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ANOMALOUS BEHAVIOUR OF A MASSIVE SPIN TWO CHARGED
PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

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A B S T R A C T

A critical study of the behaviour of a massive spin two charged particle in an external electromagnetic field is made. There is a unique Lagrangian which does not increase the number of degrees of freedom required for the description of a spin two particle. However, other unsatisfactory features are hidden in the Lagrangian equations. They possess solutions which either fail to propagate or propagate at a speed greater than the speed of light.

1. INTRODUCTION

The problem of describing the electromagnetic interactions of charged elementary particles, in particular of the hadrons, is of considerable importance. A recent review treatment of some aspects of these problems reaches conclusions valid for particles of any spin ¹⁾. However, the study of the motion of a massive charged spin $\frac{3}{2}$ particle in an external electromagnetic field ²⁾ seems to indicate that higher spin interacting particles can behave quite differently from lower spin ones. What happens is that the solutions of the interacting Rarita-Schwinger equation either do not propagate, or they propagate at a speed greater than the speed of light. Similar results are obtained in a number of other examples with interactions not restricted to the electromagnetic one ³⁾⁻⁵⁾.

The situation for a massive charged spin two particle in external electromagnetic field is more confused. The spin two Lagrangian, as it is written in Wentzel's book ^{6),*)}, with minimal interaction, has a loss of constraint, i.e., in the region where the electromagnetic field is different from zero there are six degrees of freedom instead of the five required for the spin two ^{3),7)}. This pathology has been remedied by Federbush ⁷⁾ by addition of a suitable non-minimal term to the original Lagrangian. However, no explicit proof has been given that the new Lagrangian has re-acquired the right number of degrees of freedom.

Here we want to analyze in detail some of the properties of this new Lagrangian. We will prove that the number of degrees of freedom is in fact five, but that loss of propagation and non-causal behaviour are present.

The methods employed to study the propagation properties of the equations are explained in Ref. 8), and described and used in Refs. 2)-5) to which we refer for more details. Here we summarize the procedure applied to the present situation.

For a second order system of N partial differential equations (PDE) with N unknown functions, let us denote, in matrix notation, by

$$A^{\rho\sigma} \partial_\rho \partial_\sigma u$$

the second order terms, where u is a N column vector and $A^{\rho\sigma}$ are sixteen N by N matrices, depending on the ρ, σ indices. The characteristic determinant $D(n)$ is by definition

*) This Lagrangian and others for higher spin particles was first written down by M. Fierz and W. Pauli, Proc.Roy.Soc.(London) A173, 211 (1939).

$$D(n) = |A^{\rho\sigma} n_\rho n_\sigma|$$

where n_ρ is a four-vector. To verify that the system of PDE is hyperbolic, i.e., describes propagation, one must check that all the solutions n^0 of the algebraic equation

$$D(n) = 0$$

are real, for any given real $\vec{n}$. The maximum velocity of propagation is then given by $|n^0|/|\vec{n}|$.

Unfortunately immediate application of this method to the Lagrangian equations of a spin two particle with or without interaction does not lead anywhere, because $D(n)$ is identically zero. However, the initial Lagrangian system of equations will be shown to be equivalent to a new system of PDE supplemented by constraints. The above analysis will then be successfully applied to this new system.

In Section 2 the starting equations are discussed. Section 3 contains the treatment of the constant electromagnetic field, whereas the general field case is discussed in Section 4. Most of the computations and proofs are collected in Appendices A, B and C.

2. EQUATIONS OF A SPIN TWO PARTICLE IN AN ELECTROMAGNETIC FIELD

We will adopt throughout the paper the second order formalism, and describe the free charged massive spin two particle with the following Lagrangian ⁶⁾ :

$$\begin{aligned} L = & (p_\lambda \psi_{\mu\nu})^\dagger p^\lambda \psi^{\mu\nu} - 2(p_\lambda \psi_{\mu\nu})^\dagger p^\mu \psi^{\lambda\nu} + (p^\mu \psi_{\mu\nu})^\dagger p^\nu \psi \\ & + (p_\mu \psi)^\dagger p_\lambda \psi^{\mu\lambda} - (p_\mu \psi)^\dagger p^\mu \psi - m^2 [(\psi^{\mu\nu})^\dagger \psi_{\mu\nu} - \psi^\dagger \psi] \end{aligned} \quad (2.1)$$

where $\psi^{\mu\nu}$ is a symmetric tensor,

$$\psi \equiv \psi^\rho{}_\rho$$

and $p_\mu \equiv i\partial_\mu$.

The introduction of the electromagnetic interaction into the Lagrangian (2.1) via the minimality requirement

$$p_\mu \longrightarrow \pi_\mu = p_\mu + e A_\mu \quad (2.2)$$

gives rise to an unsatisfactory theory. Two of the constraints deduced from the variational equations, coming from the Lagrangian (2.1) in which the substitution (2.2) has been performed, disappear and we are left with a description of a particle with six degrees of freedom ³⁾.

To avoid this difficulty, let us consider the following Lagrangian :

$$\begin{aligned} L = & (\pi_\lambda \psi_{,\nu})^\dagger \pi^\lambda \psi^{\nu} - 2(\pi_\lambda \psi_{,\nu})^\dagger \pi^\lambda \psi^{\lambda\nu} + (\pi^\lambda \psi_{,\nu})^\dagger \pi^\nu \psi + (\pi_\mu \psi)^\dagger \pi_\lambda \psi^{\lambda\mu} \\ & - (\pi_\mu \psi)^\dagger \pi^\mu \psi - m^2 [(\psi_{,\nu})^\dagger \psi^{\nu} - \psi^\dagger \psi] + 2\alpha (\psi_{,\nu})^\dagger F^{\mu\nu} \psi_\mu \end{aligned} \quad (2.3)$$

and try to fix the value of α in such a way to have a description of a particle with five degrees of freedom. It turns out that this is a possible way out ⁷⁾, if α has the value $-\frac{1}{2}ie$.

