SPECTRAL FUNCTION SUM RULES FROM CURRENT ALGEBRA, UNITARITY AND ANALYTICITYG. Ecker ^{*)}

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A B S T R A C T

With the assumptions of $SU(2) \times SU(2)$ current algebra, conservation of the vector currents, PCAC and finiteness of the Schwinger terms C_V and C_A , some restrictions of unitarity and analyticity on hard pion current algebra models are obtained for the specific case of the pion form factor. Among the results are a representation for the proper vertex of the vector current between pion states, the Kawarabayashi-Suzuki-Fayyazuddin-Riazuddin relation and other spectral function sum rules. In the case of polynomial models for the proper vertices, a lower bound on C_V and an upper bound for the q width are derived, and estimates are made of corrections to q pole dominance of the vector current spectral function.

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1. - INTRODUCTION

The inclusion of unitarity is one of the outstanding problems in current algebra calculations. Three methods have so far been proposed to tackle this problem ¹⁾.

- 1) One takes a renormalizable field theoretic model that incorporates the requirements of current algebra and calculates higher order perturbation theory, preferably with Padé approximation techniques that guarantee unitarity at each step ²⁾. The main drawback of this approach is the difficulty to arrive at statements that could be considered independent of the specific model.
- 2) One ignores the original motivation of effective Lagrangians and regards those Lagrangians as field theoretic models. As even invariant renormalization methods of the conventional kind fail in this case ³⁾, one is consequently led to consider unconventional methods ²⁾. At present it is not quite clear, yet, to what extent this program is successful.
- 3) One considers the Ward identities in their most general form and tries to impose unitarity on the vertices and propagators in those relations. In this way considerable progress has been made in extracting additional information from current algebra ²⁾.

It is the aim of this paper to show that rigorous constraints can be obtained by incorporating unitarity and analyticity in the hard meson current algebra method, without making any restrictions like pole dominance or elastic unitarity. The method is exhibited for the vector-axial-vector-axial-vector (VAA) current amplitudes, and in particular for the electromagnetic form factor of the pion. The basic assumptions that will be used are chiral $SU(2) \times SU(2)$ current algebra, conservation of the vector current, PCAC and the existence of the Schwinger terms C_V, C_A . The existence of the A_1 meson will not be assumed.

In Section 2, the main assumptions are stated and briefly discussed. The basic representation for the pion form factor is derived, which is a consequence of the Ward identities of the theory.

Section 3 contains the essential part of the paper. Unitarity and analyticity are used to obtain a representation for the proper $V\pi\pi$ vertex and, under additional assumptions, spectral function sum rules for higher moments of the vector current spectral function. With the additional assumption that the first moment of the vector current spectral function diverges, the KSFR relation ⁴⁾ is proved. A consequence for the phenomenological current algebra vertex is discussed.

In Section 4, the general results of the preceding section are qualitatively discussed for the case of polynomial models for the proper vertices. The validity of the polynomial approximation and its relation to Schnitzer's unitarization

program ⁵⁾ is examined. The proper $V \pi\pi$ vertex is shown to be determined up to one constant only. A lower bound for the Schwinger term C_V and an upper bound for the ρ width Γ_ρ are obtained in the cases where polynomial models are in principle applicable. Some consequences of the aforementioned spectral function sum rules for the validity of pole dominance are exhibited.

Section 5 summarizes the results and gives a short outlook on possible further applications. In Appendix A some definitions and the Ward identities for the VAA system are collected. In Appendix B, a connection between the p wave $\pi\pi$ scattering amplitude and the proper $V \pi\pi$ vertex is established, using Schnitzer's results ⁵⁾ on the unitarization of the $\pi\pi$ scattering amplitude.

2. - BASIC REPRESENTATION FOR THE PION FORM FACTOR

In this section, the assumptions which will be used throughout the paper are stated and briefly discussed, and the basic equation for the pion form factor is derived. Most of the definitions and the Ward identities for the VAA three-point functions are put together in Appendix A.

Assumption 1

The current propagators $\Delta_V^{\mu\nu}(k)$ and $\Delta_{A_1}^{\mu\nu}(q)$ have the form

$$\Delta_V^{\mu\nu}(k) = \int \frac{dm^2 \rho_V(m^2)}{k^2 - m^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \right) \quad (2.1a)$$

$$\Delta_{A_1}^{\mu\nu}(q) = \int \frac{dm^2 \rho_A(m^2)}{q^2 - m^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2} \right) \quad (2.1b)$$

with positive definite spectral functions $\rho_V(m^2)$ and $\rho_A(m^2)$. This assumption is practically equivalent to assuming the existence of the Schwinger terms C_V, C_A ; in other words, finite Schwinger terms are compatible with the representations (2.1a) and (2.1b) ⁶⁾.

Assumption 2

The charge density current commutation relations are those of chiral $SU(2) \times SU(2)$ and there are no $I = 1$ operator Schwinger terms in those commutators.

Assumption 3

The vector currents are conserved and the axial vector currents are partially conserved (PCAC). Finally, PCAC is interpreted in the sense that the spin-zero axial vector spectral function is completely given by the pion pole.

