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INVARIANT AMPLITUDES AND DOUBLE DISPERSION
RELATIONS FOR ELECTRON-MUON SCATTERING

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A B S T R A C T

For electron-muon scattering, a set of six invariant amplitudes free from kinematical singularities is determined. A simple proof of this freedom is given. Projection operators for this set are constructed. Double dispersion relations using natural subtraction points are derived for the box diagram.

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1. INTRODUCTION AND SUMMARY

We determine for electron-muon scattering a complete set of invariant amplitudes (I.A.) which are free from kinematical singularities (K.S.). K.S. are defined as the singularities introduced by spin. So, if scalar particle scattering satisfies to some order in perturbation theory the Mandelstam representation, so do our I.A.

The first proof that our I.A. are free from K.S. is simpler than the usual proofs, since it does not require the cumbersome threshold constraints between helicity amplitudes. A second proof is given which uses a remark of Hearn ¹⁾ that one can always express the matrix element for a given process into a finite over-complete set of I.A. which are free from K.S. If then, upon reducing this over-complete set into a complete set, one creates no new K.S., then the final complete set is again free from K.S. This second proof needs relations between the dependent invariant spinor operators (I.S.O.). A general method is given to construct such relations. For a number of processes de Calan and Stora ²⁾ obtained already such relations by delicately using Fierz transformations and similar identities.

Projection operators onto the set of I.S.O. are constructed and it is proved that they lie in the linear space spanned by the I.S.O. onto which they project.

Then we apply our results to the box diagram for electron-muon scattering. Indeed, the I.A. are in this case free from K.S. We want to write down double dispersion relations for the I.A.

In the literature, there exist several methods for applying dispersion relations to perturbation calculations. Wortman ³⁾ considered the box diagram for nucleon-nucleon scattering. He determined the double-discontinuity function for the I.A. according to the usual Cutcosky recipe ⁴⁾ and applied subsequently unsubtracted double dispersion relations. It can easily be seen that for the box diagram in nucleon-nucleon scattering his I.A. do indeed satisfy unsubtracted double (and a fortiori single) dispersion relations. Once it is realized that the double discontinuity of the scalar box can be obtained by cutting both in s and in t , his method for spinning particles is clear. Kubis and Gammel ⁵⁾ calculated the box diagram for pion-nucleon scattering. In this case, evaluation of the imaginary parts of the two I.A. shows that one of the I.A. must be subtracted in s in any dispersion relation. By determining the t single-variable dispersion relation in the t channel which happens to be unsubtracted, they determine the subtraction terms in s . Brunet ⁶⁾ determined natural subtraction points for the pion-nucleon box by factoring out polynomials from the single-variable discontinuities, thus improving the convergence of the double dispersion method.

Since our I.A. turn out to need subtractions both in s and in t , the methods of Wortman, Kubis and Gammel cannot be applied to the case of electron-muon scattering. It is, however, possible to write down double dispersion relations for our I.A. The subtraction points follow naturally from a consideration of a certain set of tensor integrals.

This paper is organized as follows. In Section 2, kinematics and notations are given. In Section 3, a complete set of six I.A. is given and their freedom from K.S. is proved in two ways. Also some other sets which are free from K.S. are given here. In Section 4, projection operators for this set of I.A. are constructed. In Section 5, the double dispersion relations are written down for the box diagram. The results agree with an independent calculation of the box diagram by one of us ⁷⁾. A subtle treatment concerning boundary values of the scalar box for photon mass zero is needed; the same type of difficulty, in a different context, has been remarked by Barbieri, Mignaco and Remiddi ⁸⁾. Appendix I is devoted to a property of the projection operators of Section 4 which is needed in the first proof of Section 3. Appendix II describes the general method to determine linear relations between an over-complete set of I.S.O. All tree calculations and algebraic manipulations were performed by computer ⁹⁾.

