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ON THE ORDER OF MAGNITUDE OF 8th ORDER CORRECTIONS
TO THE ANOMALOUS MAGNETIC MOMENT OF THE MUON

B. Lautrup
CERN - Geneva

ABSTRACT

It is shown that the 8th order contribution to the muon anomaly can be as large as $100-200 \times (\alpha/\pi)^4$.

With the discovery ¹⁾ that the 6th order contribution to the anomalous magnetic moment of the muon ²⁾ is of the order of magnitude of $20 \times (\alpha/\pi)^3$ the question of the order of magnitude of the 8th order corrections naturally arises. Can it possibly be so large that it will significantly influence the comparison between theory and future experiments (which may reach a precision better than 15×10^{-9}) ?

The large coefficient of the 6th order term is entirely due to the diagrams of type (a) in the Figure. These diagrams are radically different from any appearing in 4th order and require an exact calculation ¹⁾. An attempt to estimate their contribution did not succeed ³⁾. The reason for the large value of the coefficient must be sought in the large ratio of muon to electron mass. The same diagrams with only one kind of particle turn out to have a coefficient of the order of one ¹⁾. Intuitively, one can argue that the virtual photons emitted by the muon have characteristic masses of the order of m_μ^2 and therefore probe the asymptotic behaviour of the photon-photon scattering diagrams. That being unknown, we get no further in this case.

In other cases, however, such an argument leads to almost correct quantitative estimates. For the 4th order diagram (b) in the Figure, or for 6th order diagrams of the type (c), we expect that the asymptotic behaviour of second order vacuum polarization will be involved. The vacuum polarization insertion can be written as a photon propagator modified with a factor $-\Pi(k^2)$, i.e., as

$$-\frac{1}{k^2} \Pi(k^2)$$

where $\Pi(k^2)$ is the invariant amplitude from the second order renormalized photon self-energy diagram ⁴⁾. Asymptotically, this behaves as

$$\frac{1}{k^2} \propto \frac{1}{\pi} \left[\frac{1}{3} \log \frac{k^2}{m_e^2} - \frac{5}{9} \right] \quad (k^2 \gg m_e^2)$$

Because the logarithm is a slowly varying function of k^2 one can probably replace k^2 by an average value $c^2 m_\mu^2$ where c is a constant of order unity ⁵⁾. Then the effect of vacuum polarization has been reduced to a constant factor ^{*})

^{*}) This factor is actually the second order charge renormalization constant cut-off at cm_μ , i.e.,

$$Z_2 = \int_{4m_e^2}^{c^2 m_\mu^2} \frac{dt}{t} \frac{1}{\pi} \text{Im} \Pi(t)$$

$$Z_2 = \left(\frac{2}{3} \log \frac{cm_\mu}{m_e} - \frac{5}{9} \right) \frac{1}{\pi R}$$

The value of c depends on the graph(s) in question. A more rigorous derivation of this approach has been given in Ref. 6).

In 8th order no radically new types of diagrams appear involving both muons and electrons. One set of diagrams stands out, however, as the most likely candidate for a large contribution, namely, the one of type (d) in the Figure (there are actually 18 such diagrams). Using the above arguments we estimate their contribution to be

$$a^{(8)} \approx 3 Z_2 a^{(6)}$$

where $a^{(6)}$ is the 6th order contribution from diagrams (a) calculated in Ref. 1). The factor of 3 arises because the vacuum polarization loop can be inserted in three different places. Using ¹⁾ $a^{(6)} = 18.4 (\alpha/\pi)^3$, we find for $c = 1$

$$a^{(8)} \approx 166 \left(\frac{\alpha}{\pi} \right)^4 = 4.8 \times 10^{-9}$$

This number (which is gauge invariant) is of the same order of magnitude as many 6th order diagrams, as the uncertainty on the strong interaction correction, and as the weak interaction correction.

Techniques similar to these were used to calculate all the logarithmic terms in the 6th order correction ³⁾, except the diagrams (a). In the 4th order case [diagram (b)], one finds agreement with the exact calculation ⁷⁾ for

$$\log c = -5/4$$

Such a value for c tends to diminish the influence of vacuum polarization which vanishes for

$$\log c = \frac{5}{6} - \log \frac{m_\mu}{m_e} = -4.5$$

For the 6th order corrections of type (c) with one electron loop, one finds agreement with the exact calculation [using Ref. 6) or 8)] for

$$\log c \approx + \frac{5}{6}$$

which increases the influence of vacuum polarization. Both of these examples, however, suffer from the drawback that they only involve one electron loop. The only case involving two electron loops which has been calculated exactly ^{3),9)} is shown in diagram (c) in the Figure. The naïve expectation

$$Z_2^2 \frac{\alpha}{2\pi}$$

agrees with the $\log^2(m_\mu/m_e)$ term of the exact calculation. The term linear in $\log(m_\mu/m_e)$ agrees for $\log c = -5/4$ in which case the constant term becomes 625/648. The exact constant term is $\pi^2/27 + 317/324$ which is larger than the estimate by about 0.4. The value $\log c = -5/4$ reduces our estimate of the 8th order graphs (d) to $121 \times (\alpha/\pi)^4$.

