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Diffraction Radiation of Charged Rings moving
in a Corrugated Cylindrical Pipe:

A Model of Radiation Losses in an Electron Ring Accelerator

by

E. KEIL

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ABSTRACT

The energy loss of a charged ring passing an infinitely long periodic sequence of RF accelerating cavities which has recently received much attention in connection with the collective acceleration of ions in electron rings is calculated by expanding the vector potential into a series of eigenfunctions of the empty waveguide and computing their individual excitation. The frequency spectra obtained are compared to the predictions of simpler analytical models from which a set of formulae for the practical design of RF accelerating structures for electron rings can be derived. The energy loss becomes independent of γ when the electron rings have a highly relativistic speed.

1. INTRODUCTION

A bunch of charge Q travelling along an RF accelerating structure gains energy due to the accelerating field driven by an external RF source, and it loses energy by exciting additional electromagnetic fields: this is the phenomenon of "beam loading" familiar in linear accelerators¹⁾. The energy gain per unit length dU/dz takes the form

$$dU/dz = EQ - BQ^2 \quad \dots(1)$$

Here E is the average applied field along the bunch trajectory, and BQ^2 is the radiation loss or beam loading term. B is a proportionality factor whose dependence on the geometrical parameters of the accelerating structure and the bunch is one of the subjects of this paper.

Because of the quadratic increase of the radiation loss with the charge Q , the interest in an accurate evaluation of the factor B suddenly rose with the advent of the electron ring accelerator²⁾. In fact, the charge of an electron ring is a few orders of magnitude higher than that of a bunch in a typical electron or proton linear accelerator. Consequently, the radiation loss becomes an important factor in the design of an RF accelerating structure for electron rings. Although the Dubna group²⁾ indicated that the radiation loss does not present an obstacle, some authors³⁾⁴⁾ calculated an increase of the radiation loss with the axial velocity of the electron rings and concluded that there might be a practical upper energy limit for RF acceleration of electron rings.

This discrepancy is due to the different models used in the calculation. Single cavity models³⁾⁴⁾ show an increase of the radiation loss as $\gamma^{\frac{1}{2}}$. However, their validity breaks down above some value of γ when interference between successive cavities becomes an important

factor, and periodic models are more appropriate. In cylindrical geometry, they yield a radiation loss independent of γ , and in planar geometry a decrease of the radiation loss as $\gamma^{-1/2}$.⁵⁾ A variety of models was employed for the calculation of the radiation loss⁶⁾⁷⁾. We shall represent the accelerating structure (Fig. 1) by an infinitely long periodic sequence of cylindrical cavities of radius b and length g , joined by concentric tubes of radius a . The length of a period is d . The charge distribution is a uniformly filled ring with rectangular cross-section of length h and radii r_1 and r_2 . It travels through the pipe at the velocity v from $-\infty$ to $+\infty$.

After a summary of the mathematical approach, numerical results are given for a variety of accelerating structures. They are then compared with the results of other models. From this comparison we derive a set of formulae for the practical design of RF accelerating structures for electron rings.

2. MATHEMATICAL FORMALISM

2.1 Calculation of the energy loss

The mathematical formalism employed was developed by Akhiezer, Lyubarskii and Fainberg⁸⁾, its application to the present problem is described in⁹⁾. We shall give a more transparent derivation which essentially agrees with that used in¹⁰⁾.

The field created by the moving charge is obtained from the wave equation for the vector potential $\vec{A}(\vec{r}, t)$:

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \quad \dots (2)$$

where $\vec{j}$ is the current density related to the moving charge. We shall look for the solution in the form of a sum over the eigen functions of the homogeneous equation (2).

$$\vec{A}(\vec{r}, t) = \sum_{\lambda} q_{\lambda}(t) \vec{A}_{\lambda}(\vec{r}) \quad \dots(3)$$

The $\vec{A}_{\lambda}$ are normalised as follows :

$$\int_{V_N} |\vec{A}_{\lambda}|^2 dV = N/\epsilon_0 \quad \dots(4)$$

where $V_N = N.V$ is the volume of N cells. Since the boundary is periodic, the eigenfunctions have the form

$$\vec{A}_{\lambda}(\vec{r}) = e^{-i\beta_0 z} \vec{a}_{\lambda}(\vec{r}) \quad \dots(5)$$

where β_0 is a parameter in the interval $(-\pi/d, \pi/d)$ and $\vec{a}_{\lambda}(\vec{r})$ is a periodic function of z with period d . It can therefore be expanded into a Fourier series, and the eigenfunctions become:

$$\vec{A}_{\lambda}(\vec{r}) = \sum_{m=-\infty}^{+\infty} \vec{A}_{\lambda m}(r) \exp(i\beta_m z) \quad \dots(6)$$

with the propagation constant $\beta_m = \beta_0 + 2\pi m/d$. The explicit radial dependence can be obtained from (2). We find the z -component, for $r \leq a$:

$$A_{z\lambda}(r, z) = \frac{i}{\omega_{\lambda}} \sum_{m=-\infty}^{+\infty} A_{\lambda m} \frac{J_0(\chi_m r)}{J_0(\chi_m a)} \exp(-i\beta_m z) \quad (7)$$

with the radial propagation constant $\chi_m^2 = \omega_{\lambda}^2/c^2 - \beta_m^2$. The coefficients $A_{\lambda m}$ will be obtained below by field matching.