2. KINEMATICS AND NOTATION

Figure 1 illustrates the process under consideration

$$e^- + \mu^- \rightarrow e^- + \mu^- \quad (2.1)$$

m and M are electron and muon mass respectively. The energy in the c.m. system is $s = -(q_e + q_\mu)^2$, the momentum transfer is $t = -(q_\mu - q'_\mu)^2$ and the Mandelstam variable u is defined by

$$s + t + u = 2M^2 + 2m^2 \quad (2.2)$$

The pseudothreshold is $s_p = (M-m)^2$ and the threshold is $s_t = (M+m)^2$. We define $\Lambda = (s-s_p)(s-s_t)$. The boundary of the physical region of the s , t and u channel is given by

$$\Phi = t [su - (M^2 - m^2)^2] \quad (2.3)$$

The T matrix is defined in terms of the S matrix by

$$S = I + i(2\pi)^4 \delta^{(4)}(q_e + q_\mu - q'_e - q'_\mu) \prod_{i=1}^4 (2E_i)^{-\frac{1}{2}} T \quad (2.4)$$

The extraction of the energy factors in (2.3) suggests the choice of normalization of the electron and muon spinors

$$\sum_{\lambda=1}^2 u^{(\lambda)}(q_e) \bar{u}^{(\lambda)}(q_e) = -i\not{q}_e + m \quad (2.5)$$

The T matrix element can be expanded as

$$\langle e'\mu' | T | e\mu \rangle = \sum_i F_i(s,t) \bar{u}(q'_e) \bar{u}(q'_\mu) O_i u(q_e) u(q_\mu) \quad (2.6)$$

where

$$\bar{u}(q'_e) \bar{u}(q'_\mu) O_i u(q_e) u(q_\mu) = \bar{u}(q'_e) O_i^e u(q_e) \bar{u}(q'_\mu) O_i^\mu u(q_\mu) \quad (2.7)$$

The F_i are called invariant amplitudes (I.A.) ^{*}) and contain the dynamics. For further use we introduce the notation

$$O_i = O_i^e \otimes O_i^\mu$$

The O_i are called invariant spinor operators (I.S.O.) and embody the spin dependence.

For example, the T matrix for the pole diagram in Fig. 2 reads

$$\langle e'\mu' | T | e\mu \rangle = \frac{ie^2}{t} \bar{u}(q'_e) \not{\epsilon}_\mu u(q_e) \bar{u}(q'_\mu) \not{\epsilon}_\mu u(q_\mu).$$

This diagram contributes only to the vector-vector I.S.O.: $O_i = \not{\epsilon}_\mu \otimes \not{\epsilon}_\mu$ and the invariant amplitude is $(ie^2)/t$ in this case.

Since the number of independent I.A. is equal to the number of independent helicity amplitudes (H.A.), let us count the latter. Let the helicities of the incoming electron and muon be denoted by λ_1 and λ_2 , and those of the outgoing fermions by λ_3 and λ_4 . There are 16 H.A. Parity conservation, reading

$$T(\lambda_3, \lambda_4; \lambda_1, \lambda_2) = (-1)^{\sum_{i=1}^4 \lambda_i} T(-\lambda_3, -\lambda_4; -\lambda_1, -\lambda_2) \quad (2.8)$$

reduces the number of independent H.A. to 8. Time-reversal invariance

$$T(\lambda_3, \lambda_4; \lambda_1, \lambda_2) = (-1)^{\sum_{i=1}^4 \lambda_i} T(\lambda_1, \lambda_2; \lambda_3, \lambda_4) \quad (2.9)$$

^{*}) Invariant amplitudes are sometimes called form factors.

eliminates another 2. There are therefore six independent H.A., which can be chosen as

$$\begin{aligned} H_1 &= T(+,+;+,+) & H_2 &= T(+,+;+,-) & H_3 &= T(+,+;- ,+) \\ H_4 &= T(+,+;- ,-) & H_5 &= T(+,-;+,-) & H_6 &= T(+,-;- ,+) \end{aligned} \quad (2.10)$$

So, with (2.6) one has in an obvious notation

$$T = \sum_{i=1}^6 F_i(s,t) O_i \quad (2.11)$$

In the next section a set of O_i will be determined such that the corresponding $F_i(s,t)$ satisfy the Mandelstam representation, i.e., are free from kinematical singularities.