The pion form factor $F(t)$ is defined as [See (A.1a)]

$$F(t) (p_\lambda + q_\lambda) = \Delta_{\nu\lambda\eta}(k) \Gamma^\eta(q, p) \quad p^2 = q^2 = m_\pi^2, k^2 = t \quad (2.2)$$

From the Ward identities and the general solution (A.4), we find

$$\begin{aligned} \Gamma_\lambda(q, p) = & q^\mu \Delta_{\nu\mu\lambda}^{-1}(k) - C_\pi F_\pi^{-2} p^\nu \Delta_{\nu\lambda}^{-1}(k) - \\ & - F_\pi^{-2} p_\lambda + C_\pi^2 F_\pi^{-2} (g_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{k^2}) q^\mu p^\nu B_{\mu\nu}^\sigma(q, p) \end{aligned} \quad (2.3)$$

Defining a function $A(t)$ by

$$C_\pi^2 q^\mu p^\nu B_{\mu\nu}^\sigma(q, p) = \frac{1}{2} (p^\sigma + q^\sigma) F(t) \quad p^2 = q^2 = m_\pi^2, k^2 = t \quad (2.4)$$

one finally obtains

$$\begin{aligned} F(t) = & 1 - \frac{1}{2F_\pi^2} \int \frac{dm^2 \rho_V(m^2)}{m^2(t-m^2)} \{t - m^2 A(t)\} = \\ = & 1 - \frac{C_V}{2F_\pi^2} + \frac{1}{2F_\pi^2} \{A(t) - 1\} \int \frac{dm^2 \rho_V(m^2)}{t-m^2} \end{aligned} \quad (2.5)$$

This representation for the pion form factor is the basis for all the considerations in the next two sections.

Before proceeding with this equation, let us briefly reconsider the assumptions we have made so far. In order to obtain (2.5), only the three assumptions listed above have been used. One important point is that no pole dominance assumption has been made to derive (2.5); in particular, nowhere in this paper the existence of the A_1 meson will be assumed.

Probably the last substantiated of the three assumptions is the first one, concerning the finiteness of Schwinger terms, which predicts for instance that the total cross-section of e^+e^- annihilation into $I = 1$ hadrons decreases faster than s^{-2} with increasing centre-of-mass energy s . There are, of course, models where the Schwinger terms are finite, the most prominent being the algebra of fields model ⁷⁾ On the other hand, in composite models of elementary particles where the currents are bilinear in some basic fields, e.g., in the quark model, the Schwinger terms do not exist ⁸⁾ and the representations for the current propagators have to be subtracted. It is, however, well-known that in this case regularized vertices can be defined so that the Ward identities have again the same form with regularized vertices and propagators, the Schwinger terms being replaced by other constants ⁹⁾. Then an analogous equation for the pion form factor may be derived where, however, additional unknown

subtraction constants appear which have no direct physical interpretation, in the sense that they are not given by some moments of the current spectral functions. Although the analysis of the following two sections could be carried through in basically the same way, the results would have less predictive power.

3. - GENERAL RESTRICTIONS FROM UNITARITY AND ANALYTICITY

The representation (2.5) may be used in two ways. The first one is to approximate $\rho_V(m^2)$ by the 2π contribution and to solve the resulting integral equation for the pion form factor ¹⁰⁾ using a specific model for $A(t)$. Taking into account certain consistency conditions which require the introduction of subtractions in the dispersion relation for $F^{-1}(t)$ to account for higher mass contributions to $\rho_V(m^2)$, very good agreement with all existing data on $F(t)$ is obtained ¹¹⁾.

In this paper, a more general approach will be described to investigate the consequences of unitarity and analyticity for the pion form factor.

Since both $F(t)$ and $\Delta_V(t) = \int [dm^2 \rho_V(m^2) / t - m^2]$ are real analytic functions in the complex t plane with a cut from $4m_\pi^2$ to infinity, the same must be true for $A(t)$.

Therefore, $A(t)$ may be written

$$A(t) = P(t) + t^n \int_{4m_\pi^2}^{\infty} \frac{rv(x) dx}{x^n(t-x)} \quad (3.1)$$

where $P(t)$ is a polynomial in t with real coefficients.

From (2.5)

$$F(0) = 1 - \frac{C_V}{2F_\pi^2} A(0) \quad (3.2)$$

Because of vector current conservation $F(0) = 1$ and therefore

$$A(0) = 0 \quad (3.3)$$

It may seem strange that we have to impose a condition on $A(t)$ which results from vector current conservation, although the conservation has been used to derive the Ward identities which in turn led to (2.5) and (3.2). The solution of this puzzle comes from the fact that the boundary condition (A.5) already fixes $A(0)$ to be zero.

To show this, define

$$C_{\pi}^2 q^{\mu} p^{\nu} B_{\mu\nu}^{\sigma}(q, p) = \frac{1}{2} (p^{\sigma} + q^{\sigma}) F(q^2, p^2, k^2) + k^{\sigma} C(q^2, p^2, k^2) \quad (3.4)$$

with $A(m_{\pi}^2, m_{\pi}^2, k^2) = A(t)$ and $C(m_{\pi}^2, m_{\pi}^2, k^2) = 0$. Furthermore, since $B_{\mu\nu}^{\sigma}$ has no singularity for $k^2 = 0$, $A(q^2, p^2, 0)$ and $C(q^2, p^2, 0)$ are finite.

Inserting (3.4) into the boundary condition (A.5), multiplied by $C_{\pi}^2 q^{\mu} p^{\nu}$, one obtains

$$\begin{aligned} \lim_{k^2 \rightarrow 0} k_{\sigma} C_{\pi}^2 q^{\mu} p^{\nu} B_{\mu\nu}^{\sigma}(q, p) &= \lim_{k^2 \rightarrow 0} \left\{ \frac{1}{2} (p^2 - q^2) F(q^2, p^2, k^2) + \right. \\ &+ \left. k^2 C(q^2, p^2, k^2) \right\} = \frac{1}{2} (p^2 - q^2) F(q^2, p^2, 0) = \\ &= \lim_{k^2 \rightarrow 0} C_{\pi}^2 C_{\nu}^{-1} q^{\mu} p^{\nu} \left\{ \Delta_{\pi 1 \mu\nu}^{-1}(p) - \Delta_{\pi 1 \mu\nu}^{-1}(q) \right\} = \\ &= \lim_{k^2 \rightarrow 0} C_{\pi}^2 C_{\nu}^{-1} \left\{ -C_{\pi}^{-1} q^{\mu} p_{\mu} + C_{\pi}^{-1} p^{\nu} q_{\nu} \right\} = 0 \end{aligned} \quad (3.5)$$

Thus, $A(q^2, m_{\pi}^2, 0) = 0$ for all $q^2 \neq m_{\pi}^2$; as $A(q^2, m_{\pi}^2, 0)$ may be expected to be at least continuous for $q^2 \rightarrow m_{\pi}^2$, it follows that

$$F(m_{\pi}^2, m_{\pi}^2, 0) \equiv F(0) = 0 \quad (3.6)$$

Thus, (3.3) is a consequence of the Ward identities, as it should be.

