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COLLINEAR ALGEBRA FOR THE DUAL MODEL

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A B S T R A C T

The results of Del Giudice, Di Vecchia and Fubini are formulated in terms of the algebraic properties of their "photon" transition operator A_n^i for the $\alpha_0 = 1$ dual model. These are combined with the longitudinal (A_n^L) and scalar $(\bar{\Phi}_n)$ operators to form a nice closed (spectrum generating) algebra (for collinear "photons", $k_i = kn_i$) similar to the Virasoro gauge algebra for L_n . The underlying algebraic structure allows us to construct more independent physical states.

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The algebraic structure of currents is well known to lead to sum rules, and conversely, as emphasized by Weinberg ¹⁾, the zero width saturation of these sum rules leads to algebraic relations for the physical transition matrix K (or X) for $\alpha_1 + \gamma(\text{photon}) \rightarrow \alpha_2$ (or $\alpha_1 + \pi \rightarrow \alpha_2$), and the mass metric M^2 . In the corresponding hadronic problem, there is an infinite set of finite energy sum rules (FESR) to satisfy. Remarkably, the dual resonance model (DRM) provides a solution to these sum rules with a consistent internal and external spectrum (bootstrap condition). Here, we will show that the DRM solution with intercept $\alpha_0 = 1$ also implies a rich algebraic structure.

We make a direct generalization of the construction of DelGiudice, DiVecchia and Fubini (DDF) to which we shall repeatedly refer the reader ²⁾. They introduce the transition operator A_n^i for the photon-like state ($J = 1, m^2 = 0, i = 1, 2$) on the leading trajectory for the transition $\alpha_1(p_1) + \gamma(nk) \rightarrow \alpha_2(p_2)$. For the vertex $\langle p_1 | A_n^i | p_2 \rangle$ the operator A_n^i has the remarkable property that it depends only on the mass shift $m_2^2 - m_1^2 = n$. These operators (A_n^i) can be used like the original covariant harmonic oscillators $a_{(n)}^{\mu+}$ ($\mu = 0, 1, 2, 3$) to generate states. In addition to the transverse "photon" operator, we introduce a longitudinal (A_n^L) and scalar (ϕ_n) photon operators which give the complete space, and satisfy an elegant generating algebra when all photon momenta are taken collinear ($k_1 = n_1 k$).

It is highly probable that the infinite gauge algebra (L_n) for the case $\alpha_0 = 1$ cancels all negative metric states ³⁾. Although the transverse operators (A_n^i) commute with the gauge operators (L_n), they create only a subset of the physical states. They give the two-dimensional space transverse to k , of orthogonal positive definite states ^{*}). However, Brower and Thorn ³⁾ have shown that the physical space $\{|\psi\rangle; L_n|\psi\rangle = 0, \forall n > 0\}$ is equivalent (in degeneracy, spin and parity) to the space generated by the three spatial components of $a_{(n)}^{\mu+}$. We believe we can construct functionals $A_n^3(A_n^L, \Phi_s)$ that generate the remaining states.

Clearly, the complete formulation of the problem in terms of physical operators ($A_n^i, i = 1, 2, 3$) is a preliminary step to proving no ghost and gaining a deeper understanding of the algebraic structure of the physical multiplets of the DRM. Although the theory we consider has a tachyon, a similar DRM has avoided this disease ⁴⁾. If the zero mass rho can be circumvented, perhaps the vector current operator will generate the physical states and be related to A_n^i through some sort of vector dominance ⁵⁾.

^{*}) If space is given D-1 dimensions, they generate the D-2 dimensional transverse space.

1. THE CONSTRUCTION OF PHYSICAL STATES FROM PHOTONS

Consider the matrix $\langle N_1, p_1 | A_n^\dagger | N_2, p_1 \rangle$ for the on-shell transition (see the Figure) $\alpha_1(p_1) + \gamma(\lambda k) \rightarrow \alpha_2(p_2)$. For convenience, we pick the collinear frame with

$$k = (1, 0, 0, 1) \quad \text{and} \quad 2k \cdot p_1 = 2k \cdot p_2 = -1 \quad (1)$$

so that the photon momenta λk obeys the "quantization" condition,

$$\lambda = N_2 - N_1 = n \quad (2)$$

Any vector P^μ may be expressed in light cone co-ordinates as

$$P^\mu = P^+ e_1^\mu + P^- e_2^\mu + P^\perp k^\mu + (k \cdot P) k_L^\mu \quad (3)$$

where $e_i^\mu = g_i^\mu$ (for $i = 1, 2$) and $k_L = 1/2(-1, 0, 0, 1)$.