3. INVARIANT AMPLITUDES FREE FROM KINEMATICAL SINGULARITIES

Be given a complete set of six I.A. For definiteness, take the set given in (3.8). Then there are three methods to prove that these I.A. are free from K.S. :

- I. one expands the T matrix into an over-complete (finite) set of I.S.O., such that the corresponding I.A. are free from K.S. This can always be done ¹⁾. Then, if by reducing this over-complete set of I.S.O. to the given complete set no new singularities are created, one has a set of I.A. free from K.S.;
- II. inserting into (2.6) spinors with definite helicity and the explicit form for O_i , as given by (3.8), one can invert the 6×6 matrix relating the H.A. to the I.A. Regularizing the H.A. one obtains

$$F_i = \frac{a_{ij}(s,t,M,m) \hat{H}_j}{(s-s_q)^2 (s-s_p)^2} \quad (i,j = 1, \dots, 6) \quad (3.1)$$

where $\hat{H}$ are regularized H.A., (these are free from K.S.) ¹⁰⁾ and a_{ij} are polynomials in their arguments. Constraints at the thresholds now kill the poles at s_p and s_t ;

III. introducing projection operators for the I.A. (see Section 4) and projecting F_i out of (2.6), one finds

$$F_i = \frac{b_i(s, t, M, m)}{\Phi^2} \quad (i = 1, \dots, 6) \quad (3.2)$$

b_i are polynomials in their arguments, so free from K.S. (see Section 4). Φ was defined in (2.3).

Combining (3.1) and (3.2) one has

$$F_i = \frac{a_{ij} \hat{H}_j}{(s-s_t)^2 (s-s_p)^2} = \frac{b_i}{\Phi^2} \quad (3.3)$$

It follows that the F_i are free from K.S., since poles in the one equation are absent in the other and vice versa. Note that in this proof no constraints are needed.

So let us determine a set of I.A. free from K.S. according to method I. Reducing the over-complete set, mentioned under I, by means of standard Dirac algebra ²⁾ and parity conservation ^{*)}, one has in the notation of (2.11)

$$T = \sum_{i=1}^n F_i(s, t) O_i$$

where

$$\begin{aligned} O_i &= O_i^e \otimes O_i^m = \\ &= (I \text{ or } i\gamma_m) \otimes (I \text{ or } i\gamma_e), \quad \gamma_m \otimes \gamma_m \equiv V \otimes V, \\ \frac{1}{2} \gamma_{\mu\nu} \otimes \gamma_{\mu\nu} &\equiv T \otimes T, \quad i\gamma_m \gamma_5 \otimes i\gamma_m \gamma_5 \equiv A \otimes A, \\ \gamma_5 \otimes \gamma_5 &\equiv P \otimes P, \quad i\gamma_m \gamma_5 \otimes i\gamma_e \gamma_5, \\ i\gamma_m \gamma_5 \otimes \gamma_5, \quad \gamma_5 \otimes i\gamma_e \gamma_5. \end{aligned} \quad (3.4)$$

Time reversal invariance eliminates the last two I.S.O. The application of P and T invariance does not introduce new K.S. as one obtains relations like $O_i = \pm O_i$, thus implying $F_i = 0$ for the minus sign. The remaining nine I.S.O. in (3.4) are not yet independent, since it was shown in Section 2 that there are only six independent I.S.O. Three more relations must exist between them.

*) It is understood that the I.S.O. O_i have to be sandwiched between spinors when applying the Dirac equation or in relations like (3.5).

A general method to determine such relations is shown in Appendix II. One finds according to this method

$$(s-u) I \otimes I + 2M I \otimes i\cancel{P}_e + 2m i\cancel{P}_m \otimes I + 4mM V \otimes V - 2t T \otimes T + (s-u) P \otimes P = 0 \quad (3.5)$$

$$(s-u)^2 I \otimes I + 2M(s-u) I \otimes i\cancel{P}_e + 2m(s-u) i\cancel{P}_m \otimes I + 4mM i\cancel{P}_m \otimes i\cancel{P}_e + 4mMt A \otimes A - t(s-u) T \otimes T - (16m^2M^2 - (s-u)^2) P \otimes P = 0 \quad (3.6)$$

$$(s-u)(t-4m^2-4M^2) I \otimes I + 2M(t-4M^2) I \otimes i\cancel{P}_e + 2m(t-4m^2) i\cancel{P}_m \otimes I + 4mM i\cancel{P}_m \gamma_5 \otimes i\cancel{P}_e \gamma_5 - (t-4m^2)(t-4M^2) T \otimes T + (s-u)(t-4m^2-4M^2) P \otimes P = 0 \quad (3.7)$$