To see how this comes about, let us write the variational equations obtained from the Lagrangian (2.3) :

$$\begin{aligned} L^{\mu\nu} \equiv & \pi^\mu \psi^{\nu} - \pi_\lambda (\pi^\lambda \psi^{\lambda\nu} + \pi^\nu \psi^{\lambda\mu}) + \frac{1}{2} (\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) \psi \\ & + g^{\mu\nu} (\pi^\rho \pi_\rho \psi - \pi^2 \psi) - m^2 (\psi^{\mu\nu} - g^{\mu\nu} \psi) + \alpha (F^\mu_\rho \psi^{\rho\nu} + F^\nu_\rho \psi^{\rho\mu}) = 0. \end{aligned} \quad (2.4)$$

They form a second order system of PDE. Examination of Eqs. (2.4) shows that the equations

$$L^{\mu 0} = 0 \quad (2.5)$$

form four constraints; that is, equations not containing second order time derivatives. Other constraints result from the divergence of $L^{\mu\nu}$:

$$\begin{aligned} C^\nu \equiv & -m^{-2} \pi_\mu L^{\mu\nu} = \pi_\mu \psi^{\mu\nu} - \pi^\nu \psi - ie m^{-2} \left[F_{\mu\lambda} \pi^\lambda \psi^{\mu\nu} + (F^\nu_\mu \pi_\lambda + \pi_\lambda F^\nu_\mu) \psi^{\lambda\mu} \right. \\ & \left. + \left(\frac{1}{2} \pi_\mu F^{\mu\nu} + F^{\mu\nu} \pi_\mu \right) \psi \right] - \alpha m^{-2} \pi_\mu \left[F^\mu_\rho \psi^{\rho\nu} + F^\nu_\rho \psi^{\rho\mu} \right] = 0. \end{aligned} \quad (2.6)$$

In the free case ($e=0$, $\alpha=0$) the equation

$$\begin{aligned} \pi_\nu \pi_\mu L^{\mu\nu} + \frac{1}{2} m^2 L^\mu{}_\mu &\equiv (ie + 2\alpha) \pi_\nu \pi_\mu (F^{\mu\lambda} \psi^\nu) + \frac{3}{2} m^4 \psi \\ &- \frac{3}{2} (ie)^2 \left[F_{\mu\nu} F^\nu{}_\rho \psi^{\mu\rho} - \frac{1}{2} F_{\nu\mu} F^{\mu\nu} \psi \right] + ie \pi_\nu \left[(i\partial^\mu F_{\mu\lambda}) \psi^{\lambda\nu} - (i\partial^\mu F_\mu{}^\nu) \psi \right] \\ &- ie \pi_\nu \left[(i\partial_\lambda F^\nu{}_\mu) \psi^{\lambda\mu} \right] = 0 \end{aligned} \quad (2.7)$$

and its first time derivative constitute two more constraints. Together with Eqs. (2.5) and (2.6) they reduce the number of independent components to the five required to describe a spin two particle. In the interacting case, a trivial modification of the arguments given in Ref. 3) shows that any choice different from $\alpha = -\frac{1}{2}ie$ leads to the wrong number of degrees of freedom. This value fixes uniquely the Lagrangian since only the magnetic dipole interaction, with a well-defined value of the coupling constant, can cancel the unwanted second derivative terms. For future convenience, we will rewrite the equations (2.4), (2.6) and (2.7) inserting for α the value $-\frac{1}{2}ie$, and using, for simplicity, the same symbols. No confusion should arise since, from now on, we will always work with the above fixed value of α .

$$\begin{aligned} L^{\mu\nu} &\equiv \pi^2 \psi^{\mu\nu} - \pi_\lambda (\pi^\mu \psi^{\lambda\nu} + \pi^\nu \psi^{\lambda\mu}) + \frac{1}{2} (\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) \psi \\ &+ g^{\mu\nu} (\pi^\rho \pi^\lambda \psi_{\rho\lambda} - \pi^2 \psi) - m^2 (\psi^{\mu\nu} - g^{\mu\nu} \psi) - \frac{1}{2} ie (F^\mu{}_\rho \psi^{\rho\nu} + F^\nu{}_\rho \psi^{\rho\mu}) = 0 \end{aligned} \quad (2.8)$$

$$\begin{aligned} C^\nu &\equiv -m^{-2} \pi_\mu L^{\mu\nu} = \pi_\mu \psi^{\mu\nu} - \pi^\nu \psi - ie m^{-2} \left[F_{\mu\lambda} \pi^\lambda \psi^{\mu\nu} + (F^\nu{}_\mu \pi_\lambda + \pi_\lambda F^\nu{}_\mu) \psi^{\mu\lambda} \right. \\ &\left. + \left(\frac{1}{2} \pi_\mu F^{\mu\nu} + F^{\mu\nu} \pi_\mu \right) \psi \right] + \frac{1}{2} ie m^{-2} \pi_\mu \left[F^{\mu\rho} \psi_\rho{}^\nu + F^\nu{}_\rho \psi^{\rho\mu} \right] = 0 \end{aligned} \quad (2.9)$$

