Archives

DUAL MODEL SPECTRUM ANALYSIS, USING VIRASORO ALGEBRA REPRESENTATIONS

P.H. Frampton and H.B. Nielsen *)
CERN - Geneva

A B S T R A C T

The spectrum of states in the unit intercept Veneziano model is studied by noticing that there is exactly one physical state in each irreducible representation of the Virasoro algebra for spacelike momenta. The squared norms of the continued physical states are fully enumerated. A method of analytic continuation from the spacelike to the timelike domain is described.

As an illustrative example of its usefulness, the lower limit of Del Guidice, Di Vecchia and Fubini for the number of positive squared norm physical states on-mass-shell is rederived ; the states obtained in this way are of definite spin.

*) On leave of absence from Niels Bohr Institute, Copenhagen.

1. - INTRODUCTION

In 1969, Virasoro ¹⁾ discovered that if in the n point Veneziano model the intercept was taken to one, an infinite number of new gauge conditions appeared. This discovery is hoped to lead to recognition of all ghost states in the formalism as decoupled because of these gauge conditions. The gauge operators are expressed in terms of a Lie algebra

$$[L_m, L_n] = (n-m) L_{m+n} + s \delta_{m,-n} \quad (1.1)$$

It is this algebra which we are calling the Virasoro algebra.

In this paper we show that the Veneziano model Fock space (the Fock space built up from the infinite set of harmonic oscillators in the operatorial factorization of the model) considered as a Virasoro algebra representation is fully reducible for spacelike momenta. In each of the irreducible representations of which the Fock space is a direct sum, there is exactly one suitably defined physical state $[\emptyset$ state defined in Eq. (2.6)]. This is used to calculate the number of positive and negative squared norm $\emptyset$ states at each L_0 eigenvalue.

Our analysis of the Fock space may be regarded as a modification of the analysis made by Brower and Thorn ²⁾ who have described how to generate the spurious subspace of the full Fock space.

By analytic continuation in four-momentum the full reducibility is seen to occur for almost all values of the momentum. For certain discrete values of squared timelike momentum, however, one irreducible representation can jump into another one; in this case, all the states in the former representation, including its $\emptyset$ state, acquire zero norm.

As an example of the utility of the method, we derive a lower limit $d_N^{(2)}$ for the number of positive squared norm physical states ($\emptyset$ states) for timelike momenta for the level with L_0 eigenvalue N greater than that of the ground state for the same four-momentum. Here the number $d_N^{(2)}$ is defined as being the coefficient of X^N in the power series expansion for $p^2(X) = \prod_{r=1}^{\infty} (1 - X^r)^{-2}$.

Del Giudice, Di Vecchia and Fubini ³⁾ obtained the same lower limit for the number of positive squared norm physical states by a different approach.

Linear representations of the subalgebra of the Virasoro algebra corresponding to the group $SU(1,1)$ homomorphic to $SU(2,1) \approx SL(2,R)$ have been classified by Barut and Fronsdal ⁴⁾ although the full algebra does not seem to have been investigated (see, however, Guillemin and Sternberg ⁵⁾).

The organization of the article is as follows. In Section 2 the full reducibility and the result - one $\emptyset$ state in each irreducible representation - are proved for spacelike momenta. The pathologies connected with the one representation jumping into another are listed in Section 3. Section 4 is devoted to an example of an application. Finally, in Section 5, we mention our hopes for future applications of the methods put forward.

2. - SPACELIKE MOMENTA

In this section we are considering the Veneziano model Fock space as a Virasoro algebra representation. It is shown that, for spacelike momenta ^{*)}, this Fock space can be completely decomposed into irreducible representations each containing exactly one suitably defined physical state. This is done in a suitable positive definite metric (Γ). Within each of these irreducible representations all states have the same sign for the squared norm with respect to the conventional metric.

This fact enables us to count the positive norm and the negative norm "physical states" very simply for spacelike momenta.

In Fig. 1a, the clearcut manner in which the states drop into irreducible representations is pictured.

