C = 0$.

We now restrict the generality of L by assuming that $L = B + \bar{L}$, where B is an annihilation operator times a c number and $[\bar{L}, B^+] = 0$, B^+ being proportional to the corresponding creation operator. We specialize also the choice of C by taking it to be proportional to B^+ , in such a way that $[\bar{B}, \bar{C}] = [\bar{L}, \bar{C}] = 1$. We can improve the property e) by:

g) If $[\bar{B}, \bar{X}] = 0$ then, $P_{L,C}(X) = 0 \iff X = 0$.

Let us prove this. From e) $X = CY$, then $[\bar{B}, \bar{X}] = 0$ is equivalent to $C[\bar{B}, \bar{Y}] = -Y$ that is to say $X = -C^2[\bar{B}, \bar{Y}]$. One can prove that for $N > 0$, $X = (-1)^N \frac{C^{N+1}}{N!} (ad B)^N Y$. Now let $|\psi^M\rangle$ be any vector of the Fock space of degree M . Since X is of finite degree $[\bar{Y}]$ is also of finite degree due to its expression given in the proof of e) and B is an annihilation operator there exists $N(M)$ such that $(ad B)^N Y |\psi^M\rangle = 0$ for $N \geq N(M)$. Therefore, $X |\psi^M\rangle = 0$ and since $|\psi^M\rangle$ is an arbitrary vector $X = 0$.

One sees from Lemma 1 and property g) that $P_{L,C}$ preserves the product law and has an empty kernel on the operators which do not contain B and C (more precisely the corresponding annihilation and creation operators). Let us go a little further in this direction.

Suppose we can find an invertible operator Z such that $Z^{-1} L Z = B$. Then there is a one-to-one correspondence between the X 's such that $[\bar{B}, \bar{X}] = 0$ and the U 's such that $[\bar{L}, \bar{U}] = 0$. More precisely $U = Z X Z^{-1}$. But according to property c) U is of the form $P_{L,C}(V)$ for some V and we get

$$Z X Z^{-1} = P_{L,C}(V)$$

If now Z is such that $P_{L,C}(Z X Z^{-1}) = P_{L,C}(X)$ one gets that

$$P_{L,C}(X) = P_{L,C}(V) = Z X Z^{-1}$$

Therefore, for any X such that $[\bar{B}, \bar{X}] = 0$, $P_{L,C}$ is implemented as an inner automorphism. Note that $P_{L,C}(X) = U$ has a unique solution such that $[\bar{B}, \bar{X}] = 0$; this is a direct consequence of property g).

We now show that it is possible to construct a Z such that the following conditions are satisfied

- 1) $ZBZ^{-1} = L$
- 2) $ZCZ^{-1} = C$
- 3) $P_{L,C}(ZCZ^{-1}) = P_{L,C}(X)$ if $[\bar{X}, \bar{B}] = 0$

The second condition has been added in order to preserve the commutation relation $[\bar{B}, \bar{C}] = [\bar{L}, \bar{C}] = 1$.

We will prove that it is possible to choose $Z = e^{-C\bar{L}} e^Y$ with $Y = \sum_{n=1}^{\infty} C^n Y_n$ and $[\bar{C}, Y_n] = 0 \quad \forall n \geq 1$. The condition 2) follows then and the condition 3) results from the fact that with such a Z , $ZXZ^{-1} = X + CY$. Therefore, we just have to prove the existence of a Z of such a form, satisfying condition 1). This reduces to the proof of the existence of Y such that

$$\begin{aligned} e^Y B e^{-Y} &= e^{C\bar{L}} L e^{-C\bar{L}} = \bar{L} + e^{C\bar{L}} B e^{-C\bar{L}} \\ &= \sum_{n=0}^{\infty} \frac{C^n}{n!} (\text{ad } \bar{L})^n B = \sum_{n=0}^{\infty} \frac{C^n}{n!} B_n \end{aligned}$$

$$B_0 = B \quad (\text{ad } \bar{L})^n B = B_n \quad n = 1, 2, \dots$$

Lemma 2 There exists $Y_1, Y_2, \dots$ such that $[\bar{C}, Y_n] = 0, \quad n = 1, 2, \dots, \infty$ and

$$e^{\sum_{n=1}^{\infty} C^n Y_n} B e^{-\sum_{n=1}^{\infty} C^n Y_n} = \sum_{n=0}^{\infty} \frac{C^n}{n!} B_n$$

Proof: it is tedious!