$$\begin{aligned} \pi_\nu \pi_\mu L^{\mu\nu} + \frac{1}{2} m^2 L^\mu{}_\mu &\equiv \frac{3}{2} m^4 \psi - \frac{3}{2} (ie)^2 \left[F_{\mu\nu} F^\nu{}_\rho \psi^{\mu\rho} - \frac{1}{2} F_{\nu\mu} F^{\mu\nu} \psi \right] \\ &+ ie \pi_\nu \left[(i\partial^\mu F_{\mu\lambda}) \psi^{\lambda\nu} - (i\partial^\mu F_\mu{}^\nu) \psi \right] - ie \pi_\nu \left[(i\partial_\lambda F^\nu{}_\mu) \psi^{\lambda\mu} \right] = 0. \end{aligned} \quad (2.10)$$

The problem with constant external electromagnetic field looks like the free one, since in both cases in Eq. (2.10) only terms with no derivative in $\psi^{\mu\nu}$ appear. Because it is really simpler than the general case, we think it is worth while to give an independent treatment of it in the next section.

As regards the propagation properties of the Ψ waves satisfying Eq. (2.8), a simple computation shows that the characteristic determinant is identically zero. The reason is the presence of the constraints $L^{\mu 0} = 0$.

The analysis of the following two sections will allow us at the same time to determine the number of independent components and to investigate the propagation properties of the Ψ 's.

3. CONSTANT ELECTROMAGNETIC FIELD

In this section, the electromagnetic field is taken to be constant. Then Eq. (2.10) becomes :

$$\pi_\nu \pi_\mu L^{\nu\mu} + \frac{1}{2} m^2 L^\mu{}_\mu = \frac{3}{2} m^4 \psi - \frac{3}{2} (ie)^2 \left[F_{\mu\nu} F^\nu{}_\rho \psi^{\mu\rho} - \frac{1}{2} F_{\mu\nu} F^{\nu\mu} \psi \right] = 0. \quad (3.1)$$

Therefore, if we introduce the notation

$$D \equiv \frac{2}{3} m^{-4} \left(\pi_\nu \pi_\mu L^{\nu\mu} + \frac{1}{2} m^2 L^\mu{}_\mu \right) \quad (3.2)$$

the 10 equations

$$\begin{aligned} L^{\mu 0} &= 0 \\ C^\nu &= 0 \\ D &= 0 \\ \pi_0 D &= 0 \end{aligned} \quad (3.3)$$

are constraints since they do not contain any second time derivative. Consider now the following second order system of equations

$$L^{\mu\nu} + \pi^\nu C^\mu + \pi^\mu C^\nu + \frac{1}{2} (\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) D = 0 \quad (3.4)$$

which is a trivial consequence of Eqs. (2.8) and (3.3). Any solution of Eq. (2.8) satisfies Eq. (3.4). Conversely, it is shown in Appendix A that any solution of Eq. (3.4) supplemented at a given initial time by the constraints (3.3) satisfies the initial Eq. (2.8). This shifts the problem of propagation from Eq. (2.8) to Eq. (3.4), which can be written explicitly as :

$$\begin{aligned}
 & \pi^\lambda \psi^{\lambda\nu} - \frac{3}{2} i e m^{-2} \left[\pi^\mu \pi_\sigma \left(F^\nu{}_\lambda \psi^{\lambda\sigma} - F^\sigma{}_\lambda \psi^{\lambda\nu} \right) + \pi^\nu \pi_\sigma \left(F^\mu{}_\lambda \psi^{\lambda\sigma} - F^\sigma{}_\lambda \psi^{\lambda\mu} \right) \right] \\
 & - i e m^{-2} \left[\frac{1}{2} \left(\pi^\mu \pi_\lambda F^{\lambda\nu} + \pi^\nu \pi_\lambda F^{\lambda\mu} \right) \psi + \left(\pi^\mu F_\lambda{}^\nu \pi^\lambda + \pi^\nu F_\lambda{}^\mu \pi^\lambda \right) \psi \right] \quad (3.5) \\
 & - \frac{1}{2} (i e)^2 m^{-4} \left(\pi^\mu \pi^\nu + \pi^\nu \pi^\mu \right) \left(F_{\lambda\sigma} F^\sigma{}_\rho \psi^{\lambda\rho} - \frac{1}{2} F_{\lambda\sigma} F^{\sigma\lambda} \psi \right) - i e \left(F_\lambda{}^\mu \psi^{\lambda\nu} + F_\lambda{}^\nu \psi^{\lambda\mu} \right) = 0.
 \end{aligned}$$

The characteristic determinant of the system (3.5) can be computed and is equal to :

$$D(n) = (n^2)^6 \left[n^2 - (e m^{-2})^2 (F^d \cdot n)^2 \right] \left[n^2 + \left(\frac{3}{2} e m^{-2} \right)^2 (F^d \cdot n)^2 \right] \quad (3.6)$$

where

$$F^d{}_{\rho\sigma} = \frac{1}{2} \varepsilon_{\rho\sigma\tau\nu} F^{\tau\nu} \quad (3.7)$$

and

$$(F^d \cdot n)^2 = F^d{}_{\rho\sigma} n^\sigma F^d{}_{\rho\lambda} n^\lambda \quad (3.8)$$

The discussion is now the same as the one for the Rarita-Schwinger equation ²⁾. Because of the last factor in Eq. (3.6), if there is a frame of reference in which the magnitude of the magnetic field is smaller than $\frac{2}{3} m^2 e^{-1}$, the equation $D(n)=0$ has timelike solutions. If there is no such frame, $D(n)=0$ has solutions with real $\vec{n}$ and complex n^0 . In the first case causality is lost, in the second hyperbolicity is lost. The non-causal effects cannot be eliminated by the imposition of the constraints at an initial time, because they propagate causally (see Appendix A).