Using these three relations we must now eliminate three among the nine I.S.O. without introducing singularities in s and t . Inspection of (3.6) suggests to eliminate $i\cancel{P}_m \otimes i\cancel{P}_e$. Similarly, (3.7) allows $i\cancel{P}_m \gamma_5 \otimes i\cancel{P}_e \gamma_5$ to be eliminated. But (3.5) allows to be eliminated $i\cancel{P}_m \otimes I$, $I \otimes i\cancel{P}_e$ or $V \otimes V$. Each of these choices gives a set of I.S.O. with the required properties. We choose to eliminate the $V \otimes V$ I.S.O. So we find for (2.11)

$$T = F_1 I \otimes I + F_2 I \otimes i\cancel{P}_e + F_3 i\cancel{P}_m \otimes I + F_4 \frac{1}{2} \gamma_{\mu\nu} \otimes \gamma_{\mu\nu} + F_5 i\cancel{P}_m \gamma_5 \otimes i\cancel{P}_e \gamma_5 + F_6 \gamma_5 \otimes \gamma_5. \quad (3.8)$$

The invariant amplitudes $F_i(s,t)$ have no kinematical singularities. They can be calculated by a single or double dispersion relation. In the next section, projection operators for these invariant amplitudes will be constructed.

4. PROJECTION OPERATORS FOR THE INVARIANT AMPLITUDES

The projection operators P_i for the six invariant amplitudes F_i in (3.8) are defined by

$$\langle P_i, O_j \rangle = \delta_{ij} \quad (4.1)$$

where the symbol $\langle A, B \rangle$ is defined by

$$\langle A, B \rangle = (\text{Tr})_e (\text{Tr})_m [A (-i\not{p}'_e + m)(-i\not{p}'_m + M) B (-i\not{p}_e + m)(-i\not{p}_m + M)] \quad (4.2)$$

The symbols $(\text{Tr})_e$ and $(\text{Tr})_m$ indicate that the trace is taken independently for electron and muon. With (3.8) and (4.1), one has

$$\langle P_i, T \rangle = F_i \quad (4.3)$$

Given a Feynman diagram for electron-muon scattering, its contributions to the invariant amplitudes are determined by (4.3).

The construction of the P_i proceeds as follows. Expanding P_i in terms of the O_j

$$P_i = \alpha_{ij} O_j \quad (4.4)$$

one has for (4.1)

$$\langle P_i, O_k \rangle = \alpha_{ij} \langle O_j, O_k \rangle = \delta_{ik} \quad (4.5)$$

The matrix α_{ij} is therefore equal to the inverse of the matrix $\langle O_i, O_k \rangle$. The algebraic inversion of this 6×6 matrix was done by computer⁹⁾. It turns out that the α_{ij} are of the form

$$\alpha_{ij} = \frac{p_{ij}(s, t, M, m)}{\Phi^2} \quad (4.6)$$

where p_{ij} are polynomials in their arguments and Φ was defined in (2.3). Any expansion

$$P_i = \gamma_{ij} V_j \quad (4.7)$$

can be used for construction of the P_i , provided the matrix $\langle V_i, O_j \rangle$ is regular. The V_j must, however, be Lorentz invariant and have the same P and T properties as the O_i . Therefore, the V_j in (4.7) are a base in the space spanned by the O_i in (3.8) and the P_i are vectors in this six-dimensional space.

5. DOUBLE-DISPERSION RELATIONS FOR THE INVARIANT AMPLITUDES

Consider the box diagram for electron-muon scattering as given in Fig. 3. It is given in terms of the following integrals ⁷⁾

$$-i \int \frac{(\Delta_\alpha \Delta_\beta; \Delta_\gamma; 1)}{\prod_{i=1}^4 (q_i^2 + m_i^2)} d\Delta = (A_1 P_\alpha P_\beta + \dots + A_5 \delta_{\alpha\beta}; \quad (5.1)$$

$$A_6 P_\alpha + A_7 V_\alpha; A_8)$$

The functions A_i satisfy unsubtracted double dispersion relations as can be seen by rewriting the denominator in (5.1) by means of the Feynman method as $[(\Delta+c)^2+b]^4$, and subsequently shifting all (convergent) integrals $\Delta' = \Delta + c$. Let x, y and z denote the Feynman auxiliary variables. Then c is linear in x, y, z and $P = q_m + q_e$, $Q = q_m - q_m'$, and $V = q_m + q_m'$, whereas b is linear in s, t, M^2, m^2 and λ^2 and contains polynomials in x, y, z . Projecting the I.A. from the box diagram given in terms of A_i , one has