Because the $\vec{A}_{\lambda}$ are normalised and orthogonal, the field coordinates $q_{\lambda}(t)$ obey the equation:

$$\ddot{q}_{\lambda} + \omega_{\lambda}^2 q_{\lambda} = \frac{1}{N} \int_{V_N} \vec{j} \cdot \vec{A}_{\lambda}^* dV = f_{\lambda} \quad \dots(8)$$

We shall limit our calculation to currents flowing in the z direction, i.e. we consider the case of intense bunches in a linear accelerator for which experimental results¹¹⁾ are available for comparison. In electron ring accelerators, the azimuthal motion of the electrons in the ring excites H modes in addition to the E modes calculated in this paper. It is known, that for an infinite set of half planes⁵⁾ the energy loss into H modes is equal to that into E modes in the limit of $\gamma \rightarrow \infty$.

When the azimuthal motion of the electron is neglected, an electron ring with charge Q and geometrical parameters as shown in Fig. 1, travelling with the speed v , has the current density:

$$j_z = \frac{Qv}{\pi h(r_2^2 - r_1^2)} H(\frac{1}{2}h - |z - vt|) H(r - r_1) H(r_2 - r) \quad \dots(9)$$

where $H(x)$ is the Heaviside step function, $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$. Using (7), (9), the right hand side of (8) becomes :

$$f_\lambda(t) = - \frac{Qvi}{N\omega_\lambda} \sum_m A_{\lambda m} S(\frac{1}{2}\beta_m h) J(\chi_m) \exp(i\beta_m vt) \quad \dots(10)$$

The factors $S(x) = x^{-1} \sin x$ and J take into account the finite dimensions of the electron ring. J is given by :

$$J(\chi_m) = \frac{2\{r_2 J_1(\chi_m r_2) - r_1 J_1(\chi_m r_1)\}}{\chi_m (r_2^2 - r_1^2) J_0(\chi_m a)} \quad \dots(11)$$

With the N cavities centred at $z = 0$, and with j_z given by (9), $f_\lambda(t)$ vanishes for $|t| \geq T = \frac{1}{2} Nd/v$. Hence, for $t \geq T$, $q_\lambda(t)$ is given by :

$$q_\lambda(t) = \frac{1}{\omega_\lambda} \int_{-T}^{+T} f_\lambda(t') \sin \omega_\lambda(t - t') dt' \quad \dots(12)$$

which becomes:

$$q_{\lambda}(t) = \frac{Qd}{2\omega_{\lambda}^2} \sum_m A_{\lambda m} S(\frac{1}{2}\beta_m h) J(\chi_m) \{S(\phi^+) \exp(-i\omega_{\lambda} t) - S(\phi^-) \exp(i\omega_{\lambda} t)\} \dots (13)$$

where: $\phi^{\pm} = \frac{1}{2} Nd(\omega_{\lambda}/v \pm \beta_m)$ and $S(\phi) = \phi^{-1} \sin \phi$ as above.

For $N \gg 1$, the largest contributions to (13) come from the two terms in the sum with $m = \ell^{\pm}$ for which $|\omega_{\lambda} \pm \beta_{\ell^{\pm}}| < \pi/Nd$. The summation over the space harmonics m and over the N possible values of β_0 , spaced by $2\pi/Nd$, may be combined into a single sum which we write in the form of an integral:

$$q_{\lambda}(t) = \frac{Qd}{2\omega_{\lambda}^2} \frac{Nd}{2\pi} \int_{-\infty}^{+\infty} A_{\lambda m} S(\frac{1}{2}\beta_m h) J(\chi_m) \{S(\phi^+) \exp(-i\omega_{\lambda} t) - S(\phi^-) \exp(i\omega_{\lambda} t)\} d\beta_0 \dots (14)$$

Introducing the new variable $x = \omega_{\lambda} \pm \beta_m v$, we find:

$$q_{\lambda}(t) = \frac{Qdv}{2\pi\omega_{\lambda}^2} \left\{ \frac{\exp(-i\omega_{\lambda} t)}{v+d\omega/d\beta_0} \int_{-\infty}^{+\infty} A_{\lambda \ell^+} S^+ J^+ \frac{\sin x T}{x} dx - \frac{\exp(i\omega_{\lambda} t)}{v-d\omega/d\beta_0} \int_{-\infty}^{+\infty} A_{\lambda \ell^-} S^- J^- \frac{\sin x T}{x} dx \right\} \dots (15)$$