We shall now discuss the possibility of other 8th order diagrams giving very large contributions. In order to do so, we must classify all the diagrams. Let M denote a muon vertex and E an electron vertex. We can then write for the 8th order diagrams (and thereby the anomaly) in an obvious notation

$$A_9^\mu = M^9 + M^7 E^2 + M^5 E^4 + M^3 E^6$$

As M^9 does not involve electrons at all, it is independent of the mass and is equal to the 8th order electron contribution

$$M^9 = A_9^e$$

which probably ^{*)} will be of the order of $(\alpha/\pi)^4$ and can be discarded. The term $M^7 E^2$ is obtained from the 6th order diagrams without electron loops, A_7^e , by inserting an electron loop in all possible ways. Using the same arguments as before, it follows that (for $c=1$)

$$M^7 E^2 \approx 3 Z_2 A_7^e = 14 \left(\frac{\alpha}{\pi}\right)^4$$

and can thus be discarded.

*) We reason in analogy with the 6th order case where $A_7^e \approx 1.5 (\alpha/\pi)^3$. However, A_8^e contains 891 non-zero topologically distinct proper diagrams of which 93 are irreducible ¹⁰⁾. As the 72 6th order diagrams ²⁾ cancel each other to a high degree, we expect this to be the case here, too. The experimental limit on A_9^e is ²⁾:

$$-240 \left(\frac{\alpha}{\pi}\right)^4 < A_9^e < 400 \left(\frac{\alpha}{\pi}\right)^4$$

(95% confidence).

Let L_n denote an electron loop with n vertices. We can then write

$$M^5 E^4 = M^5 L_4 + M^5 L_2^2$$

The term $M^5 L_2^2$ is easy to evaluate. It is obtained from the seven 4th order diagrams by inserting two electron loops in all possible ways. We then have (for $c=1$)

$$M^5 L_2^2 \approx 3 Z_2^2 A_5^e = -9 \left(\frac{\alpha}{\pi}\right)^4$$

A 4th order loop is either a photon-photon scattering diagram or a 4th order proper vacuum polarization diagram

$$L_4 = P_4 + V_4$$

4th order vacuum polarization can be treated in the same way as second order vacuum polarization ⁹⁾ replacing it by

$$Z_4 = \left(\frac{1}{2} \log \frac{cm_\mu}{m_e} + \zeta(3) - \frac{5}{24} \right) \left(\frac{\alpha}{\pi}\right)^2$$

By comparing with an exact evaluation of 6th order diagrams ⁹⁾ we find again the "magic number"

$$\log c = -5/4$$

We shall, however, in the following also take this $c=1$. Hence

$$M^5 V_4 \approx 2 Z_4 A_5^e = -2.4 \left(\frac{\alpha}{\pi}\right)^4$$

Typical diagrams belonging to $M^5 P_4$ are shown in the Figure (f) and (g). Nothing is known about the magnitude of the contributions from such diagrams ^{*}).

Finally, we write

$$M^3 E^6 = M^3 L_6 + M^3 L_4 L_2 + M^3 L_2^3$$

By the same arguments as above, we estimate

$$M^3 L_2^3 \approx Z_2^3 A_3^e = 14 \left(\frac{\alpha}{\pi}\right)^4$$

^{*}) One might perhaps say that type (f) is a purely radiative correction to type (a) and thus α/π times smaller.

and as $L_4 = P_4 + V_4$

$$M^3 V_4 L_2 \approx 3 Z_2 Z_4 A_3^e = 16 \left(\frac{\alpha}{\pi}\right)^4$$

This estimate is probably quite uncertain because it involves diagrams of the type (h). We have already estimated

$$M^3 P_4 L_2 \approx 3 Z_2 M^3 P_4 = 166 \left(\frac{\alpha}{\pi}\right)^4$$

A 6th order loop is either a radiative correction to photon-photon scattering (i), or a proper vacuum polarization graph (k). Hence

$$L_6 = R_2 P_4 + V_6$$

The diagrams $M^3 R_2 P_4$ are not easy to estimate. The last term is

$$M^3 V_6 \approx Z_6 A_3^e$$

where Z_6 is determined by the (unknown) asymptotic behaviour of 6th order proper vacuum polarization (or the 6th order charge renormalization constant). Presumably it will not be too large.

Concluding, we can state that of the 1357 (non-zero topologically distinct proper) diagrams ¹⁰⁾ contributing to the 8th order muon anomaly, we only expect a few to give a large contribution: the 18 diagrams of type (d) which we have estimated; the 66 diagrams of type (f), the 18 diagrams of type (g), and the 60 diagrams of type (i), which have not yet been estimated. The question of whether cancellations among large terms might occur is completely open.

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