The crucial step consists in making the obvious, but important observation that the existence of C_{ν} implies an upper bound for the asymptotic decrease of $F(t)$ for $t \rightarrow \infty$.

In fact, because

$$\rho_{\nu}(t) \geq \rho_{2\pi}(t) = \frac{1}{48\pi^2 t^{\frac{1}{2}}} |F(t)|^2 (t - 4m_{\pi}^2)^{\frac{3}{2}} \theta(t - 4m_{\pi}^2) \quad (3.7)$$

and

$$\int \frac{\rho_V(m^2)}{m^2} dm^2 < \infty$$

$F(t)$ must fall off more rapidly than $t^{-\frac{1}{2}}$. For notational convenience let us write this statement as $F(t) = o(t^{-\frac{1}{2}})$ with the obvious definition for the symbol o .

On the other hand, from the representation (2.5)

$$\operatorname{Re} F(t) = 1 - \frac{C_V}{2F_\pi^2} + \frac{1}{2F_\pi^2} (\operatorname{Re} A(t) - 1) \operatorname{Re} \Delta_V(t) + \frac{\pi}{2F_\pi^2} \operatorname{Im} A(t) \cdot \rho_V(t) \quad (3.8a)$$

$$\operatorname{Im} F(t) = \frac{1}{2F_\pi^2} \operatorname{Im} A(t) \operatorname{Re} \Delta_V(t) - \frac{\pi}{2F_\pi^2} (\operatorname{Re} A(t) - 1) \cdot \rho_V(t) \quad (3.8b)$$

Since both $\operatorname{Re} F(t)$ and $\operatorname{Im} F(t)$ must go to zero faster than $t^{-\frac{1}{2}}$, it is straightforward to show, taking into account (3.1), (3.3) and (3.8a,b) and assuming that $\operatorname{Im} A(t)$ is not indefinitely oscillating, that $A(t)$ must be of the form

$$A(t) = a \cdot t + t \int \frac{w(x) dx}{x(t-x)} \quad , \quad a \text{ real} \quad (3.9)$$

With only the three assumptions of Section 2, there is no further information on $w(x)$ and the dispersion integral in (3.9). In the next section, arguments will be presented that $\operatorname{Im} A(t)$ may be expected to be small and that therefore a polynomial model for $A(t)$ should be a reasonable approximation. In the remainder of this section, however, general consequences of the representation (3.9) will be examined.

If we assume that $w(x)$ is not indefinitely oscillating and goes to zero fast enough so that $\int [w(x)/x] dx$ exists ^{*}, $A(t)$ may be written

$$A(t) = at + \int \frac{w(x) dx}{x} + c(t) \quad (3.10)$$

with $\lim_{t \rightarrow \infty} c(t) = 0$.

*) This assumption is made for notational convenience only ; the discussion can be carried through in basically the same way if $\int [w(x)/x] dx$ does not exist or if $w(x)$ is indefinitely oscillating, as long as something is known about the asymptotic behaviour of $w(x)$.

In order to discuss the high t behaviour of $F(t)$ further, it will be necessary to distinguish different cases depending on the existence or non-existence of higher moments of $\rho_V(m^2)$.

Case 1

$\int \rho_V(m^2) dm^2$ does not exist. Thus

$$\lim_{t \rightarrow \infty} t \Delta_V(t) = \infty \quad (3.11)$$

and

$$F(t) \rightarrow 1 - \frac{C_V}{2F_\pi^2} + \frac{1}{2F_\pi^2} \left(at + \int \frac{n(x)}{x} dx - 1 \right) \lim_{t \rightarrow \infty} \Delta_V(t) \quad (3.12)$$

Because $F(t) \xrightarrow[t \rightarrow \infty]{} 0$, we get $a = 0$ and

$$C_V = 2F_\pi^2 \quad (3.13)$$

The sum rule (3.13) is known as the KSFR relation ⁴⁾, although it is usually expressed in the pole-dominated version, with $\rho_V(m^2)$ given by the ρ pole.

In this case, the pion form factor may be written

$$F(t) = \frac{1}{2F_\pi^2} \left(\int \frac{n(x)}{x} dx - 1 + c(t) \right) \Delta_V(t) \quad (3.14)$$

and unless $\int [\bar{w}(x)/x] dx = 1$, $F(t)$ decreases less fast than t^{-1} , but of course faster than $t^{-\frac{1}{2}}$. Case 1 will be discussed further in Section 4 for polynomial models, where $\text{Im } A(t) = 0$.

Case 2

$\int \rho_V(m^2) dm^2$ is finite, but $\int \rho_V(m^2) m^2 dm^2$ diverges.

In this case

$$\lim_{t \rightarrow \infty} t^2 \left\{ \Delta_V(t) - \frac{1}{t} \int \rho_V(m^2) dm^2 \right\} = \infty \quad (3.15)$$

and $F(t) = o(t^{-1})$.