For spacelike momenta in a frame where p_μ has no time component (e.g., $p_\mu = p(0;0,0,1)$) we introduce the positive definite metric

*) The idea of studying the structure of the "spurious subspace" (irreducible representation) as a function of p^2 is contained in the paper of Brower and Thorn ²⁾. Here it was pointed out that for spacelike p the spurious subspace has a definite metric, and that as p^2 varies into the timelike region some norms must pass through zero and change sign. This passage of the norm through zero corresponds to the collision of representations.

$$\Gamma = (-1)^{\sum_{r=1}^{\infty} a_0^{(r)\dagger} a_0^{(r)}} \quad (2.1)$$

This commutes with $L_n(p_\mu)$,

$$[\Gamma, L_n(p_\mu)] = 0 \quad (2.2)$$

where $L_n(p_\mu)$ are the (on-shell) gauge operators defined by

$$L_n = -\sqrt{2n} \, p \cdot a^{(n)\dagger} - \frac{1}{2} \sum_{r=1}^{n-1} \sqrt{r(n-r)} \, a^{(r)\dagger} \cdot a^{(n-r)\dagger} - \\ - \sum_{r=1}^{\infty} \sqrt{r(n+r)} \, a^{(r+n)\dagger} \cdot a^{(r)} \quad (2.3)$$

for $n = 1, 2, \dots$,

$$L_{-n} = L_n^\dagger$$

and

$$L_0 = -p^2 - \sum_{r=1}^{\infty} r a^{(r)\dagger} \cdot a^{(r)}$$

which satisfy the Virasoro algebra

$$[L_m, L_n] = (n-m) L_{m+n} + s \delta_{m,-n} \quad (2.4)$$

with

$$s = \frac{1}{3} n(n^2-1) \quad (2.5)$$

We define continued physical states (or $\emptyset$ states) by the conditions

$$L_{-n} |\phi\rangle = 0 \quad (2.6)$$

for $n = 1, 2, \dots$

On-mass-shell, i.e., for

$$(L_0 - 1) | \rangle = 0 \quad (2.7)$$

the states $|\emptyset\rangle$ are accurately the physical states, Eq. (2.6) being the gauge conditions ^{*}). The other states, i.e., the states orthogonal to all the "physical states" ($\emptyset$ states) are called σ states.

We now shall show how, in the spacelike region, each $\emptyset$ state is in its own irreducible representation of the Virasoro algebra, all other of the infinite number of states in the representation being σ states. Consider the operation of an operator ²⁾

$$\sigma_z = \prod_i (L_i)^{a_i} \quad (2.8)$$

on a $\emptyset$ state. By successively using the condition (2.6) and the commutation relations (2.4) the product in the operator σ_z can be reduced so that only non-negative i are included. Further it can be shown that all of the states resulting from the operation of σ_z are linearly independent ²⁾. Therefore, by operation with all possible σ_z on a $\emptyset$ state, we obtain a tower of states; these form (as will be shown) an irreducible representation of the Virasoro algebra, all states except for the lowest lying being σ states.

^{*}) Off-shell there is strictly speaking no meaning in the term "physical state". In the literature physical states are often defined off-mass shell by the conditions

$$W_{-n} | \text{phys.} \rangle = 0$$

for $n = 1, 2, \dots$, where

$$W_{-n} = L_{-n} - L_0 - (n-1)$$

and this is necessary if one uses conventional propagators and vertices, but by modifying these operators (even only off-shell) one may make the $\emptyset$ states the physical ones.

Let $\sum_i i a_i = \xi$, then if the starting $\emptyset$ state has L_0 eigenvalue c we have

$$L_0 \sigma_\xi |\phi, c\rangle = (c + \xi) \sigma_\xi |\phi, c\rangle \quad (2.9)$$

The number of linearly independent σ_ξ is $d^{(1)}_\xi$, the partition of ξ (i.e., the number of ways in which ξ can be written as an unordered sum of positive integers).

Since the action of the generators L_n on the states of the form $\sigma_\xi |\emptyset, c\rangle$ is given completely by the group commutation rules and the L_0 eigenvalue c of the $\emptyset$ state, it is seen that the irreducible representations occurring in the Veneziano model spectrum are isomorphic when they have the same c values. For fixed momentum, of course, the c values are only equal for representations having their $\emptyset$ states at the same level above the vacuum (the ground state). By varying the momentum, the representations for different levels can become isomorphic.

Now we show that each irreducible representation of the Virasoro algebra contains only one $\emptyset$ state. We begin with the $\emptyset$ state $|\emptyset, c\rangle$ with eigenvalue of L_0 equal to c . Now operate with an σ_ξ having $\xi = 0$. Such an σ_ξ can be reduced to a combination of c numbers and L_0 so that acting on $|\emptyset, c\rangle$ gives only a c number times the same state.