The computation of the number of degrees of freedom is now simple. Our initial system of equations (2.8) is equivalent to the system (3.5), supplemented by the constraints (3.3). The system (3.5) allows 20 free initial data. They must be restricted by the 10 constraints (3.3). Therefore we have for the system (2.8) $20-10=10$ free initial data, as required for a spin two particle.

4. GENERAL ELECTROMAGNETIC FIELD

In the general case in which $F^{\mu\nu}$ is no longer constant, the system (3.4) becomes a third order system and $\pi_0 D$ acquires terms with second derivatives. Consequently a procedure different from the one of Section 3 must be employed.

We decompose the solution $\psi^{\mu\nu}$ of Eq. (2.8) in the following way :

$$\psi^{\mu\nu} = \varphi^{\mu\nu} + \pi^\mu \chi^\nu + \pi^\nu \chi^\mu + \frac{1}{2} (\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) \Omega \quad (4.1)$$

with

$$\pi \cdot \chi \equiv \pi^\mu \chi_\mu = 0 \quad (4.2)$$

$$(\pi \cdot \varphi)^\nu \equiv \pi_\mu \varphi^{\mu\nu} = 0 \quad (4.3)$$

and with $\varphi^{\mu\nu}$ symmetric in the indices μ and ν . It is easy to convince oneself that such a decomposition always exists. However, for a given $\psi^{\mu\nu}$ it is not unique. There is a rather wide gauge freedom, analyzed in Appendix B. The result is that the gauge functions are solutions of a certain system of PDE, supplemented by constraints, and that the number of independent initial gauge functions that can be given on a $t=0$ surface is 10.

Upon substitution of Eq. (4.1) into Eqs. (2.8), (2.9) and (2.10), one obtains, using Eqs. (4.2) and (4.3) the following second order system of PDE :

$$G^{\mu\nu} = 0 \quad (4.4)$$

$$S^\nu = 0 \quad (4.5)$$

$$H = 0. \quad (4.6)$$

Here

$$\begin{aligned} G^{\mu\nu} \equiv & \pi^2 \varphi^{\mu\nu} + \pi^\mu \pi^\nu \varphi - g^{\mu\nu} \pi^2 \varphi + \frac{1}{2} i e F^{\nu\mu} \varphi - m^2 (\varphi^{\mu\nu} - g^{\mu\nu} \varphi) \\ & + \frac{1}{2} i e (F^\mu{}_\rho \varphi^{\rho\nu} + F^\nu{}_\rho \varphi^{\rho\mu}) + \frac{3}{2} i e (F^\rho{}_\mu \pi_\rho \chi^\mu + F^\rho{}_\nu \pi_\rho \chi^\nu + F^\mu{}_\lambda \pi^\nu \chi^\lambda + F^\nu{}_\lambda \pi^\mu \chi^\lambda) \end{aligned}$$

$$\begin{aligned}
 & + ie \left[(i \partial_\rho F^{\rho\mu}) \chi^\nu + (i \partial_\rho F^{\rho\nu}) \chi^\mu + (i \partial^\mu F^\nu{}_\lambda) \chi^\lambda + (i \partial^\nu F^\mu{}_\lambda) \chi^\lambda \right] \\
 & + ie g^{\mu\nu} \left[3 F_\rho{}^\lambda \pi_\lambda \chi^\rho + 2 (i \partial_\lambda F_\rho{}^\lambda) \chi^\rho \right] - m^2 \left[\pi^\mu \chi^\nu + \pi^\nu \chi^\mu \right] \quad (4.7) \\
 & - m^2 \left[\frac{1}{2} (\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) - g^{\mu\nu} \pi^2 \right] \Omega + g^{\mu\nu} \left[\frac{3}{4} (ie)^2 F_{\sigma\rho} F^{\sigma\rho} + ie (i \partial^\rho F_{\sigma\rho}) \pi^\sigma \right] \Omega \\
 & + \frac{1}{2} (ie) \left[(i \partial_\rho F^{\rho\mu}) \pi^\nu + (i \partial_\rho F^{\rho\nu}) \pi^\mu + (i \partial^\nu F^\mu{}_\lambda) \pi^\lambda + (i \partial^\mu F^\nu{}_\lambda) \pi^\lambda \right] \Omega \\
 & - \frac{3}{2} (ie)^2 F^\mu{}_\lambda F^{\lambda\nu} \Omega
 \end{aligned}$$