For $T \rightarrow \infty$, the identity

$$\lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} f(x) \frac{\sin x T}{x} dx = \pi f(0)$$

may be used. Thus one obtains:

$$q_{\lambda}(t) = \frac{Qdv}{2\omega_{\lambda}^2} \left\{ \frac{\exp(-i\omega_{\lambda} t)}{v+d\omega/d\beta_0} A_{\lambda \ell^+} S^+ J^+ - \frac{\exp(i\omega_{\lambda} t)}{v-d\omega/d\beta_0} A_{\lambda \ell^-} S^- J^- \right\} \dots (16)$$

Here, $A_{\lambda\ell\pm}$ are to be evaluated for $\omega_{\lambda\pm}\beta_{\ell\pm}v = 0$. This is achieved by letting $\beta_0 = \omega_{\lambda}/v - 2\pi\ell/d$ with ℓ chosen such that $|\beta_0| < \pi/d$. We then find $\beta_{\ell} = \omega_{\lambda}/v$ and $\chi_{\ell}^2 = \chi^2 = -\omega_{\lambda}^2/v^2\gamma^2$.

Furthermore, it has been verified that $A_{\lambda\ell+} = A_{\lambda\ell-} = A_{\lambda\ell}$ which is not surprising since the change from ℓ^+ to ℓ^- just involves changing the sign of β_0 and counting the space harmonics in reversed order. The final answer is :

$$\lim_{N \rightarrow \infty} q_{\lambda}(t) = -Qd\omega_{\lambda}^{-2} A_{\lambda\ell} S(\frac{1}{2}\omega_{\lambda}h/v)J(i\omega_{\lambda}/\gamma v)\sin\omega_{\lambda}t / \left[1-v^{-2}(d\omega/d\beta_0)^2\right] \dots(17)$$

In the following, we shall work to first order in the ratio of the group velocity and the particle velocity. In this case, the term in the square bracket in (17) may be dropped.

The energy loss per period U_{λ} is obtained by calculating the stored energy in one cell of the structure and using energy conservation. With the normalisation (4) the stored energy W_{λ} is :

$$W_{\lambda} = \frac{1}{2}(|\dot{q}_{\lambda}|^2 + \omega_{\lambda}^2|q_{\lambda}|^2) \dots(18)$$

Substitution from (17) yields, with $U_{\lambda} = W_{\lambda}$

$$U_{\lambda} = \frac{1}{2} \left[Q A_{\lambda\ell} d S(\frac{1}{2}\omega_{\lambda}h/v)J(i\omega_{\lambda}/\gamma v)/\omega_{\lambda} \right]^2 \dots(19)$$

This result agrees with that given in ⁹⁾.

2.2 Calculation of the field coefficients

The suitable field expansion in the tube region $0 < r < a$ is given by (7). The propagation constants are $\beta_o = \omega/v - 2\pi\ell/d$ with ℓ chosen such that $|\beta_o| < \pi/d$, $\beta_m = \beta_o + 2\pi m/d$ and $\chi_m^2 = \omega^2/c^2 - \beta_m^2$. In this section we omit the index λ because we only consider a single eigenfunction. The tangential field components which are needed for the field matching follow from (16) and from Maxwell's equations.

$$E_z = \sum_m A_m \left[J_o(\chi_m r) / J_o(\chi_m a) \right] \exp(-i\beta_m z) \quad \dots(20)$$

$$Z_o H_\varphi = (i\omega/c) \sum_m A_m \left[J_o(\chi_m r) / J_o(\chi_m a) \right] \exp(-i\beta_m z)$$

$Z_o = 120 \pi$ ohm is the impedance of free space.

In the slot region $a \leq r \leq b$, we take radial transmission line modes which fulfil all boundary conditions except those at the slot mouth $r = a$:

$$\begin{aligned} E_z &= \sum_{s=0}^{\infty} B_s \cos(2\pi s z/g) \left[F_o(\Gamma_s r) / F_o(\Gamma_s a) \right] \\ &\quad + \sum_{s=1}^D D_s \sin((2s-1)\pi z/g) \left[F_o(\Gamma'_s r) / F_o(\Gamma'_s a) \right] \quad \dots(21) \\ Z_o H_\varphi &= (i\omega/c) \sum_{s=0}^{\infty} B_s \cos(2\pi s z/g) \left[F_1(\Gamma_s r) / \Gamma_s F_o(\Gamma_s^1 a) \right] \\ &\quad + (i\omega/c) \sum_{s=1}^D D_s \sin((2s-1) z/g) \left[F_1(\Gamma'_s r) / \Gamma_s^1 F_o(\Gamma'_s a) \right] \end{aligned}$$

with the propagation constants $\Gamma_s^2 = \omega^2/c^2 - (2\pi s/g)^2$ and $\Gamma'_s{}^2 = \omega^2/c^2 - ((2s-1)\pi/g)^2$ and the functions

$$F_i(\Gamma r) = Y_0(\Gamma b) J_i(\Gamma r) - J_0(\Gamma b) Y_i(\Gamma r) \quad \dots(22)$$