Because of the Γ metric, we have

$$\langle \phi, c | \Gamma | \phi, c \rangle > 0 \quad (2.10)$$

and, for $\xi \geq 1$,

$$\langle \phi, c | (\sigma_\xi)^\dagger \Gamma \sigma_\xi | \phi, c \rangle > 0 \quad (2.11)$$

Now $[\bar{L}_r, \Gamma] = 0$ and therefore $[\sigma_\xi, \Gamma] = 0$. We can show that no $\sigma_\xi |\emptyset, c\rangle$ state ($\xi \geq 1$) can be a $\emptyset$ state by reductio ad absurdum. Suppose that it were a $\emptyset$ state

$$|\phi', c + \xi\rangle = \sigma_\xi |\phi, c\rangle$$

and

$$L_{-r} |\phi', c + \xi\rangle = 0 \quad (2.12)$$

all $r = 1, 2, 3, \dots$ Then

$$\langle \phi', c + \xi | \Gamma | \phi', c + \xi \rangle = \langle \phi, c | (\sigma_{\xi})^{\dagger} \Gamma \sigma_{\xi} | \phi, c \rangle = 0 \quad (2.13)$$

which is in contradiction with (2.10). This proves that all $\sigma_{\xi} |\phi, c\rangle$ with $\xi \geq 1$ are σ states. To prove that any linear combination of states of the form $\sigma_{\xi} |\phi, c\rangle$ must be a σ state, we note that if

$$|g\rangle = \sum_{i=1}^q \sigma_{\xi_i}^{(i)} |\phi_i, c_i\rangle \quad (2.14)$$

were a ϕ state and if the irreducible representations generated by the $|\phi_i, c_i\rangle$ have been chosen orthogonal then $L_{-r}|g\rangle = 0$ implies that

$$L_{-r} \sigma_{\xi_i}^{(i)} |\phi_i, c_i\rangle = 0 \quad (2.15)$$

as before.

In practice, we can select a basis in the Fock space where all ϕ and σ states are eigenvectors of Γ (with eigenvalue ± 1). It is straightforward then to see that all states in the same irreducible representation have the same eigenvalue of Γ for if

$$\Gamma |\phi, c, \pm\rangle = \pm |\phi, c, \pm\rangle \quad (2.16)$$

then

$$\begin{aligned} \Gamma \sigma_{\xi} |\phi, c, \pm\rangle &= \sigma_{\xi} \Gamma |\phi, c, \pm\rangle \\ &= \pm \sigma_{\xi} |\phi, c, \pm\rangle \end{aligned} \quad (2.17)$$

States with $\Gamma = +1$ have positive norm in the conventional metric, those with $\Gamma = -1$ have negative norm. This enables us to count the positive norm states in the spacelike region, both for $\emptyset$ and σ states. The total number $d_N^{(4)}$ of states in the Fock space spanned by the four-dimensional creation and annihilation operators $a_\mu^{(r)}$, $a_\mu^{(r)+}$ is given at the N th level by the coefficient of x^N in

$$p^4(x) = \prod_{r=1}^{\infty} (1-x^r)^{-4} = \sum_{N=0}^{\infty} d_N^{(4)} x^N \quad (2.18)$$

The number $d_N^{(1)}$ of states at the $\{$ th level (i.e., with L_0 eigenvalue $c + \{$) in the relevant irreducible representations is the coefficient of x in $p(x) = \prod_{r=1}^{\infty} (1-x^r)^{-1}$. Now let $d_S^{(\emptyset)}$ be the number of $\emptyset$ states at the level S (i.e., with eigenvalue S of $-\sum_n a^{(n)+} \cdot a^{(n)}$), then counting total number of states at a level N gives

$$d_N^{(4)} = \sum_{S=0}^N d_S^{(\emptyset)} d_{N-S}^{(1)} \quad (2.19)$$

The solution of this equation (2.19) is obtained by noticing

$$\sum_{N=0}^{\infty} d_N^{(4)} x^N = \left(\sum_{S=0}^{\infty} d_S^{(\emptyset)} x^S \right) \left(\sum_{\{=0}^{\infty} d_{\{ }^{(1)} x^{\{ } \right) \quad (2.20)$$

which gives

$$\sum_{S=0}^{\infty} d_S^{(\emptyset)} x^S = \frac{p^4(x)}{p(x)} = p^3(x) = \prod_{r=1}^{\infty} (1-x^r)^{-3} \quad (2.21)$$