Writing (2.5) in the form

$$F(t) = 1 - \frac{C_V}{2F_\pi^2} + \frac{1}{2F_\pi^2} \left(at + \int \frac{n(x)}{x} dx + c(t) - 1 \right) \left\{ \frac{1}{t} \int \rho_V(m^2) dm^2 + \left[\Delta_V(t) - \frac{1}{t} \int \rho_V(m^2) dm^2 \right] \right\} \quad (3.16)$$

and using (3.15) and $\lim_{t \rightarrow \infty} t F(t) = 0$, one obtains

$$a=0, \quad C_v = 2F_\pi^2, \\ \int \frac{N(x)}{x} dx = 1 \quad (3.17)$$

Again, the KSFR relation holds, but in addition a sum rule for the absorptive part of $A(t)$ emerges. In fact, case 2 will turn out to be the only one where a non-vanishing imaginary part $w(x)$ is required by unitarity.

Case 3

$$\int \rho_V(m^2) m^2 dm^2 < \infty \Rightarrow F(t) = o(t^{-\frac{3}{2}})$$

The representation (2.5) leads to the two sum rules

$$1 - \frac{C_v}{2F_\pi^2} + \frac{a}{2F_\pi^2} \int \rho_V(m^2) dm^2 = 0 \quad (3.18)$$

$$\left(\int \frac{N(x)}{x} dx - 1 \right) \int \rho_V(m^2) dm^2 + a \int \rho_V(m^2) m^2 dm^2 = 0 \quad (3.19)$$

If one wants to continue the discussion in the same way as outlined above, some assumption about the asymptotic behaviour of the function $c(t)$ is necessary at this stage. Since information on $c(t)$ does not seem to be available, the general case will be discussed in the next section where $w(x) = 0$ will be assumed.

Before doing so, let me make a few remarks on the consequences of the representation (3.9) for the phenomenological current algebra vertex, as investigated by Schnitzer and Wise¹²⁾.

In the notation of those authors, the vertex function Γ_λ can be written in the following way^{*)}

$$\Gamma_\lambda(q, p) \Big|_{\substack{q^2 = p^2 = m_\pi^2 \\ k^2 = t}} = (p_\lambda + q_\lambda) \left[\Delta_V^{-1}(t) + F_\pi^2 m_\pi^4 t K_2(m_\pi^2, m_\pi^2, t) \right] \quad (3.20)$$

*) $K_2(q^2, p^2, k^2)$ is one of the six kinematical singularity free scalar functions referred to in Appendix A.

In particular, they have examined a polynomial model with

$$K_2(q^2, p^2, k^2) = k_{21} + (q^2 + p^2 - k^2) k_{22} \quad (3.21)$$

where k_{21} and k_{22} are constants. One interesting feature of this model is the following: if one assumes the validity of the Bjorken-Johnson-Low limit ¹³⁾ and the validity of (3.21) for large q^2 and p^2 , it can be shown ¹²⁾ that $k_{22} \neq 0$ implies

$$[\partial_\mu H_\alpha^\mu(x), \partial_\nu H_\beta^\nu(0)] \delta(x^0) \neq 0, \quad (3.22)$$

in other words, the pion field which is defined by the PCAC relation could not be the canonical pion field.

In the notation of this paper

$$\Gamma_\lambda(q, p) \Big|_{\substack{q^2=p^2=m_\pi^2 \\ k^2=t}} = - \frac{(p_\lambda + q_\lambda)}{2F_\pi^2} \left[(G - 2F_\pi^2) \Delta_V^{-1}(t) + 1 - H(t) \right] \quad (3.23)$$

Taking into account (3.9) and the positive definiteness of $\rho_V(m^2)$, which implies that $\Delta_V^{-1}(t)$ cannot increase faster than linearly with t for large t , a comparison of (3.20) and (3.23) gives

$$\lim_{t \rightarrow \infty} K_2(m_\pi^2, m_\pi^2, t) = \text{const.} \quad (3.24)$$

If we now assume that (3.21) holds not only for large q^2 or p^2 , but also for large k^2 , (3.24) implies $k_{22} = 0$. Thus, at least for the polynomial model of Schnitzer and Wise, PCAC may still define a canonical pion field.

It is obvious that the approach used in this section cannot restrict the large q^2 , p^2 , k^2 behaviour of the scalar functions $A_i(q^2, p^2, k^2)$ ($i = 1, \dots, 6$) in general. However, if one is willing to accept the ρ and the A_1 as external particles, then the same method we have used for the pion form factor can be employed to study the form factors $\langle \pi | V_\mu | A_1 \rangle$, $\langle A_1 | V_\mu | A_1 \rangle$, $\langle \pi | A_\mu | \rho \rangle$ and $\langle A_1 | A_\mu | \rho \rangle$. Unitarity and analyticity will again provide information on those form factors and therefore on $A_i(q^2, p^2, k^2)$. This program is under investigation and results will be reported elsewhere.

4. - POLYNOMIAL MODELS

In the last section, the general procedure for obtaining sum rules from unitarity and analyticity requirements within the hard pion current algebra approach has been demonstrated. In order to discuss the results more quantitatively, some assumptions about the absorptive part of $A(t)$ have to be made.

In practically all the applications of the hard pion method the proper vertex functions are assumed to vary smoothly away from their mass shell values, i.e., they are approximated by low order polynomials in the momenta. Such an approximation is expected to be reasonable at least in the resonance region, since the dominant dynamical effects in this energy region, namely the resonances, are already taken into account by the propagators.

If this approximation makes sense, then the absorptive part of $A(t)$ should be negligible in the low energy region. It is therefore tempting to assume $w(t) \simeq 0$ for all t . In this section, the consequences of such an assumption for the spectral function sum rules derived in the last section will be discussed.

Additional evidence for the soundness of such an approximation comes from the recent work of Schnitzer ⁵⁾ on the unitarization of the $\pi\pi$ scattering amplitude. It is clear from the unitarity relation for the pion form factor that in the elastic region $A(t)$ is related to the p wave projection of the $\pi\pi$ scattering amplitude. An essential ingredient of the unitarization program of Schnitzer is the assumption that the imaginary part of the so-called sea-gull term [see Ref. 5) for details] is negligible in the low energy region. It is an easy exercise to show that if the sea-gull term is real, the same must be true for the function $A(t)$, at least for $4m_\pi^2 \leq t \leq 16m_\pi^2$ (see Appendix B). For the justification of the reality condition for the contact term we refer to Ref. 5) and in particular to a recent preprint by Ford and Schnitzer ¹⁴⁾ where the sea-gull term is calculated to second order perturbation theory in the linear σ model.