We have

$$(\text{ad } \sum_{n=1}^{\infty} C^n Y_n) B = \sum_{n=1}^{\infty} C^n (\text{ad } Y_n) B - n C^{n-1} Y_n$$

which commutes with C ; then

$$\begin{aligned} (\text{ad } \sum_{n=1}^{\infty} C^n Y_n)^m B &= \sum_{n_1, n_2, \dots, n_m} C^{n_1 + n_2 + \dots + n_m} \text{ad } Y_{n_1} \text{ad } Y_{n_2} \dots \text{ad } Y_{n_m} B \\ &\quad - \sum_{n_1, n_2, \dots, n_m} n_m C^{n_1 + n_2 + \dots + n_m - 1} \text{ad } Y_{n_1} \text{ad } Y_{n_2} \dots \text{ad } Y_{n_{m-1}} Y_{n_m} \end{aligned}$$

Therefore, we must have

$$B + \sum_{n=1}^{\infty} C^n \sum_{\substack{n_1 + \dots + n_m = n \\ m \geq 1 \\ n_i \geq 1}} \frac{1}{m!} \text{ad } Y_{n_1} \dots \text{ad } Y_{n_m} B - \sum_{n=0}^{\infty} C^n \sum_{\substack{n_1 + \dots + n_m = n+1 \\ m \geq 1 \\ n_i \geq 1}} \frac{n_m}{m!} \text{ad } Y_{n_1} \dots \text{ad } Y_{n_{m-1}} Y_{n_m} = \sum_{n=0}^{\infty} \frac{C^n}{n!} B_n$$

By identification of the coefficients of C^n we get, for instance

$$Y_1 = 0 \quad -2 Y_2 = B_1 \quad -3 Y_3 = \frac{B_2}{2!} + [Y_2, B]$$

This shows, in particular, that $n_i \geq 2$. We now see how the proof can be achieved. Supposing that we know Y_n ($n \leq N$) we will show that Y_{N+1} can be expressed in terms of the Y_n ($n \leq N$), B and B_n ($n \leq N$). This is obvious by looking at the coefficient of C^N which is

$$-(N+1) Y_{N+1} - \sum_{\substack{n_1 + \dots + n_m = N+1 \\ m \geq 2 \\ n_i \geq 2}} \frac{n_m}{m!} \text{ad} Y_{n_1} \dots \text{ad} Y_{n_{m-1}} Y_{n_m} + \sum_{\substack{n_1 + \dots + n_m = N \\ m \geq 1 \\ n_i \geq 2}} \frac{1}{m!} \text{ad} Y_{n_1} \dots \text{ad} Y_{n_m} B = \frac{1}{N!} B_N$$

but the two sums are expressed in terms of Y_n 's with $n \leq N$. In the same way $[C, B_n] = 0$ for $n \geq 1$, therefore, $[C, Y_2] = [C, Y_3] = 0$ and supposing that Y_n ($n \leq N$) commute with C , one deduces that $[C, Y_n] = 0$.

Let us make two remarks:

- 1) if the B_i 's are such that $[B_i, B_j] = 0 \quad \forall i, j$ then $Y_n = -(B_{n-1}/n!)$;
- 2) if B is a temporal mode, then C , the B_i 's and the Y_i 's are temporal.

To conclude this section let us summarize what we have proved.

Theorem 2

Let L be of the form $\bar{L} + B$ and let C be such that $[B, C] = [\bar{L}, C] = 1$.

Then:

- 1) any operator X which commutes with L can be written ZUZ^{-1} where $[B, U] = 0$ and Z is defined as above;
- 2) any vector $|\psi\rangle$ which is a solution of $L|\psi\rangle = 0$ can be written as $Z|\tilde{\psi}\rangle$, where $|\tilde{\psi}\rangle$ does not contain B^+ (i.e., C).