The number of $\emptyset$ states with positive squared norm in the conventional metric is the number $d_N^{(3,+)}$ with an even number of time components. By a similar method the appropriate generating functions are proved to be

$$p^2(x) p(x^2) = \sum_{N=0}^{\infty} d_N^{(3,+)} x^N \quad (2.22)$$

for positive norm $\emptyset$ states

$$p^2(x) (p(x) - p(x^2)) = \sum_{N=0}^{\infty} d_N^{(3,-)} x^N \quad (2.23)$$

for the number $d_N^{(3,-)}$ of negative squared norm $\emptyset$ states.

It is interesting to observe that, in the spacelike region, the $\emptyset$ states are predominantly of negative squared norm since the asymptotic forms for high levels $N \rightarrow \infty$ are

$$d_N^{(3,+)} \underset{N \rightarrow \infty}{\sim} \exp\left(\sqrt{\frac{5}{3}} \pi \sqrt{N}\right) \quad (2.24)$$

$$d_N^{(3,-)} \underset{N \rightarrow \infty}{\sim} \exp\left(2 \pi \sqrt{N}\right) \quad (2.25)$$

respectively.

3. - TIMELIKE MOMENTA

The structure of the decomposition of the Fock space into irreducible representations is exactly the same as for spacelike momenta for almost all values of p^2 . For certain discrete values of p^2 , however, one irreducible representation may jump into another one. This collision of representations is intimately connected with the occurrence of zero norm $\emptyset$ states. Examples of representation collisions are shown in Figs. 1b, 1c and 1d.

In this section we prove the above statements by studying the possible pathologies for the structure of the representations. These pathologies can occur in the timelike or lightlike region because we cannot choose a metric operator Γ so that it commutes with all the group generators L_n .

In fact it is possible that the representations become reducible, but not fully reducible ^{*)} by the existence of certain new types of state, either :

- (i) a $\emptyset$ state which can be reached by group raising operators from a lower-lying $\emptyset$ state (i.e., one with lower L_0 eigenvalue) ;
- (ii) a $\emptyset$ state which cannot be reached by group raising operators from any lower-lying state.

^{*)} Reducible : there is a subrepresentation. Fully reducible : the representation can be written as a direct sum of subrepresentations that are themselves not reducible.

Now we prove that states of types (i) and (ii) can always be chosen as zero-norm states. Consider first a $\emptyset$ state of type (i), i.e.,

$$|\phi', c+\xi\rangle = \sigma_{\xi} |\phi, c\rangle \quad (3.1)$$

such that

$$L_{-r} |\phi', c+\xi\rangle = 0 \quad (3.2)$$

all $r = 1, 2, 3, \dots$. Then

$$\begin{aligned} \langle \phi', c+\xi | \phi', c+\xi \rangle &= \langle \phi, c | (\sigma_{\xi})^{\dagger} | \phi', c+\xi \rangle \\ &= 0, \end{aligned} \quad (3.3)$$

as required. Note also that all states which may be reached by operation of σ_{ξ} on the zero norm $\emptyset$ state also have zero norm.

Next consider the possible existence of a σ state $|\psi\rangle$ in the Fock space, which cannot be reached from below by raising operators, i.e., case (ii). Since $|\psi\rangle$ is a σ state we must have

$$L_{-\xi} |\psi, c+\xi\rangle = |\phi, c\rangle \quad (3.4)$$

for some ξ , where $|\phi, c\rangle$ is a $\emptyset$ state. Consider two cases :

(a) $\langle \phi, c | \phi, c \rangle \neq 0$

$$\begin{aligned} \therefore \langle \psi, c+\xi | L_{\xi} |\phi, c\rangle &= \langle \psi, c+\xi | \lambda, c+\xi \rangle \\ &\neq 0 \end{aligned} \quad (3.5)$$

where $|\lambda, c+\xi\rangle$ is a state accessible from below. If the space of states of type $\{|\lambda\rangle\}$ which can be reached by raising operators from below is non-degenerate then we can make a $|\psi, c+\xi\rangle$ state orthogonal to all

$|\lambda, c+\frac{1}{2}\rangle$ states (by, for example, Gram-Schmidt orthogonalization) which then contradicts Eq. (4.5). Therefore there must exist a zero-norm state in $\{|\lambda\rangle\}$, whose complement ^{*}) is a zero-norm $|\psi\rangle$ state.