The unknown coefficients A_m , B_s and D_s are determined by imposing boundary conditions at $r = a$: E_z must vanish along the tube wall, and E_z and H_φ must be equal across the slot mouth¹²⁾. Using the fact that the exponential function is orthogonal with period d , and that the trigonometric functions are orthogonal with period g , we obtain five equations between the unknown coefficients. Eliminating B_s and D_s , we obtain an equation for the A_m which can be written in matrix form

$$\left[\underline{C}^T \underline{H} \underline{C} \underline{M} - \underline{S}^T \underline{L} \underline{S} \underline{M} - \underline{U} \right] \underline{A} = 0 \quad \dots(23)$$

Here $\underline{A}$ is a vector with components A_m , $\underline{H}$, $\underline{L}$ and $\underline{M}$ are diagonal matrices with elements

$$\begin{aligned} H_{ss} &= K_s \Gamma_s F_0(\Gamma_s a) / F_1(\Gamma_s a) \\ L_{ss} &= 2\Gamma'_s F_0(\Gamma'_s a) / F_1(\Gamma'_s a) \\ M_{mm} &= (1/dg\chi_m) J_1(\chi_m a) / J_0(\chi_m a) \end{aligned} \quad \dots(24)$$

where $K_0 = 1$ and $K_s = 2$ for $s > 0$. $\underline{C}$ and $\underline{S}$ are matrices with elements

$$\begin{aligned} C_{st} &= (-1)^s g \frac{\beta_t g}{2} \sin \frac{\beta_t g}{2} / \left[\left(\frac{\beta_t g}{2} \right)^2 - (\pi s)^2 \right] \\ S_{st} &= -(-1)^s ig \frac{\beta_t g}{2} \cos \frac{\beta_t g}{2} / \left[\left(\frac{\beta_t g}{2} \right)^2 + \left(\frac{(2s-1)\pi}{2} \right)^2 \right] \end{aligned} \quad \dots(25)$$

$\underline{U}$ is the unit matrix and X^T is the transpose of X .

The frequencies where (23) has non-trivial solutions are those where wave travelling with the phase velocity v propagate in the waveguide. Solving (23) gives the coefficients A_m apart from a common factor which is determined by the normalisation(4).

2.3 Numerical procedure

All the calculations necessary are done by computer. The first program in the chain calculates the determinant of (23) in a given interval of frequency ω , in steps $\Delta\omega$. It finds the frequencies where the determinant vanishes, solves (23) at these frequencies, and normalises the field coefficients A_m according to (4). It issues a list of pairs of numbers (ω, A_ℓ) for use by subsequent programs. The matrices involved are truncated to finite size, which is determined by two considerations. Firstly, there must not be any eigenvalue ω which gets lost because the relevant space harmonic has been omitted; this means that there must be enough space harmonics so that the propagation constants near the border of the matrix are imaginary. Secondly, there must be enough space harmonics so that A_ℓ is computed to sufficient accuracy; this is done by trial and error. In the results, the error on the energy loss should not exceed a few percent. In order to avoid confusing the program by the K infinities ω_k of the determinant of (23) in the frequency interval considered, they are suppressed by multiplying the determinant by $\prod_{k=1}^K (1 - \omega_k/\omega)$; the infinities are at the frequencies where $k=1$

$F_1(\Gamma_s a)$, $F_1(\Gamma_s' a)$ or $J_0(\chi_m a)$ vanish, which can all be calculated directly. The step-by-step procedure misses zeros which appear in pairs within a step $\Delta\omega$. In order to find them, the expression

$$\det(\omega) \prod_{k=1}^K (1-\omega/\omega_k) / \prod_{n=1}^N (1-\omega/\omega_n)$$

is tabulated, where the ω_n are the N

zeros found so far. Missing zeros show up in sharp dips in this function which are found by inspection. These missing zeros are then found, added to the list and the new function is tabulated until it is found to be sufficiently smooth. The size of the truncated matrices is proportional to ω . This and the fact that the number of zeros in a given frequency interval is also proportional to ω ¹³⁾ results in a practical upper limit for the frequency which can be handled: beyond it the computational effort involved quickly becomes prohibitive.

The energy loss is calculated from (19) in a program which reads geometrical parameters and pairs (ω, A_ℓ) , sorts them according to geometrical parameters and frequency, and prepares tables and histograms of the frequency distribution of the energy loss, as shown in Figs. 2 & 3.