$$\begin{aligned}
 S^\nu & \equiv - \frac{3}{2} ie F^\mu{}_\lambda \pi_\mu \varphi^{\lambda\nu} + \frac{1}{2} ie \left[(i \partial^\mu F_{\lambda\mu}) \varphi^{\lambda\nu} + (i \partial_\mu F^\nu{}_\lambda) \varphi^{\lambda\mu} \right] \\
 & + \frac{3}{2} ie F_\mu{}^\nu \pi^\mu \varphi + \frac{1}{2} ie (i \partial_\mu F^{\mu\nu}) \varphi + m^2 \pi^\nu \varphi + \frac{3}{2} ie \left[F^\nu{}_\lambda \pi^2 \chi^\lambda - F^\mu{}_\lambda \pi_\mu \pi^\nu \chi^\lambda \right] \\
 & + \frac{3}{2} (ie)^2 F^\nu{}_\lambda F_\mu{}^\lambda \chi^\mu - \frac{3}{4} (ie)^2 F^\mu{}_\lambda F_\mu{}^\lambda \chi^\nu - m^2 \pi^2 \chi^\nu - ie m^2 F_\mu{}^\nu \chi^\mu \\
 & + \frac{1}{2} ie \left[(i \partial^\mu F_{\lambda\mu}) (\pi^\lambda \chi^\nu + \pi^\nu \chi^\lambda) + (i \partial_\mu F^\nu{}_\lambda) (\pi^\lambda \chi^\mu + \pi^\mu \chi^\lambda) \right] \quad (4.8) \\
 & + \frac{1}{2} ie \left[(i \partial_\rho F^{\rho\nu}) \pi^2 + (i \partial^\mu F_{\lambda\mu}) \pi^\lambda \pi^\nu + (i \partial_\rho F^\nu{}_\lambda) \pi^\lambda \pi^\rho \right] \Omega \\
 & - m^2 \left[\frac{3}{2} (ie) F^\mu{}_\lambda \pi_\mu + \frac{1}{2} ie (i \partial_\mu F^{\mu\nu}) \right] \Omega + \frac{1}{2} (ie)^2 \left[3 F^\nu{}_\lambda F^{\mu\lambda} \pi_\mu \right. \\
 & \left. + F^\nu{}_\lambda (i \partial_\mu F^{\mu\lambda}) - F^\mu{}_\lambda (i \partial_\mu F^{\nu\lambda}) - \frac{3}{2} F^\mu{}_\lambda F_\mu{}^\lambda \pi^\nu \right] \Omega
 \end{aligned}$$

$$\begin{aligned}
H \equiv & \frac{3}{2} m^4 \varphi - \frac{3}{2} (ie)^2 \left[F_{\mu\nu} F^{\nu\rho} \varphi^{\mu\rho} - \frac{1}{2} F_{\nu\mu} F^{\mu\nu} \varphi \right] - ie \left[(i\partial_\lambda F^\nu{}_\mu) \pi_\nu \varphi^{\lambda\mu} \right. \\
& + (i\partial^\mu F_\mu{}^\nu) \pi_\nu \varphi \left. \right] + ie \left[(i\partial^\mu F_{\mu\lambda}) \pi^2 \chi^\lambda + (i\partial_\lambda F_\mu{}^\nu) \pi_\nu \pi^\lambda \chi^\mu \right] \\
& - \frac{3}{2} (ie)^2 F_{\mu\nu} F^{\nu\rho} (\pi^\mu \chi^\rho + \pi^\rho \chi^\mu) + \frac{1}{2} (ie)^2 (i\partial_\lambda F_\mu{}^\nu) F^{\lambda\mu} \chi^\nu \\
& + (ie)^2 (i\partial^\mu F_{\mu\lambda}) F_\nu{}^\lambda \chi^\nu + \left[\frac{3}{2} m^4 \pi^2 - \frac{3}{2} (ie)^2 \left(F_{\mu\nu} F^{\nu\rho} \pi^\mu \pi^\rho - \frac{1}{2} F_{\nu\mu} F^{\mu\nu} \pi^2 \right) \right] \Omega \\
& + (ie)^2 \left[\frac{3}{2} (i\partial^\mu F_{\mu\lambda}) F_\nu{}^\lambda \pi^\nu - \frac{3}{2} (i\partial_\lambda F^\nu{}_\mu) F^{\lambda\mu} \pi_\nu \right. \\
& \left. + \frac{1}{2} (i\partial^\mu F_{\mu\lambda}) (i\partial^\nu F_\nu{}^\lambda) - \frac{1}{2} (i\partial_\lambda F^\nu{}_\mu) (i\partial_\nu F^{\lambda\mu}) \right] \Omega.
\end{aligned} \tag{4.9}$$

Every solution $\psi^{\mu\nu}$ of Eq. (2.8) determines, through the relation (4.1), a solution of Eqs. (4.4), (4.5) and (4.6) which satisfies the following constraints :

$$\pi_\mu \varphi^{\mu\nu} = 0 \tag{4.10}$$

$$\pi_\mu \chi^\mu = 0 \tag{4.11}$$

$$G^{0\nu} - \pi^0 \pi_\mu \varphi^{\mu\nu} = 0 \tag{4.12}$$

$$S^0 + m^2 \pi^0 \pi_\mu \chi^\mu = 0. \tag{4.13}$$

Equations (4.12) and (4.13) are easily checked to be constraints, since they do not contain second time derivatives. Conversely it is shown in Appendix C that every solution of Eqs. (4.4)-(4.6), satisfying the constraints (4.10)-(4.13) at the time $t=0$, provides a solution $\psi^{\mu\nu}$ of Eq. (2.8), given by Eq. (4.1).

We can now compute the number of degrees of freedom. The system formed by Eqs. (4.5)-(4.7) is a second order system of 15 PDE for the 15 unknown functions $\varphi^{\mu\nu}, \chi^\nu, \Omega$. The number of initial value functions is 30. They must satisfy at $t=0$ the 10 constraints (4.10)-(4.13). Moreover, since the $\varphi^{\mu\nu}, \chi^\nu, \Omega$ are determined by the ψ 's modulo a gauge transformation, there is the extra freedom of the 10 gauge arbitrary functions to be given at $t=0$. Therefore $30-10-10=10$ is the number of initial data to be given at $t=0$ in order to determine uniquely the motion of the ψ 's. This corresponds to a correct description of a spin two particle.