(b) $\langle \phi, c | \phi, c \rangle = 0.$

In this case we consider the complement $|\phi, c\rangle_{\perp}$ of $|\phi, c\rangle$ such that

$$\langle \phi, c | \phi, c \rangle_{\perp} = 0$$

$$\langle \phi, c | \phi, c \rangle_{\perp} \neq 0$$

$$\therefore \langle \psi, c+\frac{1}{2} | L_{\frac{1}{2}} | \phi, c \rangle_{\perp} = \langle \psi, c+\frac{1}{2} | \lambda, c+\frac{1}{2} \rangle \neq 0 \quad (3.6)$$

which is similar to Eq. (3.5) so that again $|\psi\rangle$ can be chosen as a zero-norm state.

Zero-norm states of both types (i) and (ii) occur in the Veneziano model spectrum. The simplest examples occur at the first excited level of the unit-intercept amplitude. Taking momentum $p_{\mu} = p(1,0,0,1)$, then the four states contained in $a_{\mu}^{(1)+} |0\rangle$ can be chosen as :

$$\left. \begin{array}{l} a_x^{(1)+} |0\rangle \\ a_y^{(1)+} |0\rangle \end{array} \right\}$$

positive-norm physical states

$$(a_z^{(1)+} - a_t^{(1)+}) |0\rangle$$

zero-norm physical state [case (i)]

$$(a_z^{(1)+} + a_t^{(1)+}) |0\rangle$$

zero-norm spurious state [case (ii)]

This example is illustrated in Fig. 1b.

^{*}) We define the complement of a lightlike vector as that vector obtained by reversing the sign of the time component.

4. - PRELIMINARY STUDY OF SQUARED NORMS FOR TIMELIKE MOMENTA

In this section we prove ^{*}) the existence of at least $d_N^{(2)}$ positive squared norm $\emptyset$ states for timelike momenta (in particular physical states on-shell) at the level $-\sum_{n=1}^{\infty} n a^{(n)+} \cdot a^{(n)} = N$ where $d_N^{(2)}$ is the coefficient of x^N in the power series

$$p^2(x) = \prod_{r=1}^{\infty} (1-x^r)^{-2} = \sum_{N=0}^{\infty} d_N^{(2)} x^N \quad (4.1)$$

These positive squared norm physical states all have definite spin.

A - An algorithm for spacelike momenta

We first shall give an algorithm for construction of the $\emptyset$ states in the spacelike region. Take $p_\mu = (0,0,0,p_z)$, and let c be the L_0 eigenvalue of the ground state vacuum. At level $L_0 = (c+N)$ we may choose a basis of states in the Fock space such that all basis vectors are eigenstates of Γ

^{*}) This result can be proved more directly. Consider the frame in which

$p_\mu = p(1;0,0,0)$. We define operators $\mathcal{L}_n$ by :

$$\mathcal{L}_n = \frac{1}{2} \sum_{r=1}^{n-1} \sqrt{r(n-r)} \underline{a}^{(r)+} \cdot \underline{a}^{(n-r)+} + \sum_{r=1}^{\infty} \sqrt{r(n+r)} \underline{a}^{(r+n)+} \cdot \underline{a}^{(r)}$$

$$\mathcal{L}_{-n} = \mathcal{L}_n^\dagger$$

$$\mathcal{L}_0 = -p^2 - \sum_{r=1}^{\infty} r \underline{a}^{(r)+} \cdot \underline{a}^{(r)}$$

where $\underline{a}^{(r)+}$, $\underline{a}^{(r)}$ denote the three-vector creation, annihilation operators respectively. These operators can be considered the Virasoro algebra generators in a Veneziano model in a three-dimensional Euclidean space for zero momentum. Since the arguments in section 2 for spacelike momentum do not depend on the dimension (and Γ is redundant for definite metric) we can fully reduce the Fock space and obtain the number $d_N^{(2)}$ of physical states. The corresponding states (with no time excitations) in the four-dimensional Minkowski space are physical, because annihilation of such states by $\mathcal{L}_{-n}$ implies annihilation by L_{-n} .