3. NUMERICAL RESULTS

All results are compiled in Table I. All structures are given with dimensions as used in the computation. In principle, the number of geometrical parameters can be reduced by one if the following scaling law is applied: If all geometrical dimensions are scaled by a factor α , the resonant frequencies and energy losses are scaled by a factor $1/\alpha$. The cases 23 to 40 were computed in connection with an electron ring accelerator design study¹⁴⁾ at the Lawrence Radiation Laboratory, Berkeley, and the cases 41 to 46 in connection with an experiment on the radiation loss¹¹⁾ at the Stanford Linear Accelerator Center.

4. ANALYTICAL MODELS

A few models give closed expressions for the radiation loss. They are described below. In Chapter 5, they will be compared with our computational results, and will form the basis of practical formulae for the radiation loss.

4.1 Lawson's diffraction model

Lawson has calculated the energy loss using diffraction theory³⁾. He considers diffraction at the up-stream hole of radius a in a cavity of length g , and calculates the fraction of energy which is diffracted to radii $r \geq a$ at the downstream end wall of the cavity. His result is for a point charge

$$U = \frac{0.44 (2g/\pi c^5)^{\frac{1}{2}} Q^2 a \gamma^{-2}}{4\pi\epsilon_0} \int_{c/a}^{\infty} \omega^{3/2} K_1^2(\omega a/c\gamma) d\omega \quad \dots(26)$$

The lower cut-off is introduced in order to allow for the limited validity of the diffraction model. The formula is claimed to hold for $\gamma \gg 1$. The integrand will be used below to draw frequency spectra. Evaluating (26) numerically yields:

$$U = \frac{0.6}{4\pi\epsilon_0} \frac{Q^2 g^{\frac{1}{2}} \gamma^{\frac{1}{2}}}{a^{3/2}} \quad \dots(27)$$

The magnitude of the radiation loss and its increase as $\gamma^{\frac{1}{2}}$ led Lawson to the suggestion that there is a severe practical upper limit to the energy which can be reached by RF acceleration of electron rings.

4.2 The closed cavity model

The resonant frequency ω is for the E_{0mn} mode in a simple cylindrical cavity of length g and radius b :

$$\omega = c \left[\mu_m^2 + (n\pi/g)^2 \right]^{\frac{1}{2}} \quad \dots(28)$$

and the energy loss into the $E_{15) \text{ omn } 16) 17) 18)}$ mode is for an electron ring of the dimensions shown in Fig. 1 :

$$U = \frac{8Q^2 \beta^2}{4\pi\epsilon_0 g b^2} \left(\frac{1 + \delta}{2}\right)^{n_0} \left[1 - (-)^n \cos \frac{\omega g}{v}\right] \left[J(\mu_m) S(\omega h/2v)\right]^2 \left[1 + (n\pi/g\mu_m \gamma)\right]^{-2} \dots (29)$$

μ_m is the m - th solution of $J_0(\mu_m b) = 0$. J is defined as in (11) with a replaced by b and $S(x) = x^{-1} \sin x$ as before.

Kolpakov and Kotov¹⁵⁾ have argued that (29) holds for $\lambda \geq a$ where a is the hole radius. In order to find any modes at all which fulfil this condition, the condition $b/a > j_1$ must be satisfied. They have summed (29) over all modes with $\lambda \geq a$ in an approximate manner which also requires $g \leq 2\beta a$. Using the same cut-off and the same approximations, Kolpakov, Kotov and Om-San-Kha¹⁹⁾ have considered the case shown in Fig. 1, and found for a point charge

$$U = \frac{g^2}{4\pi\epsilon_0} \frac{Q^2}{2a^2 d} \dots (30)$$

This is a factor g/d smaller than the result of Kolpakov and Kotov.¹⁵⁾ The contributions of all modes were summed up by Winterbon¹⁶⁾ and by Levine and Sessler¹⁸⁾. They found that a finite result is only obtained for finite minor ring radius; the result diverges logarithmically for a ring charge with vanishing minor radius, and even more strongly for a point charge. In the case where the axial minor ring dimension is constant in its rest frame, and hence Lorentz contracted in the laboratory frame, the total radiation loss increases without limit with increasing γ , but less rapidly than γ .

4.3 The optical resonator model

Sessler²⁰⁾ developed a formula for the radiation loss based on the analogy between a set of infinite plates with circular holes and a pair of circular mirrors. The latter case has been treated extensively

in conjunction with optical resonators²¹⁾. The argument is as follows :

The fractional energy loss α per traversal of the wave between the mirrors, due to the diffraction of the first radial mode, is, according to Vainshtein²¹⁾

$$\alpha = 8j_1^2 \sigma(M + \sigma) / [(M + \sigma)^2 + \sigma^2]^2 \quad \dots(31)$$

where $\sigma = \zeta(\frac{1}{2})/\sqrt{\pi} = 0.824$, $j_1 = 2.405$ is the first zero of J_0 and $M = (4\omega a^2/cd)^{\frac{1}{2}}$.