If we therefore assume that the absorptive part of $A(t)$ may be neglected, $A(t)$ reduces to

$$A(t) = a.t \quad (4.1)$$

Therefore all possible hard pion models where the scalar functions $A_i(q^2, p^2, k^2)$ are approximated by polynomials can only differ in the value of the constant a , as far as $A(t)$ is concerned. In this respect, all polynomial hard pion models are equivalent to the original σ model of Schnitzer and Weinberg ¹⁵⁾.

Let us now look at the various cases as discussed in the last section.

Case 1

$$\int \rho_V(m^2) dm^2 = \infty$$

Because of $a = 0$ and $C_V = 2F_\pi^2$ the pion form factor has in this case the especially simple representation

$$F(t) = - \frac{1}{2F_\pi^2} \Delta_V(t) \quad (4.2)$$

One of the consequences of this representation is the absence of zeros of $F(t)$ for all t in the complex t plane except possibly on the cut. This implies that $F(t)$ for space-like t and the phase of $F(t)$ are uniquely determined by the modulus of $F(t)$ on the cut¹⁶⁾, which is measurable in an e^+e^- colliding beam experiment.

Moreover, if one saturates the KSFR relation by the ρ pole only and takes into account $a = 0$, the Ward identities determine uniquely the ρ width Γ_ρ . For $m_\rho = 765$ MeV, $F_\pi = 95$ MeV, Γ_ρ comes out to be 133 MeV. This point will be further clarified at the end of this section.

Case 2

$$\int \rho_V(m^2) dm^2 < \infty, \quad \int \rho_V(m^2) m^2 dm^2 = \infty$$

As already mentioned, in this case a simple polynomial model is not sufficient. As a matter of fact, one obtains a sum rule for the absorptive part of $A(t)$ [see (3.17)], which must therefore be non-zero.

Case 3 (general case)

$$\int dm^2 m^{2N} \rho_V(m^2) < \infty \quad (N \geq 1, \text{ integer})$$

The representation (2.5) together with $w(t) = 0$ leads to $F(t) = o(t^{-(N+2)/2})$ and the following sum rules

$$1 - \frac{C_V}{2F_\pi^2} + \frac{a}{2F_\pi^2} \int dm^2 \rho_V(m^2) = 0 \quad (4.3)$$

$$- \int dm^2 m^{2n} \rho_V(m^2) + a \int dm^2 m^{2(n+1)} \rho_V(m^2) = 0 \quad n=0,1,\dots, [N/2] \quad (4.4)$$

where $[N/2]$ denotes the largest integer that is smaller or equal to $N/2$. If we know in addition that $\int dm^2 \rho_V(m^2) m^{2(N+1)}$ diverges, we get the same equations and the additional information that $F(t)$ must decrease less fast than $t^{-(N+1)}$.

Equations (4.4) imply $a > 0$, which because of (4.3) gives

$$C_V > 2F_\pi^2 \quad (4.5)$$

Moreover, simple ρ pole dominance is excluded. If $\rho_V(m^2) = g_\rho^2 \delta(m^2 - m_\rho^2)$, (4.4) implies $a = m_\rho^{-2}$ which is in contradiction with (4.3). Thus, for polynomial models, unitarity is incompatible with a ρ pole dominated spectral function.

To get some feeling for the importance of contributions other than from the ρ pole, let us write

$$\int dm^2 m^{2n} \rho_V(m^2) = g_\rho^2 m_\rho^{2n} (1 + z_n) \quad (4.6)$$

and assume that C_V is pole dominated, i.e., $C_V = (g_\rho^2/m_\rho^2)$, and, furthermore, that $z_0 \geq 0$.

Defining $b = am_\rho^2$, (4.3) gives

$$1 - \frac{g_\rho^2}{2F_\pi^2 m_\rho^2} + \frac{b}{2F_\pi^2 m_\rho^2} g_\rho^2 (1 + z_0) = 0 \quad (4.7)$$

or

$$b = (1 + z_0)^{-1} \left(1 - \frac{2F_\pi^2 m_\rho^2}{g_\rho^2} \right) < (1 + z_0)^{-1} \leq 1 \quad (4.8)$$

using (4.5) and $z_0 \geq 0$.

This implies from the set of equations (4.4)

$$1 + z_n = b(1 + z_{n+1}) < 1 + z_{n+1}$$

or

$$z_{n+1} > z_n \quad (4.9)$$

Therefore, the equations are in agreement with the almost evident conjecture that non ρ pole contributions become more and more important as one goes to higher moments of $\rho_V(m^2)$.

The Ward identities determine the ρ width Γ_ρ [which may be assumed to be equal to $\Gamma(\rho \rightarrow 2\pi)$] to be

$$\Gamma_\rho = \frac{1}{y} \frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{96\pi F_\pi^2} [1-b]^2 = \frac{1}{y} (1-b)^2 \cdot 133 \text{ MeV} \quad (4.10)$$

with

$$y = 2F_\pi^2 m_\rho^2 / g_\rho^2$$

Taking for y the Orsay value ¹⁷⁾ ~ 0.7 and determining b from (4.10) and the experimental ρ width, one can calculate z_0 and z_1 , for instance. For the sake of presenting some numbers assume that the background to the ρ pole in the spectral function integrals can be represented by a particle with mass $m_{\rho'}$ and coupling constant $g_{\rho'}$ to the vector current. Then $g_{\rho'}^2 = z_0 g_\rho^2$ and $m_{\rho'}^2 = z_1 z_0^{-1} m_\rho^2$; for Γ_ρ between 110 and 125 MeV, z_0 varies from 0.3 to 0.6, but $z_1 z_0^{-1}$ varies much less between 16 and 13. Thus, the background may be represented by a particle with a mass around $4m_\rho$. The corrections to the pole dominated Schwinger term C_V are, in the same approximate sense only, then given by $z_0^2 z_1^{-1}$, which varies between 0.02 and 0.05, being smaller for smaller Γ_ρ .