$$\Gamma |\lambda, N+c, \pm\rangle = \pm |\lambda, N+c, \pm\rangle \quad (4.2)$$

The vacuum is a $\emptyset$ state $|\emptyset, c, +\rangle$ and acting on it with all σ_z operators we form σ states $|\sigma, c+\xi, +\rangle$. At level $L_0 = c+1$ the states are $|\sigma, c+1, +\rangle = L_{+1}|\emptyset, c, +\rangle$, two $\emptyset$ states $|\emptyset, c+1, +\rangle$ and one $|\emptyset, c+1, -\rangle$. Similarly we proceed at the level $L_0 = (c+N)$ to construct $\emptyset$ states which are eigenstates of Γ as follows. Construct all σ states by acting on lower $\emptyset$ states by

$$|\sigma, c+N, \pm\rangle = \sigma_{N-\xi} |\emptyset, c+\xi, \pm\rangle \quad (4.3)$$

Take a Fock space state $|\lambda, c+N, +\rangle_1$ and by a Gram-Schmidt orthogonalization procedure successively orthogonalize to all states $|\sigma, c+N, +\rangle$, by means of

$$|\lambda, c+N, +\rangle_1 \rightarrow \left\| |\sigma, c+N, +\rangle_i \right\|^2 |\lambda, c+N, +\rangle_1 - \sum_i \langle \sigma, c+N, + | \lambda, c+N, + \rangle_i |\sigma, c+N, +\rangle_i \quad (4.4)$$

The remainder after orthogonalization is $|\emptyset, c+N, +\rangle_1$, a $\emptyset$ state. Repeat with all $|\lambda, c+N, +\rangle_1$ until there is a complete set of $\emptyset$ states in the form

$$|\emptyset, c+N, +\rangle_i = \sum_j a_{ij}^{(+)}(p) |\lambda, c+N, +\rangle_j \quad (4.5)$$

Similarly the $\Gamma = -1$ subspace can be treated to give analytic expressions for the states in that subspace :

$$|\emptyset, c+N, -\rangle_i = \sum_j a_{ij}^{(-)}(p) |\lambda, c+N, -\rangle_j \quad (4.6)$$

The coefficients $a_{ij}^{\pm}(p)$ are polynomials in the momentum p , so that if we analytically continue in the momentum we can have linear dependences of the states $|\emptyset, c+N, \pm\rangle_i$ only for a finite number of discrete values of p [where $\det(a_{ij}^{\pm}(p)) = 0$].

B - Continuation to timelike momenta

Having fully understood all the norms of $\emptyset$ and σ states for spacelike momenta (see Section 2), we shall now continue the $\emptyset$ states constructed by our algorithm. This enables us to construct states for timelike momentum.

The transition to the timelike region is performed in two stages :

- (i) $p_{\mu} = (0,0,0,p_z) \rightarrow p_{\mu} = (0,0,0,ip_z)$ which takes spacelike $p^2 < 0$ into timelike $p^2 > 0$;
- (ii) $p_z \rightarrow -ip_t, p_t \rightarrow ip_z$ (a complex Lorentz transformation) making finally $p_{\mu} = (p,0,0,0)$.

During the first step no norms of states are changed, since the Γ operator still provides us with a redefinition of the scalar product which is positive definite. For momentum $p_{\mu} = (0,0,0,ip_z)$ the number of positive and negative norm states thus remains equal to those cited for the spacelike region at the end of Section 3.

On step (ii), however, the norms of $|\emptyset, c+N, \pm \rangle_i$ flip according to whether there is an odd or even number of z excitations in the spacelike region. The new norms of the $\emptyset$ states $|\emptyset, c+N, \pm \rangle_i$ thus depend in detail on the $a_{ij}^{(\pm)}(p)$ coefficients.

We may focus attention on those $\emptyset$ states which have no z excitations in the spacelike region, but only t, x, y excitations. These become positive norm physical states in the timelike region because they have only x, y and z but not t excitations. This is true also on-mass-shell since we may begin with $(p_z)^2 = m^2$ for the level concerned. The number of such states is given by the generating function $p^2(x)$.

The result of the Section 3 was that linear dependence of states from different irreducible representations can only occur when there is a zero norm state. By construction the present states have positive squared norm. So no linear dependence is possible.