3. CONSTRUCTION OF THE SOLUTIONS OF $L_1|\psi\rangle = 0$ AND $L_2|\psi\rangle = 0$

As we have recalled in the Introduction the physical states are solutions of an infinite set of equations ²⁾

$$L_N |\psi\rangle = 0 \quad N = 1, 2, \dots, \infty \quad (1)$$

where in the rest frame of the resonances ⁶⁾

$$L_N = \gamma^{-1} b_4^{(N)} - \sum_{n=1}^{\infty} b^{+(n)} b^{(n+N)} + \frac{1}{2} \sum_{n=1}^{N-1} b^{(N-n)} b^{(n)} \quad \text{with} \quad \gamma^{-1} = i\sqrt{2}M$$

As it is well known, the equations (1) are not all independent due to the commutation relations

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{12} m(m^2-1) \delta_{m+n,0} \quad (2)$$

where d is the dimension of the space of the oscillators $(b_\mu^{(n)})$ $\mu = 0, 1, \dots, d-1$. Therefore, a physical state is a solution of $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$ only. As it has been shown in Ref. 3), and according to our general Theorem 2 of the previous Section, there exists an operator

$$Z_1 = e^{-C_1 \bar{L}_1} e^{\chi_1}$$

with

$$C_1 = -\gamma b_4^{+(1)}, \quad \bar{L}_1 = \sum_{n=1}^{\infty} b^{+(n)} b^{(n+1)}, \quad \chi_1 = -\sum_{n=2}^{\infty} \frac{C^n B_{n-1}}{n!} \quad (B_n = \gamma^{-1} n! b_4^{(n)})$$

such that

$$Z_1^{-1} L_1 Z_1 = \gamma^{-1} b_4^{(1)}$$

Therefore, the space of the solutions of $L_1|\psi\rangle = 0$ is $\tilde{H} = Z_1 H^{(1)}$ where $H^{(1)}$ is the initial Fock space where the oscillator $b_4^{+(1)}$ has been suppressed. This is only true if $\gamma^{-1} \neq 0$ (i.e., $M \neq 0$)⁷⁾. Next, we want to solve

$$L_2 \tilde{H} = 0 \quad (3)$$

or, equivalently

$$Z_1^{-1} L_2 Z_1 H^{(1)} = 0 \quad (3')$$

namely, a new operator $\hat{L}_2 = Z_1^{-1} L_2 Z_1$ on a smaller space $H^{(1)}$. The transformations laws for the creation and destruction operators are known series in $\gamma b_4^{+(1)}$, we need not record here. Therefore, we can compute $\hat{L}_2$ and put it under the form of a suitable expansion

$$\hat{L}_2 = \sum_{n=0}^{\infty} L_{2,n} (C_1)^n \quad (4)$$

Analogously to (4) we can define an expansion for each $\hat{L}_N = Z_1^{-1} L_N Z_1$

$$\hat{L}_N = \sum_{m=0}^{\infty} L_{N,m} (C_1)^m \quad (5)$$

Now, Equation (3') is equivalent to an infinite set of equations

$$L_{2,n} H^{(1)} = 0 \quad n = 0, 1, 2, \dots, \infty \quad (6)$$

and from $[\bar{L}_1, L_N] = (N-1)L_{N+1}$ we get

$$m L_{N,m} = (N-1) L_{N+1, m-1} \quad (7)$$

which shows the equivalence between the two sets of operators $\{L_{2,n}; n=0,1,\dots,\infty\}$ and $\{L_{N,0}; N=2,3,\dots,\infty\}$. On the other hand, Eq. (2) leads trivially to the commutation relations

$$[L_{N,0}, L_{N',0}] = (N-N') L_{N+N',0} + \frac{d}{12} N(N^2-1) \delta_{N+N',0} \quad (8)$$