The only conclusion which one might want to draw from this little piece of arithmetics is that the experimental data seem to indicate that pole dominance is probably a reasonable approximation for the Schwinger term, but not necessarily for the higher moments of $\rho_V(m^2)$. In particular, pole dominance for Weinberg's second sum rule ¹⁸⁾ may be misleading.

To summarize briefly the contents of this section, in the cases where polynomial models are applicable at all (Cases 1 and 3), the following two bounds were obtained

$$C_V \geq 2F_\pi^2 \quad (4.11)$$

$$0 < \Gamma_\rho \leq y^{-1} \cdot 133 \text{ MeV} \quad (4.12)$$

with $y \sim 0.7$ from the Orsay data and $y = 1$ from the ρ pole saturated KSFR relation. Assuming the existence of higher moments of $\rho_V(m^2)$, one gets further constraints in the form of spectral function sum rules. All the results are in agreement with present experimental information, thereby giving additional justification to the approximation made in this section, concerning the absorptive part of $A(t)$.

5. - CONCLUSION AND OUTLOOK

With a straightforward approach some consequences of unitarity and analyticity for current algebra calculations have been obtained for the specific case of the electromagnetic form factor of the pion. Among the general results is the KSFR relation which was derived from the usual current algebra postulates, finiteness of Schwinger terms and the assumption that $\int \rho_V(m^2) m^2 dm^2$ diverges. The various spectral function sum rules are especially restrictive for polynomial models for the proper vertices. In particular, the proper $V \pi \pi$ vertex is determined up to one constant. If this parameter is calculated by putting in the experimental ρ width then, for example, the magnitude of the shift of the peak of the pion form factor with respect to the resonance position is uniquely given and in agreement with the predictions of phenomenological fits to $F(t)$ ¹¹⁾. Furthermore, a lower bound for C_V and an upper bound for Γ_ρ are obtained, both again in agreement with experimental data. The spectral function sum rules are used to estimate the corrections to pole dominance of the spectral function.

If further progress along these lines in the VAA system is to be made, one has to introduce the ρ and the A_1 as external particles. Then additional constraints for the phenomenological current algebra vertex follow from a study of the various form factors.

The investigation of quark type models of currents with infinite Schwinger terms is another possibility. Although a priori more unknown quantities are introduced, compared to models with finite Schwinger terms, such an analysis would be interesting from a principal point of view, since apparently the quark models are more successful in predicting the high energy behaviour of various amplitudes. In addition, models with infinite Schwinger terms are interesting from the point of view of anomalous Ward identities in the case of VVA and AAA vertex functions ⁹⁾.

It need hardly be emphasized that the approach presented in this paper is in no way limited to $SU(2) \times SU(2)$ or VAA vertex functions. For instance, one interesting application is in $K_{\ell 3}$ decays where the same method may be used for the vertices of $SU(3) \times SU(3)$ currents ¹⁹⁾.

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APPENDIX A : DEFINITIONS AND WARD IDENTITIES ¹⁵⁾

In order to separate the spin-one and spin-zero parts of the axial vector current, one defines proper, diagonalized vertices $\Gamma_\lambda(q,p)$, $\Gamma_{\nu\lambda}(q,p)$, $\Gamma_{\mu\nu\lambda}(q,p)$ in the following manner :

$$\begin{aligned} & \int d^4x d^4y e^{iqx - ipy} \langle 0 | T \{ \partial_\mu \Pi_a^\mu(x), \partial_\nu \Pi_a^\nu(y), V_c^\lambda(0) \} | 0 \rangle = \\ & = i \epsilon_{abc} F_\pi^2 m_\pi^4 (m_\pi^2 - q^2)^{-1} (m_\pi^2 - p^2)^{-1} \Delta_v^{\lambda\eta}(k) \Gamma_\eta(q,p) \end{aligned} \quad (A.1a)$$

$$\begin{aligned} & \int d^4x d^4y e^{iqx - ipy} \langle 0 | T \{ \partial_\mu \Pi_a^\mu(x), \Pi_a^\nu(y), V_c^\lambda(0) \} | 0 \rangle = \\ & = \epsilon_{abc} F_\pi m_\pi^2 (m_\pi^2 - q^2)^{-1} \Delta_{\pi 1}^{\nu\sigma}(p) \Delta_v^{\lambda\eta}(k) \Gamma_{\sigma\eta}(q,p) + \\ & + \epsilon_{abc} F_\pi^2 m_\pi^2 (m_\pi^2 - q^2)^{-1} (m_\pi^2 - p^2)^{-1} p^\nu \Delta_v^{\lambda\eta}(k) \Gamma_\eta(q,p) \end{aligned} \quad (A.1b)$$

$$\begin{aligned} & \int d^4x d^4y e^{iqx - ipy} \langle 0 | T \{ \Pi_a^\mu(x), \Pi_a^\nu(y), V_c^\lambda(0) \} | 0 \rangle = \\ & = i \epsilon_{abc} \Delta_{\pi 1}^{\mu\tau}(q) \Delta_{\pi 1}^{\nu\sigma}(p) \Delta_v^{\lambda\eta}(k) \Gamma_{\tau\sigma\eta}(q,p) + \\ & + i \epsilon_{abc} F_\pi \{ (m_\pi^2 - p^2)^{-1} p^\nu \Delta_{\pi 1}^{\mu\sigma}(q) \Delta_v^{\lambda\eta}(k) \Gamma_{\sigma\eta}(p,q) + \\ & + (m_\pi^2 - q^2)^{-1} q^\mu \Delta_{\pi 1}^{\nu\sigma}(p) \Delta_v^{\lambda\eta}(k) \Gamma_{\sigma\eta}(q,p) \} + \\ & + i \epsilon_{abc} F_\pi^2 (m_\pi^2 - q^2)^{-1} (m_\pi^2 - p^2)^{-1} q^\mu p^\nu \Delta_v^{\lambda\eta}(k) \Gamma_\eta(q,p) \end{aligned} \quad (A.1c)$$

