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A NUMERICAL STUDY OF ANALYTICITY IN THE COUPLING CONSTANT

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A B S T R A C T

For two simple field theories, both of zero-space dimension, the Riemann surface of the energy as a function of the square of the coupling constant is studied. The convergence of perturbation theory is established and a comparison is made between the exact energies and their Padé approximations.

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1. - INTRODUCTION

The pioneering work of Bender and Wu ¹⁾ on the analyticity in the coupling constant for the anharmonic oscillator has very much stimulated the research in this direction ²⁾. Most of it, however, has been restricted to the anharmonic oscillator, which is the zero-space analogue of a ϕ^4 field theory. In the present paper, a numerical investigation will be made of two theories with a $\psi^2\phi$ interaction, where ψ and ϕ represent a fermion and a boson field respectively. The first model of this type to be considered is one with a fixed fermion of isospin one half, which is capable of emitting and absorbing bosons of isospin one ³⁾. In the second model the fermions can only be created or annihilated in pairs, with the simultaneous emission or absorption of a boson of isospin one. Although any number of bosons can be present a further restriction is introduced by limiting the number of fermion pairs to zero and one. The precise formulation of this model will be given in Section 2.

The main result of the present paper, like in the work of Bender and Wu ¹⁾, is that the energy levels $E_n(g^2)$ are the values of a single analytic function on the different branches of its Riemann surface. An important difference is, however, that for each energy level $E_n(g^2)$ the power series expansion in g^2 has a finite radius of convergence r_n , which approaches zero for n going to infinity. This finite radius of convergence also occurs in the ladder approximation of the Bethe-Salpeter equation ⁴⁾, as was shown by Mattioli ⁵⁾, but not in the Peres model ^{6),7)}, which is of the $\phi_1^2\phi_2$ type, where both ϕ_1 and ϕ_2 are boson fields.

In the two models, presented in this paper, the expansions of $E_n(g^2)$ are not Stieltjes series. The $[N+1, N]$ Padé approximants nevertheless show a very good convergence to the exact solutions, except when level crossing occurs.

It is furthermore shown that the poles of the Padé approximants simulate cuts, of which some are actually present, but some are spurious.

For fixed g^2 the energy levels are calculated as a function of the isospin, considered as a continuous variable. Repulsion and attraction of levels occur in much the same way as has been found for Regge trajectories ^{8),9)}. The formulation (3.5) of the first model shows that it also describes an harmonic oscillator, in which the orbital motion is coupled to the spin of a central infinitely heavy particle. The variable J , which is actually the isospin of the meson field, then plays the role of orbital angular momentum and the curves showing the energy levels as a function of J are the Regge trajectories for such an oscillator.

One can, however, also consider the functions $E_n(J)$ as isospin Regge trajectories. This can be seen by comparison with potential theory ¹⁰⁾. There the amplitude for large momentum transfer is determined by the leading trajectory. For the present models it is seen in the transition from (2.1) to (3.5) that J plays the role of the

angular momentum and the meson field (or rather its canonical conjugate) is the equivalent of the momentum in potential theory. If the field strength is assumed to be proportional to the observed multiplicity n , one could then speculate by analogy that for high multiplicity events and as long as n is less than the average multiplicity, the amplitude as a function of n is proportional to

$$f(E, n) \sim C(E) n^{J(E)},$$

where $J(E)$ is the leading isospin trajectory.

The oversimplified models of the present paper give of course insufficient support to this idea.

2. - EQUATIONS FOR THE TWO MODELS

The Hamiltonian for the model with one fixed fermion of isospin one half is taken to be

$$H = \sum_{\ell=1}^3 a_{\ell}^* a_{\ell} + g \sum_{\ell=1}^3 (a_{\ell} + a_{\ell}^*) \tau_{\ell}, \quad (2.1)$$

where τ_{ℓ} are the Pauli (iso)spin operators, a_{ℓ}^* and a_{ℓ} are the creation and annihilation operators of the mesons and g is the coupling constant. It may be of interest to note that this Hamiltonian has also been used ¹¹⁾ to describe the Jahn-Teller effect, in which degenerate electron states interact with the nuclear vibrations. For each value of the total isospin $j = J - \frac{1}{2}$ ($J = 1, 2, \dots$) the eigenvalue problem can be written in the form ³⁾

$$H \vec{X} = E \vec{X}, \quad (2.2)$$

where H is the following Hermitian matrix

$$H = \begin{vmatrix} J-1 & g\sqrt{2J+1} & & & \\ g\sqrt{2J+1} & J & g\sqrt{2} & & \\ & g\sqrt{2} & J+1 & g\sqrt{2J+3} & \\ & & g\sqrt{2J+3} & J+2 & g\sqrt{4} \\ & & & g\sqrt{4} & J+3 & g\sqrt{2J+5} \\ & & & & & \ddots \end{vmatrix} \quad (2.3)$$

For real values of g , the components X_n of the vector $\vec{X}$ are real and the normalization condition, i.e., the equation for the eigenvalue spectrum, can be written with the squares (and not the absolute squares) of X_n as

$$\sum_n X_n^2 = 1. \quad (2.4)$$

This has the advantage that now Eqs. (2.2), (2.3) and (2.4) define the analytic continuation of the spectrum also for complex g . In the numerical calculation, the Hamiltonian was truncated and therefore Eq. (2.4) actually was not used.