$$F_i = r_{ij}(s, t, M, m) A_j(s, t) \quad (j=1, \dots, 8) \quad (5.2)$$

with r_{ij} polynomials in their arguments. With

$$A_j(s, t) = \frac{1}{\pi^2} \int \frac{ds' dt'}{(s'-s)(t'-t)} p_j(s', t') \quad (5.3)$$

we have found a double dispersion relation for the I.A. The r_{ij} are the natural subtractions. The boundary for $p_j(s', t')$ is of course the well-known Mandelstam boundary for the double spectral function of the scalar box. A subtle treatment is required in order to prove that the function with $Q_\alpha Q_\beta$ in (5.1) does not have a pole at $t = 0$. This can be proved by noting that

$$\text{Im } A_0(t \neq 0, \lambda \neq 0) = \frac{2\pi^3}{-t} \ln\left(\frac{-t}{\lambda^2}\right) \frac{1}{\sqrt{\lambda}} \quad (5.4)$$

but

$$\text{Im } A_0(t \neq 0, \lambda \neq 0) = \frac{\pi^3 \sqrt{\lambda}}{\lambda^2} \cdot \frac{1}{\lambda + \lambda^2 s} \quad (5.5)$$

with

$$\Lambda = (s-s_p)(s-s_t) \quad (5.6)$$

Taking the limit $\lambda \downarrow 0$ too early would give (5.4) instead of (5.5) and would erroneously give a pole at $t = 0$. The spectral functions for the double dispersion relations for $a_i(s,t)$ are now easily obtained by multiplying the corresponding spectral function for the scalar box ("the Mandelstam ρ ") with the projection of $\Delta_\alpha \Delta_\rho$ onto $P_\alpha P_\rho$ etc., under the restrictions that all four propagators are on the mass shell. The proof for this assertion is as follows. After projecting out a_i and cutting in s , requiring two propagators to be zero, the numerator is expanded into the two propagators left. Only the term with two propagators in the denominator has a discontinuity in t , giving the Mandelstam ρ , multiplied by the residue of the numerator which was obtained after cutting in s .

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APPENDIX I

In this appendix we will show that the projection operators P_i onto our set of I.A. are of the form

$$P_i = \frac{p'_{ij}(s, t, M, m)}{\Phi^k} O_j \quad (A.1)$$

where p'_{ij} is a polynomial in its arguments. It turns out in the actual calculation that

$$p'_{ij} = p_{ij}(s, t, M, m) \cdot \Phi^k$$

where p_{ij} is again a polynomial in its arguments, but for the proof of freedom of K.S. in Section 3 we need only that some power of Φ occurs in (A.1). So consider the set

$$O_i = I \otimes I, I \otimes i\gamma_e, i\gamma_m \otimes I, \sum \gamma_{\mu\nu} \otimes \gamma_{\mu\nu}, i\gamma_m \gamma_r \otimes i\gamma_m \gamma_r, \gamma_r \otimes \gamma_r \quad (A.2)$$

and another set

$$V_i = I \otimes I, I \otimes i\gamma_e, i\gamma_m \otimes I, i\gamma_m \otimes i\gamma_e, i\gamma_m \gamma_r \otimes i\gamma_e \gamma_r, \gamma_r \otimes \gamma_r \quad (A.3)$$

As shown in Section 4, the projection operators P_i are given by

$$P_i = \alpha_{ij} O_j \quad \text{with} \quad (\alpha^{-1})_{ij} = \langle O_i, O_j \rangle \quad (A.4)$$

The singularities in P_i are therefore given by $\det(\alpha^{-1})$. We recall the definition in (4.2) and define

$$\langle C, D \rangle_e = (\text{Tr})_e [C (-i\gamma'_e + m) D (-i\gamma_e + m)] \quad (A.5)$$

where C and D are ordinary γ matrices.