F_π is the pion decay constant ($F_\pi \simeq 95$ MeV), $k = p - q$, and Δ_V, Δ_{A_1} are the propagators for the vector current and the spin-one part of the axial vector current, respectively. Bose symmetry implies

$$\Gamma_\lambda(q,p) = \Gamma_\lambda(p,q) \quad , \quad \Gamma_{\mu\nu\lambda}(q,p) = \Gamma_{\nu\mu\lambda}(p,q)$$

The assumptions listed in Section 1 lead to the Ward identities ¹⁵⁾

$$k^\lambda \Gamma_\lambda(q, p) = C_V^{-1} (q^2 - p^2) \quad (\text{A.2a})$$

$$k^\lambda \Gamma_{\nu\lambda}(q, p) = -F_\pi C_V^{-1} k^\mu \Delta_{\pi 1 \mu\nu}^{-1}(p) \quad (\text{A.2b})$$

$$k^\lambda \Gamma_{\mu\nu\lambda}(q, p) = C_V^{-1} \{ \Delta_{\pi 1 \mu\nu}^{-1}(p) - \Delta_{\pi 1 \mu\nu}^{-1}(q) \} \quad (\text{A.2c})$$

$$p^\nu \Gamma_{\nu\lambda}(q, p) = -F_\pi C_\pi^{-1} \Gamma_\lambda(q, p) + F_\pi C_\pi^{-1} q^\mu \Delta_{\nu\mu\lambda}^{-1}(k) \quad (\text{A.2d})$$

$$q^\mu \Gamma_{\mu\nu\lambda}(q, p) = -F_\pi C_\pi^{-1} \Gamma_{\nu\lambda}(q, p) + C_\pi^{-1} \{ \Delta_{\nu\nu\lambda}^{-1}(k) - \Delta_{\pi 1 \nu\lambda}^{-1}(p) \} \quad (\text{A.2e})$$

and the first Weinberg sum rule ¹⁸⁾

$$C_V = C_\pi + F_\pi^2 \quad (\text{A.3})$$

The Schwinger terms C_V, C_A are defined as

$$C_V = \int \frac{\rho_V(m^2)}{m^2} dm^2$$

where $\rho_V(m^2), \rho_A(m^2)$ are the spectral functions for the vector current and the spin-one part of the axial vector current.

The most general solution of the Ward identities may be written in the following form

$$\begin{aligned} \Gamma_{\mu\nu\lambda}(q, p) = & \frac{k_\lambda}{k^2} C_V^{-1} \{ \Delta_{\pi 1 \mu\nu}^{-1}(p) - \Delta_{\pi 1 \mu\nu}^{-1}(q) \} + \\ & + (g_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{k^2}) B^\sigma_{\mu\nu}(q, p) \end{aligned} \quad (\text{A.4})$$

with the boundary condition

$$\lim_{k^2 \rightarrow 0} k_\sigma B^\sigma_{\mu\nu}(q, p) = C_V^{-1} \{ \Delta_{\pi 1 \mu\nu}^{-1}(p) - \Delta_{\pi 1 \mu\nu}^{-1}(q) \} \quad (\text{A.5})$$

in order to eliminate the kinematical singularity at $k^2 = 0$.

The most general form for $B_{\lambda\mu\nu}(q,p)$ is ^{*)}

$$\begin{aligned} B_{\lambda\mu\nu}(q,p) = & H_1(q^2, p^2, k^2) g_{\mu\nu} Q_\lambda + H_2(q^2, p^2, k^2) g_{\mu\lambda} Q_\nu + \\ & + H_2(p^2, q^2, k^2) g_{\nu\lambda} Q_\mu + H_3(q^2, p^2, k^2) g_{\mu\lambda} k_\nu - \\ & - H_3(p^2, q^2, k^2) g_{\nu\lambda} k_\mu + H_4(q^2, p^2, k^2) Q_\lambda Q_\mu Q_\nu + \\ & + H_5(q^2, p^2, k^2) Q_\lambda Q_\mu k_\nu - H_5(p^2, q^2, k^2) Q_\lambda Q_\nu k_\mu + \\ & + H_6(q^2, p^2, k^2) Q_\lambda k_\mu k_\nu \end{aligned} \quad (A.6)$$

with the symmetry relations

$$H_1(q^2, p^2, k^2) = H_1(p^2, q^2, k^2) \quad (A.7a)$$

$$H_4(q^2, p^2, k^2) = H_4(p^2, q^2, k^2) \quad (A.7b)$$

$$H_6(q^2, p^2, k^2) = H_6(p^2, q^2, k^2) \quad (A.7c)$$

and $Q = p + q$.

If $\Gamma_{\nu\lambda}(q,p)$ and $\Gamma_\lambda(q,p)$ are now calculated from (A.2e) and (A.2d), the remaining Ward identities are automatically fulfilled, taking into account (A.3). Thus, the most general solution of the Ward identities contains six arbitrary scalar functions with the symmetry requirements (A.7a,b,c) and the boundary conditions (A.5). One can define new scalar functions which take the boundary conditions into account and are then not further constrained ¹²⁾.