To study the propagation properties of the system (4.5)-(4.7), let us compute its characteristic determinant $D(n)$. Since second derivatives of $\varphi^{\mu\nu}$ are present only in Eq. (4.5), the determinant $D(n)$ splits into the product $D_1(n)D_2(n)$, where $D_1(n)$ is the characteristic determinant constructed with the coefficients of the second derivatives of $\varphi^{\mu\nu}$ coming from Eq. (4.5) and $D_2(n)$ is the characteristic determinant constructed with the coefficients of the second derivatives of χ^μ and Ω coming from Eqs. (4.6) and (4.7).

$D_1(n)$ is easily computable and one obtains

$$D_1(n) = -2(n^2)^{10}. \quad (4.14)$$

Therefore the propagation of the $\varphi^{\mu\nu}$ is a normal one. $D_2(n)$ can be explicitly written :

$$D_2(n) = \begin{vmatrix} A & x_0 & x_1 & x_2 & x_3 \\ y^0 & a^0_0 & a^0_1 & a^0_2 & a^0_3 \\ y^1 & a^1_0 & a^1_1 & a^1_2 & a^1_3 \\ y^2 & a^2_0 & a^2_1 & a^2_2 & a^2_3 \\ y^3 & a^3_0 & a^3_1 & a^3_2 & a^3_3 \end{vmatrix}$$

where

$$A = \frac{3}{2} m^4 n^2 - \frac{3}{2} (ie)^2 \left(F_{\mu\nu} F^{\nu\rho} n^\mu n^\rho - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} n^2 \right)$$

$$\begin{aligned}
 a^\mu{}_\nu &= -m^2 n^2 g^\mu{}_\nu + \frac{3}{2} i e (n^2 F^\mu{}_\nu - n^\mu n_\nu F^\rho{}_\rho) \\
 x_\sigma &= i e \left[(i \partial^\mu F_{\mu\sigma}) n^2 + (i \partial_\lambda F_\sigma{}^\lambda) n^\lambda n_\nu \right] \\
 y^\sigma &= \frac{1}{2} i e \left[(i \partial^\lambda F_{\lambda\sigma}) n^2 + (i \partial_\mu F^{\lambda\mu}) n_\lambda n^\sigma + (i \partial_\mu F^\sigma{}_\lambda) n^\mu n^\lambda \right].
 \end{aligned}$$

The computation yields

$$\begin{aligned}
 D_2(n) &= m^4 (n^2)^3 \left[\frac{3}{2} m^4 n^2 - \frac{3}{2} e^2 (F^d \cdot n)^2 \right] \left[m^4 n^2 + \left(\frac{3}{2} e \right)^2 (F^d \cdot n)^2 \right] \\
 &\quad + \frac{1}{2} m^2 (n^2)^2 \left\{ m^4 \left[x^2 n^2 - (x \cdot n)^2 \right] + \left(\frac{3}{2} e \right)^2 (n \cdot F^d \cdot x)^2 \right\}. \quad (4.15)
 \end{aligned}$$

The structure of $D_2(n)$ suggests the possible existence of anomalies, for which the modes χ^μ and Ω are responsible. Let us analyze it in simple situations.

If $F^{\mu\nu}$ is constant on space-time, then $x_\sigma = 0$ and we recover the results of Section 3.

If $F^{\mu\nu} = 0$ and $\partial_\mu F^{\mu\nu} = 0$ at a point of space time, then $D_2(n)$ at that point becomes :

$$D_2(n) = \frac{1}{2} m^6 (n^2)^2 \left[3 m^6 (n^2)^3 + e^2 q^2 \right] \quad (4.16)$$

where $q_\mu = \partial_\nu F_{\rho\mu} n^\nu n^\rho$. Then the analysis of the solutions n_μ of

$$D_2(n) = 0 \quad (4.17)$$

with $D_2(n)$ given by Eq. (4.16) can be easily performed. In a static electric field the equations are hyperbolic, and non-causal propagation is present, whereas in a magnetostatic field with sufficiently strong gradients, the hyperbolic character is lost since Eq. (4.17) develops solutions with $\vec{n}$ real and n^0 complex [see for more details Ref. 3].

In the case in which both the electromagnetic field and its derivatives are different from zero, if there exists a frame of reference in which the magnetic field and its curl are sufficiently small, non-causal modes of propagation occur. We must show that Eq. (4.17) with $D_2(n)$ given by Eq. (4.15) admits timelike n^μ solutions. For $n = (1, \vec{0})$, Eq. (4.15) yields

$$D_2((1, \vec{0})) = \left[m^4 - \left(\frac{3}{2} e \right)^2 (\vec{B})^2 \right] \left[m^4 \left(\frac{3}{2} m^4 + \frac{3}{2} e^2 (\vec{B})^2 \right) - \frac{1}{2} m^2 (\vec{s})^2 \right] + \frac{1}{2} m^2 \left(\frac{3}{2} e \right)^2 [(\vec{B} \cdot \vec{s})^2 - (\vec{B})^2 (\vec{s})^2] \quad (4.18)$$

where $\vec{B}$ is the magnetic field and

$$\vec{s} = e \nabla \times \vec{B}. \quad (4.19)$$

If there exists a frame in which

$$m^4 \left[m^4 - \left(\frac{3}{2} e \right)^2 (\vec{B})^2 \right] \left[\frac{3}{2} m^4 + \frac{3}{2} e^2 (\vec{B})^2 \right] - \frac{1}{2} m^6 e^2 (\nabla \times \vec{B})^2 > 0 \quad (4.20)$$