To illustrate the method we construct the first excited levels of this type by the analytic continuation method.

We start with $N = 0$ which is the vacuum state and build up the $\emptyset$ and σ states at each level. For $N = 1$ the four-momentum is lightlike on-mass-shell so we start with spacelike $p_\mu = (0; 0, 0, \epsilon)$. The σ state at this level is

$$|\sigma, c+1, +\rangle = a_z^{(0)\dagger} |\phi, c, +\rangle \quad (4.7)$$

and the three $\emptyset$ states obtained are

$$|\phi, c+1, -\rangle = a_t^{(1)\dagger} |\phi, c, +\rangle \quad (4.8)$$

$$|\phi, c+1, +\rangle_1 = a_x^{(1)\dagger} |\phi, c, +\rangle \quad (4.9)$$

$$|\phi, c+1, -\rangle_2 = a_y^{(1)\dagger} |\phi, c, +\rangle \quad (4.10)$$

The states (4.9) and (4.10) are those which become the positive norm physical states on-mass shell. As mentioned already the other two Fock space states at this level become a complementary pair of zero-norm states on-mass shell.

The $N = 2$ level is less trivial and the rotational symmetry becomes more evident. At this level the five σ states obtained by raising operators from below become [for spacelike momentum $p_\mu = (0; 0, 0, 1)$]

$$|\sigma, c+2, +\rangle_1 = (4 a_z^{(2)\dagger} - a_t^{(1)\dagger} a_z^{(1)\dagger}) |\phi, c, +\rangle \quad (4.11a)$$

$$|\sigma, c+2, +\rangle_2 = (a_x^{(2)\dagger} + a_x^{(1)\dagger} a_z^{(1)\dagger}) |\phi, c, +\rangle \quad (4.11b)$$

$$|\sigma, c+2, +\rangle_3 = (a_y^{(2)\dagger} + a_y^{(1)\dagger} a_z^{(1)\dagger}) |\phi, c, +\rangle \quad (4.11c)$$

$$|\sigma, c+2, +\rangle_4 = (a_z^{(2)\dagger} + a_z^{(1)\dagger} a_z^{(1)\dagger}) |\phi, c, +\rangle \quad (4.11d)$$

$$|\sigma, c+2, -\rangle = (a_t^{(2)\dagger} + a_t^{(1)\dagger} a_z^{(1)\dagger}) |\phi, c, +\rangle \quad (4.11e)$$

which span the $\Gamma = \pm \sigma$ subspaces respectively. Gram-Schmidt orthogonalization for $\Gamma = \pm$ gives the $\emptyset$ states

$$|\phi, c+2, +\rangle_1 = a_x^{(1)+} a_y^{(1)+} |\phi, c, +\rangle \quad (4.12a)$$

$$|\phi, c+2, +\rangle_2 = (a_x^{(1)+} a_x^{(1)+} - a_y^{(1)+} a_y^{(1)+}) |\phi, c, +\rangle \quad (4.12b)$$

$$|\phi, c+2, +\rangle_3 = (a_x^{(1)+} a_x^{(1)+} + a_y^{(1)+} a_y^{(1)+} - 2a_z^{(1)+} a_z^{(1)+}) |\phi, c, +\rangle \quad (4.12c)$$

$$|\phi, c+2, +\rangle_4 = (a_x^{(1)+} a_z^{(1)+} - a_z^{(2)+}) |\phi, c, +\rangle \quad (4.12d)$$

$$|\phi, c+2, +\rangle_5 = (a_y^{(1)+} a_z^{(1)+} - a_z^{(2)+}) |\phi, c, +\rangle \quad (4.12e)$$

$$|\phi, c+2, +\rangle_6 = (8a_z^{(2)+} - 4a_x^{(1)+} a_x^{(1)+} + 5a_x^{(1)+} a_x^{(1)+} + 5a_y^{(1)+} a_y^{(1)+} + 10a_z^{(1)+} a_z^{(1)+}) |\phi, c, +\rangle \quad (4.12f)$$

$$|\phi, c+2, -\rangle_1 = a_x^{(1)+} a_t^{(1)+} |\phi, c, +\rangle \quad (4.12g)$$

$$|\phi, c+2, -\rangle_2 = a_y^{(1)+} a_t^{(1)+} |\phi, c, +\rangle \quad (4.12h)$$

$$|\phi, c+2, -\rangle_3 = (a_z^{(1)+} a_t^{(1)+} - a_t^{(2)+}) |\phi, c, +\rangle \quad (4.12i)$$

Of these the states (4.12a,b,c,g,h) contain no z excitation and may be continued to the mass shell. We recognize the five components of a spin-two traceless tensor. The missing positive norm physical state of the full spectrum at this level is a spin-zero state ⁶⁾.