We obtain

$$L_{2,0} = \gamma_2^{-1} b_4^{(2)} + \sum_{n=2}^{\infty} b_4^{+(n)} b_4^{(n+2)} - \sum_{n=1}^{\infty} \sum_{i=1}^{d-1} b_i^{+(n)} b_i^{(n+2)} - \frac{\gamma^2}{2} (\bar{L}_1^{(1)})^2 + \frac{1}{2} \sum_{i=1}^{d-1} b_i^{(1)} b_i^{(1)} \quad (9)$$

with

$$\gamma_2^{-1} = (2-\gamma^2)/2\gamma \quad \bar{L}_1^{(1)} = \sum_{n=2}^{\infty} b_4^{+(n)} b_4^{(n+1)} - \sum_{n=1}^{\infty} \sum_{i=1}^{d-1} b_i^{+(n)} b_i^{(n+1)}$$

$$(N \geq 3) \quad L_{N,0} = [\gamma^{-1} - (N-1)\gamma] b_4^{(N)} + \sum_{n=2}^{\infty} b_4^{+(n)} b_4^{(n+N)} + \gamma \bar{L}_1^{(1)} b_4^{(N-1)} + \sum_{i=1}^{d-1} b_i^{(1)} b_i^{(N-1)} - \frac{1}{2} \sum_{k=2}^{N-2} b_k b_k^{(N-k)} \quad (9')$$

The important remark to be made here is that this new set of operators does not contain $b_4^{+(1)}$ and then, instead of (6) we have to solve

$$L_{N,0} H^{(1)} = 0 \quad N=2,3,\dots,\infty$$

with the same algebra (8) for the $L_{N,0}$'s but where the time component of the mode $n=1$ has completely disappeared. We also want to stress that the new operators $L_{N,0}$ have the same form as the initial L_N apart from a modification of the coefficient of the linear term $b_4^{(N)}$ and the fact that $b_4^{(1)}$ has been replaced by $\gamma \bar{L}_1^{(1)}$. This last operator does not commute with $b_4^{(N-1)}$ and we have chosen to write $\gamma \bar{L}_1^{(1)}$ on the left of $b_4^{(N-1)}$. The term $-(N-1)\gamma b_4^{(N)}$ comes from the commutation of $b_4^{(1)}$ with $\gamma_1^{-1} b_4^{(N-1)} \gamma_1$ but we have

$$[b_4^{(1)}, \gamma_1^{-1} b_4^{(N-1)} \gamma_1] = \gamma_1^{-1} [\gamma_2 b_4^{(1)} \gamma_1^{-1}, b_4^{(N-1)}] \gamma_1 = \gamma_1^{-1} [\gamma L_1, b_4^{(N-1)}] \gamma_1$$

which gives $-(N-1)\gamma b_4^{(N)}$ to zero order in $b_4^{(1)}$.

Now, if $\gamma_2^{-1} \neq 0$ we know that there exists an operator γ_2 such that $\gamma_2^{-1} L_{2,0} \gamma_2 = \gamma_2^{-1} b_4^{(2)}$ so we can apply exactly the same method to eliminate completely the time-component of the mode $n=2$ and so on. By repeating this procedure m times if we call ${}^{(m)}L_N$, ($N > m$) the operator corresponding to L_N transformed successively by $\gamma_1, \gamma_2, \dots, \gamma_m$ we obtain ⁸⁾:

$$\boxed{N=2p}$$

$$(m < p) \quad {}^m L_N = (\gamma^{-1} + \sum_{k=1}^m \gamma_k \alpha_k^{N-k}) b_4^{(N)} + \sum_{n=m+1}^{\infty} b_4^{+(n)} b_4^{(n+N)} + \sum_{k=1}^m \gamma_k \bar{L}_k^{(m)} b_4^{(N-k)} - \frac{1}{2} \sum_{k=m+1}^{N-m-1} b_k b_k^{(N-k)} + (s.p.) \quad (10)$$

$$(m=p) \quad {}^{(p)}L_N = \gamma_N^{-1} b_4^{(N)} + \sum_{n=p+1}^{\infty} b_4^{+(n)} b_4^{(n+N)} + \sum_{k=1}^{p-1} \gamma_k \bar{L}_k^{(p)} b_4^{(N-k)} - \frac{(\gamma_p)^2}{2} (\bar{L}_p^{(p)})^2 + (s.p.)$$