Sessler found the energy flux $Pd\omega$ in the frequency interval $d\omega$ in the beam by evaluating the Poynting vector at the beam edge $r = a$ and multiplying it by the cross-section of the beam :

$$Pd\omega = \frac{Q^2 a^2 \omega^2}{4\pi^2 \epsilon_0 \beta^3 \gamma^2 c^3} K_1^2 (\omega a / \beta \gamma c) d\omega \quad \dots(32)$$

K_1 is the modified Bessel function of the first kind. He obtained the total energy loss from the product of (31) and (32) by integrating over all ω :

$$U = \frac{2j_1^2 \sigma Q^2 a^2}{\pi^2 \epsilon_0 \beta^3 \gamma^2 c^3} \int_{2\pi c/d}^{\infty} \omega^2 (M + \sigma) K_1^2 (\omega a / \beta \gamma c) [(M + \sigma)^2 + \sigma^2]^{-2} d\omega \quad \dots(33)$$

He introduced a lower frequency cut-off to allow for the limited validity of diffraction formulae. We have integrated (33) numerically for comparing our computed results with this model. We have also used the integrand for plotting frequency spectra. Closed expressions have been derived by making simplifying assumptions.

For high values of γ , $K_1(X)$ may be replaced by $1/X$. We then find :

$$U = \frac{j_1^2 \sigma d Q^2}{2\pi^2 \epsilon_0 a^2 \beta} \left\{ \frac{2a\sqrt{2\pi}/d}{(2a\sqrt{2\pi}/d + \sigma)^2 + \sigma^2} + \frac{1}{\sigma} \tan^{-1} \left(\frac{\sigma}{2a\sqrt{2\pi}/d + \sigma} \right) \right\} \dots (34)$$

This formula can in turn be simplified in two different ways :

i) If $2a\sqrt{2\pi}/d \gg \sigma$, i.e. for short periods d and large hole radii a , Sessler found :

$$U = \frac{j_1^2 \sigma}{4\pi^2 \epsilon_0} \sqrt{\frac{2}{\pi}} \frac{d^2 Q^2}{a^3 \beta} \dots (35)$$

ii) In the opposite case $2a\sqrt{2\pi}/d \ll \sigma$, i.e. for long periods d and small radii a , we find :

$$U = \frac{j_1^2}{8\pi \epsilon_0} \frac{d Q^2}{a^2 \beta} \dots (36)$$

All these formulae are for point charges. In this model, the energy loss does not depend on γ when γ is large; this property and the fact that it describes an infinite periodic structure distinguishes it from the two previous models. To take account of the finite length h of the electron rings or bunches, we introduce an upper frequency cut-off at $\omega = 2\pi c/h$. We obtain for high values of γ :

$$U = \frac{j_1^2 \sigma d Q^2}{2\pi^2 \epsilon_0 a^2 \beta} \left\{ \frac{\frac{2a}{d} \sqrt{2\pi}}{(\frac{2a}{d} \sqrt{2\pi} + \sigma)^2 + \sigma^2} - \frac{2a\sqrt{\frac{2\pi}{dh}}}{(2a\sqrt{\frac{2\pi}{dh}} + \sigma)^2 + \sigma^2} + \frac{1}{\sigma} \tan^{-1} \frac{\sigma(2a\sqrt{\frac{2\pi}{d}})(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{d}})}{\sigma^2 + (\frac{2a\sqrt{2\pi}}{d} + \sigma)(2a\sqrt{\frac{2\pi}{dh}} + \sigma)} \right\} \dots (37)$$

5. CONCLUSIONS

5.1 Comparison with an experiment

An experiment on the radiation loss has been performed in the electron linear accelerator at SLAC¹¹⁾. In this case the bunch length h was about 1.5 mm. This lead to a reduction of the energy loss, compared to a point charge, by about 40%.

Table II shows a comparison of the computed energy loss and that given in (37). The SLAC structure does not consist of a large number of identical periods, but rather of 10-foot sections inside which the hole radius a and outer radius b vary²²⁾. This fact is ignored in the computation. The first two entries in Table II show an "average" cell, the remaining entries show actual cells at equal intervals along a 10-foot section. The computed energy loss is obtained as follows : The computer results are taken up to $\omega = 4$ or $6 \times 10^{11} \text{ s}^{-1}$, and for higher frequencies up to $\omega = c/h$, the same integral as (33) is taken with the appropriate limits, and added to the computer results. The results obtained with the simple formula (37) and by computer are in good agreement, and the experimental result, 38 MeV¹¹⁾, is in very good agreement with the calculation.