We now introduce the basic operators of the DRM,

$$Q^\mu(z) = q_0 - i\phi_0 \ln z + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \{ a_n z^n + a_n^\dagger z^{-n} \} \quad (4)$$

$$P^\mu(z) = i z \frac{dQ^\mu}{dz}$$

which obey the (local) commutation relations

$$[Q^\mu(z_1), P^\nu(z_2)] = -ig^{\mu\nu} \delta(\theta_1 - \theta_2), \quad z_j = e^{2\pi i \theta_j} \quad (5)$$

In addition, we shall need the vertex and gauge operators

$$V(z, nk) = \exp \{ i\sqrt{2} nk \cdot Q(z) \} \quad (6)$$

$$L_n = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} : P^2(z) :$$

where the normal ordering for V is not needed because $k^2 = 0$ and, for L_n , is only needed to define L_0 .

$$L_0 = \frac{1}{2} p_0^2 + \sum n a_n^\dagger a_n = \frac{1}{2} p_0^2 + H \quad (7)$$

H is the excitation energy of the resonance state,

$$H |N, p\rangle = N |N, p\rangle \quad (8)$$

We can now define the "photon" transition matrix ($i = 1, 2$).

$$\langle N_1, p_1 | A_n^i | N_2, p_2 \rangle = \frac{1}{\sqrt{2}} \langle N_1, p_1 | P^i(1) V(1, nk) | N_2, p_2 \rangle \quad (9)$$

Inserting the Cauchy expression for the Kronecker delta

$$\delta_{n, N_2 - N_1} = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} z^{-N_1} z^{N_2} \quad (10)$$

and using the on-shell condition

$$z^{-n} e^{i\sqrt{2} nk \cdot q_0} |N_2, p_2\rangle = \exp \{ i\sqrt{2} nk (q_0 - ip_0 \ln z) \} |N_2, p_2\rangle \quad (11)$$

we have the matrix element of the operator of DDF.

$$A_n^i = \frac{1}{2\pi i} \oint \frac{dz}{\sqrt{2} z} z^{-H} P^i(1) V(1, nk) z^H \quad (12)$$

The single valuedness of the integrand is guaranteed on shell by setting $\sqrt{2} k \cdot p_0 = -1$.

A state $|\psi\rangle$ is physical if

$$(L_0 - 1)|\psi\rangle = 0 \quad \text{and} \quad L_n|\psi\rangle = 0, \quad \text{for } n > 0 \quad (13)$$

As a consequence of the algebraic properties of A_n^i ,

$$[A_m^i, A_n^j] = \frac{m}{2} \delta_{m+n,0} \delta_{ij}, \quad (14)$$

$$[L_m, A_n^j] = 0,$$

$$A_{-n}^i = (A_n^i)^\dagger \quad \text{and} \quad A_n^i |0, p\rangle = 0, \quad n > 0$$

one recovers the result of DDF that

$$(A_{-1}^{i_1})^{\lambda_1} (A_{-2}^{i_2})^{\lambda_2} \dots (A_{-N}^{i_N})^{\lambda_N} |0, p\rangle, \quad \sum_{j=1}^N j \lambda_j = N \quad (15)$$

are independent, orthogonal physical states. Our proof of orthogonality is the new feature of this approach. The commutator $[A_n^i, A_m^j]$ takes such a simple form because the vertices $V(z, nk)$, $V(z', mk)$ commute with each other and P^i .

$$[a^i, k \cdot a^\dagger] = [k \cdot a, k \cdot a^\dagger] = 0. \quad (16)$$

Only $[\bar{P}^i(z), P^i(z')]$ gives rise to a derivative of a delta function. More details of these derivations, particularly the more subtle commutators involving normal ordering of the next section will be given in a subsequent paper.

2. SPECTRUM GENERATING ALGEBRA

The number of states $(T^2(N))$ generated by A_{-n}^i at level N corresponds to a two-dimensional harmonic oscillator. To enlarge this space, one could relax the condition that all the photon momenta (k_i) be collinear $(k_i \cdot k_j = 0)$. However, the essential algebraic simplicity is lost, by the need to use the difficult (non-local) commutator relation

$$[k_1 \cdot Q(z_1), k_2 \cdot Q(z_2)] = -i\pi k_1 \cdot k_2 \frac{\theta_1 - \theta_2}{|\theta_1 - \theta_2|}, \quad z_j = e^{2\pi i \theta_j} \quad (17)$$

for $-1/2 < \theta_1 - \theta_2 < 1/2$. Moreover, covariance suggests we consider the full vector

$$A_n^r = \frac{1}{2\pi i} \oint \frac{dz}{z} \frac{1}{2} : \left\{ \frac{1}{\sqrt{2}} P^r(z), V(z, nk) \right\}_+ \quad (18)$$