$$(2p-1 \geq m > p) \quad {}^{(m)}L_N = \gamma_N^{-1} b_4^{(N)} + \sum_{n=m+1}^{\infty} b_4^{+(n)} b_4^{(n+N)} + \sum_{k=1}^{N-m-1} \gamma_k \bar{L}_k^{(m)} b_4^{(N-k)} \quad (11)$$

$$- \sum_{k=N-m}^{p-1} \gamma_{N-k} \gamma_k \bar{L}_k^{(m)} \bar{L}_{N-k}^{(m)} - \frac{(\gamma_p)^2}{2} (\bar{L}_p^{(m)})^2 + (s.p.) \quad (12)$$

$$\boxed{N=2p+1}$$

$$(m \leq p) \quad {}^{(m)}L_N \quad \text{is also given by Eq. (10)} \quad (10')$$

$$(2p \geq m > p) \quad {}^{(m)}L_N = \gamma_N^{-1} b_4^{(N)} + \sum_{n=m+1}^{\infty} b_4^{+(n)} b_4^{(n+N)} + \sum_{k=1}^{N-m-1} \gamma_k \bar{L}_k^{(m)} b_4^{(N-k)} - \sum_{k=N-m}^p \gamma_{N-k} \gamma_k \bar{L}_k^{(m)} \bar{L}_{N-k}^{(m)} + (s.p.) \quad (11')$$

To make these formulae understandable we need the following definitions. We have

$$a) \quad \bar{L}_N^{(N)} = (N-1) \bar{L}_N - \gamma_N^{-1} b_4^{(N)}$$

$$b) \quad \bar{L}_N^{(m)} \quad \text{with } m > N \text{ is the same operator as } \bar{L}_N^{(N)} \text{ but where the time-components of the models } 1, 2, \dots, m \text{ have been suppressed;}$$

$$c) \quad [{}^{(k-1)}L_k, b_4^{(m)}] = \alpha_k^m b_4^{(m+k)} + \dots$$

It is easy to check that we have the following relations which then completely determine the parameters we have introduced:

$$\gamma_{2p}^{-1} = \gamma^{-1} + \sum_{k=1}^{p-1} \gamma_k \alpha_k^{2p-k} + \frac{\gamma_p}{2} \alpha_p^p \quad (13a)$$

$$\gamma_{2p+1}^{-1} = \gamma^{-1} + \sum_{k=1}^p \gamma_k \alpha_k^{2p+1-k} \quad (13b)$$

$$\alpha_{2k}^n = - \left(n + \sum_{i=1}^{k-1} \gamma_i \gamma_{2k-i} \alpha_i^n \alpha_{2k-i}^{n+i} + \frac{(\gamma_k)^2}{2} \alpha_k^n \alpha_k^{n+k} \right) \quad (14a)$$

$$\alpha_{2k+1}^n = - \left(n + \sum_{i=1}^k \gamma_i \gamma_{2k+1-i} \alpha_i^n \alpha_{2k+1-i}^{n+i} \right) \quad (14b)$$

To conclude this section, we have shown the following result,

Any state $|\psi_N\rangle$ of degree N solution of $L_1|\psi_N\rangle=0$ and $L_2|\psi_N\rangle=0$ is of the form

$$|\psi_N\rangle = z_1 z_2 \dots z_N |\psi_N^{sp}\rangle \quad (15)$$

where $|\psi_N^{sp}\rangle$ is a state of degree N having the same parity, spin, etc... as $|\psi_N\rangle$ but constructed only with spatial components of the oscillators. This is only true if the existence of the z_i ($i=1,2,\dots,N$) is guaranteed, namely, if $\gamma_i^{-1} \neq 0$ ($i=2,3,\dots,N$), We shall discuss this point in the last section.