(smallness condition), there $D_2((1, \vec{0}))$ is positive. From inequality (4.20) it follows that $m^4 > (3e/2)^2 \vec{B}^2$. Then we can find a frame of reference where $D_2((1, \vec{0}))$ is negative, because by a Lorentz transformation we can make $m^4 = (3e/2)^2 \vec{B}^2$ and because the quantity $(\vec{B} \cdot \vec{s})^2 - \vec{B}^2 \vec{s}^2$ is always negative. By continuity there must be a frame where

$$D_2((1, \vec{0})) = 0. \quad (4.21)$$

The normal to the spacelike plane representing such a frame is the timelike four-vector for which we were looking.

In all these cases the constraints cannot eliminate the non-causal rays, because they propagate along the light cone (see Appendix C).

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A P P E N D I X A

We prove that Eq. (3.4) supplemented by the constraints (3.3) at $t=0$ implies Eq. (2.8). Rewrite Eq. (3.4) as follows :

$$L^{\mu\nu} + \pi^\nu \left(C^\mu + \frac{1}{2} \pi^\mu D \right) + \pi^\mu \left(C^\nu + \frac{1}{2} \pi^\nu D \right) = 0. \quad (A.1)$$

Since $L^{\mu 0} = 0$ at $t=0$ it follows that

$$\pi^\mu \left(C^0 + \frac{1}{2} \pi^0 D \right) + \pi^0 \left(C^\mu + \frac{1}{2} \pi^\mu D \right) = 0 \quad (A.2)$$

at $t=0$. From Eqs. (3.3) and (A.2) considered at $t=0$ we deduce that

$$\pi^0 \left(C^\mu + \frac{1}{2} \pi^\mu D \right) = 0 \quad (A.3)$$

for all μ , at $t=0$. On the other hand, because of the identities $C^\nu \equiv -m^{-2} \pi_\mu L^{\mu\nu}$ and $D \equiv -\frac{2}{3} m^{-4} ((m^2/2) L^\mu{}_\mu - m^2 \pi \cdot C)$ [see Eqs. (2.6) and (3.2)] we can rewrite the system (A.1) and its trace in terms of the functions $C^\nu + \frac{1}{2} \pi^\nu D$ and D :

$$\begin{aligned} \pi^2 \left(C^\nu + \frac{1}{2} \pi^\nu D \right) + \pi_\mu \pi^\nu \left(C^\mu + \frac{1}{2} \pi^\mu D \right) - m^2 \left(C^\nu + \frac{1}{2} \pi^\nu D \right) + \frac{m^2}{2} \pi^\nu D &= 0 \\ -\pi^2 D + 4\pi \cdot \left(C + \frac{1}{2} \pi D \right) + 3m^2 D &= 0. \end{aligned} \quad (A.4)$$

This is a second order system of PDE for the five functions $C^\nu + \frac{1}{2} \pi^\nu D$ and D , which propagates causally, since its characteristic determinant $D(n)$ is equal to $-2(n^2)^5$. By the constraints (3.3) and (A.3) at $t=0$, we know that $C^\nu + \frac{1}{2} \pi^\nu D$ and D are zero with their first time derivatives at $t=0$. By the system (A.4), they are identically zero. This implies, by Eq. (A.1), that $L^{\mu\nu} = 0$.

A P P E N D I X B

We analyze here the gauge freedom connected with Eq. (4.1). Suppose there exist two sets of functions $\varphi^{\mu\nu}, \chi^\nu, \Omega$ and $\varphi'^{\mu\nu}, \chi'^\nu, \Omega'$ such that :

$$\varphi'^{\mu\nu} + \pi^\mu \chi'^\nu + \pi^\nu \chi'^\mu + \frac{1}{2}(\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) \Omega' = \varphi^{\mu\nu} + \pi^\mu \chi^\nu + \pi^\nu \chi^\mu + \frac{1}{2}(\pi^\mu \pi^\nu + \pi^\nu \pi^\mu) \Omega \quad (\text{B.1})$$

with

$$\pi \cdot \chi' = \pi \cdot \chi = 0 \quad (\text{B.2})$$

$$(\pi \cdot \varphi')^\nu = (\pi \cdot \varphi)^\nu = 0. \quad (\text{B.3})$$

Define

$$M \equiv \Omega' - \Omega \quad (\text{B.4})$$

$$N^\nu \equiv \chi'^\nu - \chi^\nu. \quad (\text{B.5})$$

Then contraction of Eq. (B.1) with π_μ and $\pi_\nu \pi_\mu$ and use of the equation $\pi \cdot N = 0$ yield, respectively

$$\pi^2 N^\nu + ie F_\mu{}^\nu N^\mu + \frac{3}{2} ie F_\mu{}^\nu \pi^\mu M + \frac{1}{2} ie (i\partial_\rho F^{\rho\nu}) M + \pi^\nu \pi^2 M = 0 \quad (\text{B.6})$$

$$ie F^{\nu\mu} \pi_\mu N_\nu + 2ie \pi_\nu F_\mu{}^\nu N^\mu + \frac{3}{2} ie \pi_\nu F^{\mu\nu} \pi_\mu M + \frac{1}{2} ie \pi_\nu (i\partial_\rho F^{\rho\nu}) M + (\pi^2)^2 M = 0. \quad (\text{B.7})$$