Defining

$$O_i = \beta_{ij} V_j \quad (\text{A.6})$$

one has

$$(\alpha^{-1})_{ij} = \langle O_i, O_j \rangle = \beta_{ik} \langle V_k, V_e \rangle \beta_{ej}^T \quad (\text{A.7})$$

so

$$\det(\alpha^{-1}) = \det(\langle V_k, V_e \rangle) \cdot (\det \beta)^2. \quad (\text{A.8})$$

The matrix $\gamma_{kl} = \langle V_k, V_l \rangle$ is given by

$$\gamma = \left(\begin{array}{c|c} c & 0 \\ \hline 0 & g\left(\frac{\Phi}{t}\right)^2 \quad 0 \\ & 0 \quad 4t^2 \end{array} \right) \quad (\text{A.9})$$

and from (A.3) it is seen that the 4×4 matrix c is the dual product of two 2×2 matrices c^e and c^m

$$c = \left(\begin{array}{c|c} c_{11}^e I & c_{12}^e I \\ \hline c_{21}^e I & c_{22}^e I \end{array} \right) \left(\begin{array}{c|c} c^m & 0 \\ \hline 0 & c^m \end{array} \right) \quad (\text{A.10})$$

with

$$c_{ij}^e = \langle O_i^e, O_j^e \rangle_e \quad \text{and} \quad O_1^e = I, \quad O_2^e = i\cancel{F}_m \quad (\text{A.11})$$

and similarly c^m .

The 2x2 matrix c^e is readily evaluated to be

$$c^e = \begin{pmatrix} 4m^2 - t & 2m(u-s) \\ 2m(u-s) & t(4m^2 - t) + (u-s)^2 \end{pmatrix} \quad (A.12)$$

so

$$\det c = 16\Phi = 16t[su - (M^2 - m^2)^2] \quad (A.13)$$

Collecting the results of (A.13), (A.10) and (A.9), one has

$$\det (\langle V_u, V_e \rangle) = (16\Phi)^6 \quad (A.14)$$

On the other hand, the matrix β in (A.6) has the structure :

$$\beta = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & B & \\ & & & & & 0 \\ & & & & & & 1 \end{pmatrix} \quad (A.15)$$

For completeness we give in Table I this matrix. For $\det \beta$ we need only the 2x2 submatrix b of the 6x2 matrix B , given by the projections of $0_4, 0_5$ onto V_4, V_5 . The projection operator onto V_5 is equal to

$$\pi_5 = \left(\frac{t}{s\Phi} \right)^2 V_5 \quad (A.16)$$

as follows from (A.9). The projection operator onto V_4 is the product of two projection operators as follows from (A.3)

$$\pi_4 = P_2^e \otimes P_2^m \quad (A.17)$$

where P_j^e are defined by

$$\langle P_j^e, O_\ell^e \rangle_e = \delta_{j\ell} \quad (j, \ell = 1, 2) \quad (\text{A.18})$$

and similarly P_j^m . Expanding

$$P_j^e = \alpha_{jk}^e O_k^e \quad (\text{A.19})$$

one has with (A.11)

$$\langle \alpha_{jk}^e O_k^e, O_\ell^e \rangle_e = \alpha_{jk}^e c_{k\ell}^e = \delta_{j\ell} \quad (\text{A.20})$$

So

$$P_i^e = (c^{e^{-1}})_{ij} O_j^e \quad (\text{A.21})$$

Inserting (A.21) and (A.16) into b

$$b = \begin{pmatrix} \langle \pi_u, O_u \rangle & \langle \pi_s, O_u \rangle \\ \langle \pi_u, O_s \rangle & \langle \pi_s, O_s \rangle \end{pmatrix} \quad (\text{A.22})$$

one has

$$b = \begin{pmatrix} \frac{mM(u-s)}{\Phi} & \frac{mMt}{\Phi} \\ -\frac{(t-4m^2)(t-4M^2)}{4\Phi} & \frac{(s-u)t}{4\Phi} \end{pmatrix} \quad (\text{A.23})$$

Incidentally, the fact that $\det b \neq 0$ guarantees that the set O_i is linearly independent. Finally, one has the desired result

$$\det(\alpha^{-1}) = (16 \Phi)^6 \cdot \left(\frac{mM}{\Phi} \right)^2 \propto \Phi^4 \quad (\text{A.24})$$

Next we prove the result given in (3.1). Expanding the T matrix into the set V_i and inserting spinors with definite helicity, as given in (2.10), one obtains the result given in Table II.

The inversion of the relations in Table II is simple since the 4×4 matrix, operating on the first four G_i , is a dual product of two 2×2 matrices. The determinant of this 4×4 matrix is proportional to $\wedge^4$. Finally, expressing G_i into F_i by means of (A.6) one obtains relation (3.1).