5.2 Comparison between computed and analytic results

Since the energy loss increases as $\gamma^{\frac{1}{2}}$ in Lawson's model (27), and is constant in the Sessler-Vainshtein model (34), there must be a value of γ where the two models yield the same result. Using (27) and (34) we find for the critical γ_c :

$$\gamma_c = \frac{25 d^2}{ag} \left[\frac{2a\sqrt{2\pi}/d}{(2a\sqrt{2\pi}/d)^2 + \sigma^2} + \frac{1}{\sigma} \tan^{-1} \frac{\sigma}{2a\sqrt{2\pi}/d + \sigma} \right]^2 \dots(38)$$

Lawson²³⁾ suggested that his model applies for $\gamma < \gamma_c$, and that the Sessler-Vainshtein model applies for $\gamma > \gamma_c$, because at low γ and large ratio d/a the electromagnetic fields surrounding the electron ring are matched to the pipe between successive cavities, i.e. there is no interference between cavities. In the opposite case, the interference between successive cavities causes a levelling-off of the energy loss with increasing γ . Evidence for this was also given by Hemp²⁴⁾, using diffraction theory. The computed results shown in Table I confirm this hypothesis. When $\gamma < \gamma_c$ there is an increase of the energy loss with increasing γ , in the opposite case, $\gamma > \gamma_c$, the energy loss is approximately independent of γ . Further checks on the models are possible by comparing frequency spectra. The most significant checks can be made when γ and γ_c are very different.

Figure 2 shows two examples with $\gamma > \gamma_c$; the frequency spectrum is independent of γ . Similarly, Figure 3 shows two examples with $\gamma < \gamma_c$; here the frequency spectrum depends on γ . It is also seen that the two models describe the frequency spectra rather well in their respective range of validity.

The good agreement between computed and analytic frequency spectra even at low frequencies allows us to use the analytic frequency spectra in the whole frequency spectrum. We have adopted the following procedure for calculating the radiation loss from analytic formulae :

- i) if $b/a \geq j_1$, the low frequency contribution is calculated from (30) and the low frequency limit for the remainder of the spectrum is set at $\omega = c/a$. For $b/a < j_1$, this limit is taken to be $\omega = cj_1/b$, i.e. at the frequency of the lowest closed cavity mode (28).
- ii) If $\gamma > \gamma_c$ as calculated from (38), the optical resonator model (33) is used with the above lower limit. If $\gamma < \gamma_c$, Lawson's model (26) is used with the above limit.

The energy losses calculated according to this procedure are also shown in Table I. The errors between computed and analytic results are usually below 30%. The only visible trend in the errors is that Lawson's formula (26) does not seem to hold for very small γ .

Therefore, the formulae should be sufficiently accurate for a rough parameter survey. Once the range of likely parameters has been narrowed down, more accurate figures for the radiation loss can be obtained by computation for the lower frequency region, and by using analytic formulae beyond that region, choosing the switch-over frequency such that the ratio between the losses in the two ranges becomes reasonable. It may be seen from Table I that in most of our cases the computation could have been stopped at a lower frequency.

Table I

Comparison between computed radiation losses and analytic formulae

a, b, d, g are geometrical dimensions shown in Fig. 1. γ_c is calculated from (38). U_c is the computed energy loss up to a circular frequency ω , U_{es} is the estimated energy loss for frequencies above ω . $U = U_c + U_{es}$. U_{th} is the analytic energy loss obtained as described in 5.2. All energy losses are in keV/electron/cell for point charges of 10^{13} particles. The letter code indicates the formulae used for U_{es} and U_{th} : KKO means that the low frequency contribution (30) has been included, L stands for Lawson's diffraction model (26), and S for the optical resonator model (33). The last column gives the relative difference between U and U_{th} in per cent.

	a [m]	b [m]	d [m]	g [m]	γ	γ_c	$\left[10^9 \frac{\omega}{s}\right]$	U_c	U_{es}	U	U_{th}		ERROR
1	.05314	.08314	.15314	.07314	5	108	300	398	0	398	183	L	54
2	.05314	.08314	.15314	.07314	20	108	300	738	4	742	621	L	16
3	.05314	.08314	.15314	.07314	50	108	300	856	135	991	1142	L	13
4	.05314	.16628	.15314	.07314	5	108	100	443	0	444	319	L+KKO	28
5	.05314	.16628	.15314	.07314	20	108	200	759	22	782	763	L+KKO	2
6	.05314	.16628	.15314	.07314	50	108	100	695	525	1220	1284	L+KKO	5
7	.04000	.25000	.02000	.02000	50	1	200	286	12	299	281	S+KKO	6
8	.06000	.25000	.02000	.02000	50	1	200	130	2	132	116	S+KKO	12
9	.05000	.20000	.02000	.02000	50	1	200	181	5	186	173	S+KKO	7
10	.05000	.30000	.02000	.02000	50	1	200	189	5	194	173	S+KKO	11
11	.05000	.25000	.04000	.02000	50	4	200	309	13	322	304	S+KKO	6
12	.05314	.16628	.15314	.03657	20	216	200	557	16	573	499	L+KKO	13
13	.05314	.16628	.07657	.03657	20	19	200	437	3	440	463	S+KKO	5
14	.05000	.25000	.02000	.02000	20	1	300	182	0	182	154	S+KKO	16
15	.05000	.25000	.02000	.02000	50	1	300	191	2	193	173	S+KKO	10
16	.05000	.25000	.02000	.02000	200	1	300	193	11	204	191	S+KKO	6
17	.02657	.16628	.15314	.07314	20	403	200	2149	337	2486	2263	L+KKO	9
18	.05000	.25000	.01000	.01000	50	1	200	96	2	98	75	S+KKO	23
19	.05000	.25000	.03000	.03000	50	1	200	274	9	282	279	S+KKO	1
20	.05000	.25000	.10000	.10000	50	21	300	905	19	925	1087	S+KKO	15
21	.05000	.25000	.20000	.02000	50	1031	250	591	119	711	691	L+KKO	3
22	.05000	.25000	.10000	.02000	50	107	300	517	87	604	696	L+KKO	13