The normal ordering is unnecessary for $k_r A_n^r$, so we easily calculate ($n \neq 0$)

$$k_r A_n^r \propto \oint dz \frac{d}{dz} (\exp \{i\sqrt{2}nk \cdot Q(z)\}) = 0 \quad (19)$$

Since $A_0^r = p_0^r / \sqrt{2}$, $k_r A_n^r = -1/2 \delta_{n,0}$ in general. We have introduced one new component, the "longitudinal photon",

$$A_n^L = k_L \cdot A_n, \quad A_{-n}^L = (A_n^L)^\dagger, \quad A_n^L |0, p\rangle = 0, \quad n > 0. \quad (20)$$

This time the normal ordering is non-trivial procedure to avoid infinities when A_n^L is applied to a state $|N_1, p\rangle$ and it also destroys the commutativity with L_n . Consequently, we introduce a scalar "photon",

$$\Phi_n = \frac{1}{2\pi i} \oint \frac{dz}{z} V(z, nk) \quad (21)$$

which (as in electrodynamics) may be useful in cancelling the spurious transitions allowed by A_n^L .

The operators A_n^i, A_m^L, ϕ_k generate the entire spectrum of states (real and spurious) and close algebraically.

$$\begin{aligned}
 [A_m^i, A_n^j] &= \frac{m}{2} \delta_{m+n,0} \delta_{ij} \\
 [A_m^L, A_n^L] &= (m-n) A_{m+n}^L + m^3 \delta_{m+n,0} \\
 [A_m^L, A_n^j] &= -n A_{m+n}^j \\
 [A_m^L, \Phi_n] &= -n \Phi_{m+n} \\
 [\Phi_m, A_n^j] &= [\Phi_m, \Phi_n] = 0
 \end{aligned} \tag{22}$$

This beautiful algebra resembles the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12} D m(m^2-1) \delta_{m+n,0} \tag{23}$$

particularly for $[A_m^L, A_n^L]$. However, the c number, resulting from normal ordering, does not depend on the dimension of space D ($a_{(n)}^{\mu}$, $\mu = 0, 1, 2, \dots, D-1$).

In a covariant language the general four-vector operator $\alpha_n^{\mu} = A_n^{\mu} + k_L^{\mu} \Phi_n$, plays the same role as a_n^{μ} in generating the states. Indeed, expanding $Q(z, \epsilon) = q_0 - ip_0 \ln z + \epsilon \sum (a_n z^n + a_n^{\dagger} z^{-n}) / \sqrt{n}$ around $\epsilon = 0$ (the no interaction point), one can easily prove that $(\alpha_{-1}^{\mu})^N \dots (\alpha_{-N}^{\mu})^N |0, p\rangle$ spans the entire space ($\sum i \lambda_i = N$). Just as in Ref. 3) for L_n^+ , the closure of the algebra allows one to construct

$$\begin{aligned}
 [\alpha_m^{\mu}, \alpha_n^{\nu}] &= m \alpha_{m+n}^{\mu} k^{\nu} - n \alpha_{m+n}^{\nu} k^{\mu} \\
 &+ \frac{m}{2} \delta_{n,-m} [g^{\mu\nu} - k_L^{\mu} k^{\nu} - k^{\mu} k_L^{\nu} + 2m^2 k^{\mu} k^{\nu}]
 \end{aligned} \tag{24}$$

a linearly independent basis by giving the operators α_n^{μ} a canonical ordering.

Perhaps, it is helpful to make more explicit the intuitive picture reflected in the case of an internal "space" (θ) and "time" (τ) co-ordinate ($z = \exp[2\pi(\tau + i\theta)]$). The spatial average over P

$$\langle P^r \rangle = \frac{1}{2\pi i} \oint \frac{dz}{z} P^r(z) \equiv \int_0^1 d\theta P^r(\theta, \tau) \quad (25)$$

gives the physical momentum operator ^{*)} p_0 . Hence P^r is a generalized density operator for the excited system $(|P, N\rangle)$. To relate the dual "Hamiltonian" L_0 (displacements in τ) to the mass operator $p_0^2/2$, Ramond ⁶⁾ considers the replacement

$$\frac{1}{2} p_0^2 = \frac{1}{2} \langle P^2 \rangle \rightarrow L_0 = \frac{1}{2} \langle : P^2 : \rangle \quad (26)$$

By the same token $V = \exp(i\sqrt{2} kQ)$ is a generalized translation in momentum space, and the classical gauge invariant coupling $(p_1^r + p_2^r = \langle p_1, N | (e^{i\sqrt{2}kq_0} p_0^r + p_0^r e^{i\sqrt{2}kq_0}) | p_2, N \rangle / \sqrt{2})$ goes into A_n^r .