4. DISCUSSION AND CONCLUDING REMARKS

If one tries to compute explicitly the γ_i^{-1} as functions of γ , $i=2,3,\dots,\infty$ from Eqs. (13a,b) and (14a,b), the analytic expressions one obtains are quite simple up to $i=7$ and one sees that $\gamma_i^{-1} \neq 0$ except for some unphysical values of γ^2 (i.e., off the mass shell). For $i \geq 8$ the expressions become more complicated and it is necessary to use a computer. The results are indicated in Figs. 1 and 2: in the plots γ^2 versus N the two solid curves represent the boundaries of the shaded regions where the values of γ^2 for which $\gamma\gamma_N^{-1}=0$ occur; for each level N the dashed curve represents the corresponding physical value of γ^2 , $\gamma^2 = 1/(2(N-1))$. This extreme regularity up to $N=100$ ⁹⁾ is very encouraging and suggests that by our method we do obtain all the solutions of $L_1|\psi\rangle=0$ and $L_2|\psi\rangle=0$, this space having, therefore, the dimension of the space constructed with the spatial components of the oscillators.

We note that our method of construction of the solution is independent of the dimension d of the space, but d will play a role when we compute the norms of the solutions due to the commutation relations (2). As an illustration we give the only scalar state which occurs at $N=2$, namely¹⁰⁾

$$\begin{aligned} |\psi\rangle &= z_1 z_2 \sum_{i=1}^{d-1} b_i^{+(1)} b_i^{+(1)} |0\rangle \\ &= \left\{ \sum_{i=1}^{d-1} b_i^{+(1)} b_i^{+(1)} + \frac{\gamma_2}{2} (d-1) \tilde{b}_4^{+(2)} \right\} |0\rangle \end{aligned}$$

whose norm is

$$\langle \psi | \psi \rangle = 2(d-1) \left[1 - \frac{18^2 (d-1)}{(2-8^2)^2} (1-18^2) \right]$$

In this case $\gamma^2 = -\frac{1}{2}$, therefore, $\langle \psi | \psi \rangle = 0$ when $d-1=25$ and if $d>26$ this physical state becomes a ghost¹¹⁾. There are also states having a zero norm whatever the dimension d is; an example of such a scalar state is for $N=3$,

$$\begin{aligned} |\psi\rangle &= z_1 z_2 z_3 \sum_{i=1}^{d-1} b_i^{+(1)} b_i^{+(2)} \\ &= \int \sum_{i=1}^{d-1} b_i^{+(1)} \tilde{b}_i^{+(2)} + \frac{2}{3} (d-1) \delta_3 \tilde{b}_4^{+(3)} \rangle |0\rangle, \quad \delta_3 = \delta (1-2\delta^2)^{-1} \end{aligned}$$

whose norm is

$$\langle \psi | \psi \rangle = 2(d-1)(1-4\delta^2) \left(1 - \frac{2(d-1)\delta^2}{3(1-2\delta^2)^2}\right)$$

which is zero since for $N=3$, $\gamma^2 = -\frac{1}{4}$.

Finally, it is readily seen that the operator $Z = z_1 z_2 \dots z_N$ is not unitary so we cannot conclude on the sign of the norm of the solutions. In fact, from the presence of zero norm states in the solutions we did not expect Z to be unitary but one could ask whether it is now possible to characterize the zero norm states and, in particular, to check if the conjecture about their form which was made in Ref. 4) is correct.

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- 5) This will be an implicit condition from now on.
- 6) From now on we shall use the notations of Ref. 3).
- 7) This situation corresponds to the state $|\psi\rangle = b_4^{+(1)}|0\rangle$ which is a solution of $L_1|\psi\rangle = 0$ for $M=0$ but at the same time a spurious state of zero norm.
- 8) As we have seen in the previous example the spatial part of these operators does not play any role, so, to simplify the notations we have recorded its presence by the symbol (s.p.).
- 9) The computation time is so small that we could easily push the analysis further on.
- 10) Recall that like in Ref. 3), $\tilde{b}^{+(n)} = \beta_1 b^{+(n)} \beta_1^{-1}$.
- 11) This strange value was also pointed out in a different context by C. Lovelace in "Pomeron Form Factors and Dual Regge Cuts", Rutgers University preprint (1971).

FIG.1

FIG. 2