Similarly at $N = 3$ the analytic continuation gives 10 states, corresponding to the components of one spin 3 and one spin 1 particle.

5. - DISCUSSION

A quite different technique has been used by Del Guidice, Di Vecchia and Fubini ³⁾ to obtain a set of positive squared norm physical states. Del Guidice et al. construct their physical states by operating by the vertex for a zero-mass vector particle ("photon"). They obtain the same number of states per level as we have done. The states obtained in their way are not of definite spin.

Some of the content of the present article can be extracted from the excellent paper of Brower and Thorn ²⁾. The analysis of our Section 2 closely parallels their Appendix (our term "irreducible representation" corresponds to their "spurious subspace").

Although we have up to now only used our method to rederive a lower bound on the number of positive norm physical states (and even the states we obtained can be found more directly), we are optimistic that the approach using Virasoro algebra representations will prove fruitful in attacking the central problem of showing that there are no negative squared norm physical states in the model. More generally we are tempted to speculate that any covariant formulation of a dual model may have the feature that each physical state may carry with it a tower of spurious states belonging to a unique representation of the Virasoro algebra or a similar algebra. In this sense the Virasoro algebra is not an essential feature of the physical states of the model, but is merely a reflection of the redundancy of the formalism ^{*)}. Although the fully covariant formulation may remain the most elegant, it is quite possible to envisage an alternative one where spurious states never enter from the beginning and where the Virasoro algebra plays no role.

ACKNOWLEDGEMENTS

The authors would like to thank their colleagues of the CERN Theoretical Study Division, especially Peter Goddard and David Olive, for stimulating and challenging discussions.

^{*)} It seems suggested by Ramond's work ⁷⁾ that the redundancy arises because of the want to have a covariant formalism for particles with spin.

REFERENCES

- 1) M.A. Virasoro - Phys.Rev. D1, 2933 (1970).
- 2) R.C. Brower and C.B. Thorn - Nuclear Phys. B31, 183 (1971).
- 3) E. Del Giudice, P. Di Vecchia and S. Fubini - MIT Preprint CTP 209 (1971).
- 4) A.O. Barut and C. Fronsdal - Proc.Roy.Soc. A287, 532 (1965).
- 5) V.W. Guillemin and S. Sternberg - Bull.Amer.Math.Soc. 70, 16 (1964).
- 6) P. Di Vecchia and E. Del Giudice - Nuovo Cimento 70A, 579 (1970).
- 7) P. Ramond - Nuovo Cimento 4A, 544 (1971).

FIGURE CAPTIONS

Figure 1a Illustration of the clean reduction of the Veneziano model Fock space for spacelike momenta. The $\emptyset$ states ("physical states", i.e., $L_{-n}|\emptyset\rangle = 0$) are represented by squares ($\blacksquare$), and the ∇ states ("spurious states") by triangles ($\blacktriangledown$). Solid lines delimit the irreducible subspaces. The signs within the symbols denote the signs of the squared norms defined by the conventional metric. The ordinate is the L_0 eigenvalue and the numerical coefficients denote how many times the given irreducible representation of a particular signature occurs at that L_0 eigenvalue.

Figure 1b Illustration of the reduction of the Veneziano model Fock space for $p^2 = 0$ (lightlike).
A wiggly arrow indicates that one can go the way indicated by the arrow, but not the opposite way, by operating with the group generators.
Comparing with Fig. 1a, we can say that the negative squared norm state at level $N = 1$ has jumped into the representation belonging the $N = 0$ $\emptyset$ state (the ground state). It has been replaced by a zero-norm $|\psi\rangle$ state.

Figure 1c Same as Fig. 1b but for timelike momentum $p^2 = 1$ and $p^2 = 2$ respectively. [We note that the pathologies (and the associated zero-norm states) discussed in Section 3 of the text can occur also for non-integer positive values of p^2 .]

55044

FIG.1