^{*)} Of course, one could add terms with k_λ , but those terms are spurious since they do not contribute to $\Gamma_{\mu\nu\lambda}$ and therefore do not appear in the Ward identities.

APPENDIX B : CONNECTION BETWEEN THE p WAVE $\pi\pi$ SCATTERING AMPLITUDE AND THE FUNCTION $A(s)$

The p wave projection $T^{J=1}(s)$ of the $\pi\pi$ scattering amplitude, as given by current algebra, has the general form ⁵⁾

$$T^{J=1}(s) = R_1(s) - 2K_1\Gamma(s) + K_1^2\Delta_V(s)^{-1} + F(s)\Gamma(s) \quad (B.1)$$

where $K_1 = 1 - (C_V/2F_\pi^2)$, $R_1(s)$ = p wave projection of $\left\{ \begin{array}{l} \text{sea-gull } t_c(s,t) + \\ \text{+ polynomials + (t) and (u) channel "trees"} \end{array} \right\}$, $\Gamma(s) = F(s)\Delta_V(s)^{-1}$, and $F(s)$ is the pion form factor.

Inserting our representation (2.5) for $F(s)$ into (B.1) gives

$$T^{J=1}(s) = R_1(s) + \frac{1}{4F_\pi^4} \Delta_V(s) [H(s) - 1]^2 \quad (B.2)$$

We now make use of Schnitzer's Lemma 1 ⁵⁾ :

if $\text{Im } t_c(s) = 0$ for $4m_\pi^2 \leq s \leq 16m_\pi^2$, where $t_c(s)$ is the p wave projection of the sea-gull term $t_c(s,t)$, then $F(s)$ can be written

$$F(s) = K_1 + K_1^{-1} R_1(s) \Delta_V(s) \quad (B.3)$$

in the same energy interval.

Comparing (B.3) with (2.5) gives

$$K_1^{-1} R_1(s) = \frac{1}{2F_\pi^2} [H(s) - 1] \quad (B.4)$$

Furthermore, since by definition $\text{Im } R_1(s) = \text{Im } t_c(s)$ for $s \geq 4m_\pi^2$ and $\text{Im } t_c(s) = 0$ by assumption, it follows that

$$\text{Im } H(s) = 0 \quad \text{for} \quad 4m_\pi^2 \leq s \leq 16m_\pi^2 \quad (B.5)$$

Finally, from (2.5)

$$\text{Im } \Gamma(s) = K_1 \pi \frac{\rho(s)}{|\Delta_V(s)|^2} + \frac{1}{2F_\pi^2} \text{Im } H(s) \quad (B.6)$$

and we immediately obtain Lemma 2 of Ref. 5) :

with the assumptions of Lemma 1 and assuming

$$\text{Im } \Gamma(s) = 0 \quad \text{for } 4m_\pi^2 \leq s \leq s_0 \quad (s_0 > 4m_\pi^2)$$

one gets

$$C_v = 2F_\pi^2 \tag{B.7}$$

REFERENCES

- 1) B. Zumino - Rapporteur's talk at the 15th International Conference on High Energy Physics, Kiev (1970).
- 2) See Ref. 1) for a list of references.
- 3) G. Ecker and J. Honerkamp - CERN Preprint TH. 1348 (1971), to be published in Nuclear Phys.B.
- 4) K. Kawarabayashi and M. Suzuki - Phys.Rev.Letters 16, 255 (1966) ;
Fayyazuddin and Riazuddin - Phys.Rev. 147, 1071 (1966).
- 5) H.J. Schnitzer - Phys.Rev.Letters 24, 1384 (1970) ; Phys.Rev. D2, 1621 (1970).
- 6) G. Källén - Helv.Phys.Acta 25, 417 (1952) ;
H. Lehmann - Nuovo Cimento 11, 342 (1954).
- 7) T.D. Lee, S. Weinberg and B. Zumino - Phys.Rev.Letters 18, 1029 (1967) ;
T.D. Lee and B. Zumino - Phys.Rev. 163, 1667 (1967).
- 8) V.N. Gribov, G.L. Ioffe and I. Ya. Pomeranchuk - Phys.Letters 24B, 554 (1967).
- 9) K. Wilson - Phys.Rev. 181, 1909 (1969) ;
I.S. Gerstein and R. Jackiw - Phys.Rev. 181, 1955 (1969) ;
W.A. Bardeen - Phys.Rev. 184, 1848 (1969).
- 10) J.J. Brehm, E. Golowich and S.C. Prasad - Phys.Rev.Letters 23, 666 (1969) ;
J.J. Brehm and E. Golowich - Phys.Rev. D2, 1668 (1970).
- 11) G. Ecker - CERN Preprint TH. 1312 (1971), to be published in Nuovo Cimento A.
- 12) H.J. Schnitzer and M.L. Wise - Ann.Phys.(N.Y.) 59, 129 (1970).
- 13) J.D. Bjorken - Phys.Rev. 148, 1467 (1966) ;
K. Johnson and F.E. Low - Progr.Theor.Phys.Suppl.(Japan) 87-88, 74 (1966).
- 14) P. Ford and H.J. Schnitzer - Brandeis University Preprint (April 1971).
- 15) H.J. Schnitzer and S. Weinberg - Phys.Rev. 164, 1828 (1967).
- 16) T.N. Truong, R. Vinh Mau and Pham Xuan Yem - Phys.Rev. 172, 1645 (1968) ;
T.N. Truong and R. Vinh Mau - Phys.Rev. 177, 2494 (1969).
- 17) J.E. Augustin et al. - Phys.Letters 28B, 503 (1969).
- 18) S. Weinberg - Phys.Rev.Letters 18, 507 (1967).
- 19) I.S. Gerstein, H.J. Schnitzer and S. Weinberg - Phys.Rev. 175, 1873 (1968) ;
I.S. Gerstein and H.J. Schnitzer - Phys.Rev. 175, 1876 (1968).