APPENDIX II

In this appendix we will derive in a general way relations like (3.5), (3.6) and (3.7). Suppose one has nine I.S.O., O_k of which the first six are linearly independent

$$O_{6+j} = \tau_{jk} O_k \quad (j=1,3 \text{ and } k=1,6) \quad (B.1)$$

The coefficients τ_{jk} are now given by the projection operators P_k on the I.S.O., O_k :

$$\langle \pi_k, O_{6+j} \rangle = \tau_{jk} \quad (B.2)$$

The construction of P_k proceeds in two steps. First projection operators π_k are constructed for such a set V_k of I.S.O. which has simple π_k , but need not be free of K.S. In our case, this is the set V_1 in Appendix I. Then, by relating the actual set O_1 to the set V_k by means of (A.6), the projection operators are constructed:

$$O_i = \beta_{ik} V_k \quad \langle \pi_k, V_e \rangle = \delta_{ke} \quad \langle P_i, O_j \rangle = \delta_{ij} \quad (B.3)$$

So

$$\beta_{ik} = \langle \pi_k, O_i \rangle \quad (B.4)$$

and expanding

$$P_j = \lambda_{jk} \pi_k \quad (B.5)$$

one has, using (B.4) and (B.5)

$$\langle P_i, O_j \rangle = \lambda_{ik} \langle \pi_k, O_j \rangle = \lambda_{ik} \beta_{jk} = \delta_{ij} \quad (B.6)$$

So P_j is given by

$$P_j = (\beta^T)^{-1}_{je} \pi_e \quad (B.7)$$

Relations (3.5), (3.6) and (3.7) were obtained by calculating⁸⁾ by computer (B.2).

TABLE I

The Matrix β^T defined in (A.6)

$$\beta^T = \begin{pmatrix} 1 & 0 & 0 & \frac{z}{4\phi} (t(t-4m^2-4M^2)-z^2) - \frac{mMz^2}{\phi} & 0 \\ 0 & 1 & 0 & \frac{M}{2\phi} (t(t-4M^2)-z^2) - \frac{mz(t-4M^2)}{2\phi} & 0 \\ 0 & 0 & 1 & \frac{m}{2\phi} (t(t-4m^2)-z^2) - \frac{Mz(t-4m^2)}{2\phi} & 0 \\ 0 & 0 & 0 & -\frac{mMz}{\phi} - \frac{(t-4m^2)(t-4M^2)}{4\phi} & 0 \\ 0 & 0 & 0 & \frac{mMt}{\phi} & \frac{zt}{4\phi} & 0 \\ 0 & 0 & 0 & \frac{z}{t} & \frac{4mM}{t} & 1 \end{pmatrix}$$

$z = s-u; \quad \phi = t(su-(M^2-m^2)^2)$

TABLE II

$$T = \sum_{i=1}^6 G_i V_i$$

$$\begin{aligned} \frac{1}{2}(\hat{H}_5 + \frac{\hat{H}_1}{+st}) &= G_1 - \frac{d}{m}G_2 - \frac{d}{m}G_3 + \frac{d^2}{mM}G_4 \\ \hat{H}_2 &= \frac{e'}{2M}G_1 - eG_2 - \frac{de'}{2mM}G_3 + \frac{de}{m}G_4 \\ \hat{H}_3 &= \frac{e}{2m}G_1 - \frac{de}{2mM}G_2 - e'G_3 + \frac{de'}{M}G_4 \\ -\frac{1}{2}(\hat{H}_6 + \frac{\hat{H}_4}{t}) &= \frac{ee'}{4mM}G_1 - \frac{e^2}{2m}G_2 - \frac{(e')^2}{2M}G_3 + ee'G_4 \\ \frac{1}{2}(\hat{H}_6 - \frac{\hat{H}_4}{t}) &= \frac{\Delta}{4mM}G_5 \\ -\frac{1}{2}(\hat{H}_5 - \frac{\hat{H}_1}{\Delta+ts}) &= \frac{\Delta}{mM}G_6 \end{aligned}$$

$d = s-m^2-M^2, \quad e = s+m^2-M^2, \quad e' = s-m^2+M^2, \quad \Delta = (s-s_p)(s-s_t)$

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FIG. 1 $e^- \mu$ Scattering.

FIG. 2 Born Term.

FIG. 3 Box Diagram.

$$P = q_e + q_m ; Q = q_m - q'_m ; V = q_m + q'_m$$