Table I (cont'd)

NO	a [m]	b [m]	d [m]	g [m]	Y	γ_c	ω_s^{-1} [10 ⁹ s ⁻¹]	U _c	U _{es}	U	U _{th}		ERROR
23	.19000	.50000	.15500	.03200	40	9	100	66	0	66	73	S+KKO	9
24	.19000	.50000	.07600	.01900	40	1	100	36	0	36	30	S+KKO	17
25	.19000	.50000	.21000	.03200	40	29	100	75	1	75	103	S+KKO	27
26	.24000	.50000	.15500	.03200	40	5	100	41	0	41	40	S	3
27	.24000	.50000	.07600	.01900	40	1	100	21	0	21	16	S	26
28	.24000	.50000	.21000	.03200	40	15	100	50	0	50	58	S	13
29	.14000	.50000	.15500	.03200	40	21	100	112	2	114	142	S+KKO	19
30	.14000	.50000	.07600	.01900	40	2	100	63	1	64	60	S+KKO	6
31	.14000	.50000	.21000	.03200	40	66	100	130	15	146	164	L+KKO	11
32	.19000	.50000	.15500	.03200	20	9	100	58	0	58	62	S+KKO	7
33	.19000	.50000	.07600	.01900	20	1	100	33	0	33	26	S+KKO	21
34	.19000	.50000	.21000	.03200	20	29	100	64	0	64	67	L+KKO	5
35	.24000	.50000	.15500	.03200	20	5	100	35	0	35	34	S	4
36	.24000	.50000	.07600	.01900	20	1	100	19	0	19	14	S	29
37	.24000	.50000	.21000	.03200	20	15	100	43	0	43	49	S	12
38	.14000	.50000	.15500	.03200	20	21	100	100	1	101	107	L+KKO	5
39	.14000	.50000	.07600	.01900	20	2	100	59	0	59	52	S+KKO	11
40	.14000	.50000	.21000	.03200	20	66	100	114	1	115	106	L+KKO	8
41	.01152	.04167	.03500	.02911	999	70	600	4619	2563	7183	8950	S+KKO	20
42	.01152	.04167	.03500	.02911	4000	70	400	4162	3524	7686	9343	S+KKO	18
43	.01294	.04198	.03500	.02911	4000	54	400	3455	2546	6001	7288	S+KKO	18
44	.01226	.04179	.03500	.02911	4000	61	400	3762	2962	6724	8180	S+KKO	18
45	.01142	.04158	.03500	.02911	4000	71	400	4212	3610	7823	9518	S+KKO	18
46	.01039	.04135	.03500	.02911	4000	88	400	4835	4689	9524	11638	S+KKO	18

Table II

Comparison of calculated energy losses in
MeV for the SLAC experiment: 10^9 electrons,
 $r=h=1.5\text{mm}$, 86000 cells, $d=3.5\text{ cm}$, $g=2.9112\text{ cm}$.

a (cm)	b (cm)	γ	computed energy loss	energy loss from (37)
1.152	4.1665	999	37.9	41.3
1.152	4.1665	4000	38.5	41.3
1.294	4.198	4000	31.7	31.7
1.226	4.179	4000	34.7	35.9
1.142	4.158	4000	39.0	42.2
1.039	4.135	4000	45.2	52.0

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FIGURE CAPTIONS

- Fig. 1 Three periods of the waveguide with geometrical parameters. The geometrical parameters of the electron ring are also indicated.
- Fig. 2 Energy loss spectrum for $\gamma > \gamma_c$ as given in (38). The left graph shows case no. 15, the right graph shows case no. 16 in Table I. L is the energy loss spectrum (26) according to Lawson, S-V that according to Sessler-Vainshtein (33).
- Fig. 3 Energy loss spectrum for $\gamma < \gamma_c$ as given in (38). The left graph shows case no. 40, the right graph shows case no. 31 in Table I. L is the energy loss spectrum (26) according to Lawson, S-V that according to Sessler-Vainshtein (33).

Figure 1

Figure 2

Figure 3