Consider now Eqs. (B.6) and (B.7) as a system of PDE in the unknowns N^ν and M . To get a unique solution for the initial value problem, one needs to assign on a $t=0$ surface $M, \partial_0 M, \partial_0^2 M, \partial_0^3 M, N^\nu, \partial_0 N^\nu$, i.e., 12 functions of the space variables. However, in order to give rise to a gauge transformation, the N^ν must satisfy the equation $\pi \cdot N = 0$. This relation can be obtained by supplementing Eqs. (B.6) and (B.7) with the following constraints at $t=0$:

$$\pi \cdot N = 0 \quad (\text{B.8})$$

and

$$\pi^2 N^0 + ie F_\mu^0 N^\mu + \frac{3}{2} ie F_\mu^0 \pi^\mu M + \frac{1}{2} ie (i \partial_\rho F^{\rho 0}) M + \pi^0 \pi^2 M - \pi^0 \pi \cdot N = 0. \quad (B.9)$$

In fact comparison of Eq. (B.9) at $t=0$ with Eq. (B.6) yields

$$\pi_0(\pi \cdot N) = 0 \quad (B.10)$$

at $t=0$. By subtracting Eq. (B.7) from the contraction of π_μ with Eq. (B.6) one obtains

$$\pi^2(\pi \cdot N) = 0 \quad (B.11)$$

at all times. By Eqs. (B.8) and (B.10) at $t=0$ one concludes that $\pi \cdot N = 0$ is satisfied at all times.

It follows that the gauge freedom depends on 10 functions given at $t=0$: the 12 coming from the Cauchy problem of the system (B.6), (B.7) are reduced to 10 by the 2 constraints (B.8) and (B.9).

A P P E N D I X C

We want to prove that if $\varphi^{\mu\nu}, \chi^\nu, \Omega$ satisfy the system of PDE (4.5)-(4.7) supplemented at $t=0$ by the constraints (4.10)-(4.12) the $\psi^{\mu\nu}$ obtained from Eq. (4.1) is a solution of Eq. (2.8). Comparison of Eq. (4.4) with Eq. (4.12), and Eq. (4.5) with Eq. (4.13) implies that

$$\pi^0(\pi_\mu \varphi^{\mu\nu}) = 0 \quad (C.1)$$

$$\pi^0(\pi \cdot \chi) = 0 \quad (C.2)$$

at $t=0$. Define

$$f^\nu \equiv \pi_\mu \varphi^{\mu\nu}$$

$$d \equiv \pi \cdot \chi.$$

Then, using the explicit expressions (4.7)-(4.9) for $G^{\mu\nu}, S^\nu$ and H , one obtains :

$$\pi_\mu G^{\mu\nu} - S^\nu \equiv (\pi^2 - m^2) f^\nu + \frac{1}{2} i e F^\nu{}_\rho f^\rho + \frac{3}{2} i e F^\nu{}_\rho d + i e (i \partial_\rho F^\rho{}_\nu) d - m^2 \pi^\nu d = 0 \quad (C.3)$$

and

$$\begin{aligned} \pi_\nu S^\nu + \frac{1}{2} m^2 G^\mu{}_\mu - H \equiv & -\frac{3}{2} i e F^\mu{}_\lambda \pi_\mu f^\lambda + \frac{1}{2} i e (i \partial_\mu F^\mu{}_\lambda) f^\lambda - m^2 (\pi^2 + m^2) d \\ & + \frac{1}{2} i e (i \partial^\lambda F_{\lambda\mu}) \pi^\mu d - \frac{3}{4} (i e)^2 F^\mu{}_\lambda F^\lambda{}_\mu d = 0. \end{aligned} \quad (C.4)$$

Therefore f^ν and d satisfy a second order PDE, with zero initial conditions by Eqs. (C.1), (C.2), (4.10) and (4.11). They are identically zero. Then one can go trivially back from Eq. (4.5) to Eq. (2.8). Moreover, the structure of Eqs. (C.3) and (C.4) implies that f^ν and d propagate causally. As a consequence also the constraints (4.12) and (4.13) propagate in a causal way.

R E F E R E N C E S

- 1) S. Weinberg, in "Lectures on Elementary Particles Physics and Quantum Field Theory", 1970 Brandeis Summer Institute in Theoretical Physics, Vol.1, p.284. Edited by S. Deser, M. Grisaru and H. Pendleton, M.I.T. Press Publisher, Cambridge, Mass., and London, UK.
- 2) G. Velo and D. Zwanziger, Phys.Rev. 186, 1337 (1969).
- 3) G. Velo and D. Zwanziger, Phys.Rev. 188, 2218 (1969).
- 4) G. Velo, Nuovo Cimento Letters 3, 80 (1972).
- 5) G. Velo and D. Zwanziger, in "Lectures from the Coral Gables Conference on Fundamental Interactions at High Energy", January 1971, Vol.4, p.8. Edited by M. Dal Cin, G.J. Iverson and A. Perlmutter, Gordon and Breach Science Publishers, N.Y., London, Paris.
- 6) G. Wentzel, "Quantum Field Theory", p.205, Wiley-Interscience Inc., N.Y., (1969).
- 7) P. Federbush, Nuovo Cimento 19, 572 (1961).
- 8) R. Courant and D. Hilbert, "Methods of Mathematical Physics", Vol.2, Chap.VII, Wiley-Interscience Inc., N.Y., (1962).