$$\frac{1}{2} \{ \langle \frac{1}{\sqrt{2}} P^r \rangle, \langle V \rangle \}_+ \rightarrow \alpha_n^r = \frac{1}{2} \langle : \{ \frac{1}{\sqrt{2}} P^r, V \}_+ : \rangle \quad (27)$$

A_n^r is conserved for all k^2 and is closely related to the current of Ref. 5). Fubini has also noted that amplitudes are vacuum expectations values of τ ordered products of $V(\theta = 0, \tau, k)$, so Φ_n^+ , A_n^r are analogous to creation operators for these interacting "fields".

3. PHYSICAL STATES IN THE LONGITUDINAL SPACE

If we can combine A_m^L and Φ_n to get a one-dimensional physical space $\{|\psi_1\rangle\}$, the full space can be generated by applying the A_n^i operators. The pertinent commutation relations are

$$\begin{aligned} [L_1, A_n^r] &= -\frac{1}{2} \delta^{rn} \Phi_{n+1}^r \\ [L_1, \Phi_m^r] &= (n-m-1) \Phi_{m+1}^r \end{aligned} \quad (28)$$

*) Actually $(1/\sqrt{2})p_0$ and $\sqrt{2} q_0$ are physical operators, hence the many factors of $\sqrt{2}$.

where we define

$$\Phi_m^n = \frac{1}{2\pi i} \oint \frac{dz}{z} z^{n-m} V(z, nk) \quad (29)$$

Clearly, not all Φ_m^n are independent, since they can all be related to Φ_n^1 through the Fourier expansion ($z = e^{2\pi i\theta}$)

$$V(z, nk) = \{V(z, k)\}^n = \sum_{m=-\infty}^{\infty} \Phi_{m+n}^n z^m \quad (30)$$

The operators Φ_m^n are convenient since they commute with themselves and A_m^i , and close algebraically with A_n^L :

$$[A_m^L, \Phi_r^n] = -n \Phi_{m+r}^{n+n} \quad (31)$$

$$(\Phi_r^n)^+ = \Phi_{-r}^{-n} \quad \text{and} \quad \Phi_r^n |0\rangle = 0 \quad \text{for} \quad r > 0.$$

One can immediately solve the problem of finding all the states $\{|\hat{\psi}_1\rangle\}$ satisfying $(L_0 - 1)|\hat{\psi}_1\rangle = L_1|\hat{\psi}_1\rangle = 0$, by introducing the operators

$$\hat{A}_n^L = A_n^L - \frac{n}{2} \Phi_n$$

$$\hat{\Phi}_n = \sum_{l=1}^{\infty} l \Phi_{l-1}^{-1} \Phi_{n+1-l}^{n+1} \quad (32)$$

To show that $[L_1, \hat{\Phi}_n] = 0$, note that (on shell) the series $\hat{\Phi}_{-n}^0 |N, P\rangle$ terminates and $\hat{\Phi}_n^{n+1}$ commutes with L_1 . Noting that $\hat{\Phi}_{-1} \equiv 0$, we can show the number ^{*)} of independent states generated by $\hat{A}_{-n}$ and $\hat{\Phi}_{-n}$ is $T^2(N) - T^2(N-1)$ as expected.

Our attempts at finding a basis for the subset of states are at a preliminary stage. We need only to enforce $L_2|\psi\rangle = 0$, since all L_n are multiple commutators of L_1 and L_2 . At present, we are able to construct physical states containing one $\hat{A}_n^{L+}$. They are given by a sort of coherent state expression,

*) The essential identity in all the counting is $\sum_{\ell=0}^N T^k(N-\ell) T(\ell) \equiv T^{k+1}(N)$.

$$\exp \{-z^m \hat{A}_{-n}\} \{\Psi_{-m}(z)\}^{-\frac{1}{2}} |0, p\rangle \quad (33)$$

where

$$\Psi_{-m}(z) = \sum_{r=0}^{\infty} (m z^m)^r \hat{\Phi}_{-m,r}$$

The proof depends on relations between Fourier coefficients.

We are optimistic that a deeper study of the algebra, perhaps focussing on the "fields" $\Psi_m(z)$ and $\pi(z) = \sum z^n \hat{A}_n$, will lead to a solution of this problem. The problem of checking the norm of all the physical states is still more difficult. Particularly, as the diagonal elements with the A_m^i modes effect the metric tensor. Indeed, this mixing causes a ghost on the second daughter for $D-1 > 25$ spatial dimensions ³⁾.

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