Multiplying the n th row of (2.3) by a certain number and dividing the n th column by the same number, in such a way that all elements in the upper diagonal become equal to one, the spectrum is not changed and H takes the form

$$H = \begin{vmatrix} J-1 & 1 & & & & \\ g^2(2J+1) & J & 1 & & & \\ & 2g^2 & J+1 & 1 & & \\ & & g^2(2J+3) & J+2 & 1 & \\ & & & 4g^2 & J+3 & 1 \\ & & & & g^2(2J+5) & J+4 & 1 & \dots \end{vmatrix} \quad (2.5)$$

Since the square roots are removed, this form shows clearly that the energy levels of this Hamiltonian H_T , when truncated, are functions only of g^2 and J . The function

$$F(g^2, J, E) = \text{Det}(H_T - E)$$

is a polynomial in E , where each coefficient itself is a polynomial in g^2 and J . For such a case it is shown in the theory of algebraic functions ¹²⁾ that the solutions $E_n(g^2, J)$ of the equation $F(g^2, J, E) = 0$ are the values of one holomorphic function on the different Riemann sheets in the g^2 variable for fixed J , or in the J variable for fixed g^2 . Branch points occur for those values of g^2 (with J fixed) for which two or more eigenvalues become equal. Since in this paper higher than two-fold degeneracies are never encountered, all branch points are of the square root type.

The Hamiltonian for the second model is derived from the σ_5 pion-nucleon theory by letting only s wave pions with zero momentum interact with the nucleons. In this way one obtains $H = H_0 + H_1$, where $H_0 = a_l^* a_l + m(c^* c + d^* d)$ and

$$H_1 = -\frac{1}{2} g \left[c^* \tau_l (a_l + a_l^*) \sigma_y d^* + d \tau_l (a_l + a_l^*) \sigma_y c \right] \quad (2.6)$$

A summation over ℓ from 1 to 3 is understood. The pseudoscalar pions are annihilated by a_ℓ , the nucleons and antinucleons by c and d respectively. The spin and isospin indices carried by c and d are not written explicitly. In H_0 the meson mass is put equal to one and the nucleon mass is indicated by m . This Hamiltonian is clearly invariant under isospin rotations. It also conserves spin and parity and is invariant under particle-antiparticle conjugation. In order to get a theory for which numerical solutions can be obtained without too much difficulty, a further restriction is imposed by putting equal to zero all matrix elements of H_1 , for which the initial and final state differ by more than one nucleon-antinucleon pair. The number of mesons in any state remains unrestricted and the invariance under the above-mentioned symmetry transformations is not violated.

The simplification of the eigenvalue problem for this reduced Hamiltonian is performed in the same way as for the first model ³⁾: an eigenstate is written as a sum of terms, each of which is the product of a known nucleon state with definite spin, isospin, parity and G parity and an unknown mesonic state with matching values for these quantum numbers. The mesonic states are then written as a product of a known and simple tensor operator and an unknown isoscalar operator acting on the nucleon state. This isoscalar is then expanded in powers of $G^* = a_\ell^* a_\ell^*$ and the expansion coefficients are considered as the components of a vector in Hilbert space. It is in this space that the Hamiltonian will be written down. Only the sector with baryon number $B = 0$ will be discussed. It turns out that in this sector the states with spin one are not coupled to each other and therefore have the unperturbed spectrum. This is of course related to the fact that with s wave pions no pure mesonic state with spin one can be built. The only non-trivial states therefore have spin zero, but arbitrary integer isospin J . The parity and G parity are both equal to $P = G = (-1)^J$.

The equations which are finally obtained for the above-mentioned expansion coefficients P_n, Q_n, R_n ($n=0,1,2,\dots$) read as follows:

$$\begin{aligned} (2n+J-1+2m-E)P_n + g\sqrt{\frac{J}{2J+1}} [\sqrt{2n} Q_{n-1} + \sqrt{2n+2J+1} Q_n] &= 0 \\ (2n+J-E)Q_n + g\sqrt{\frac{J}{2J+1}} [\sqrt{2n+2J+1} P_n + \sqrt{2n+2} P_{n+1}] + g\sqrt{\frac{J+1}{2J+1}} [\sqrt{2n} R_{n-1} + \sqrt{2n+2J+3} R_n] &= 0 \\ (2n+J+1+2m-E)R_n + g\sqrt{\frac{J+1}{2J+1}} [\sqrt{2n+2J+3} Q_n + \sqrt{2n+2} Q_{n+1}] &= 0 \end{aligned} \quad (2.7)$$

with as normalization condition

$$\sum_{n=0}^{\infty} \{P_n^2 + Q_n^2 + R_n^2\} = 1. \quad (2.8)$$

With $X_{3n+1} = P_n$, $X_{3n+2} = Q_n$, $X_{3n+3} = R_n$ ($n=0,1,2,\dots$) these equations take the form

$$H \vec{X} = E \vec{X}, \quad (2.9)$$

where H is a real symmetric band matrix of width five, while (2.8) becomes

$$\sum_{n=1}^{\infty} X_n^2 = 1. \quad (2.10)$$

The same trick as was used in the first model for removing the square roots from the matrix elements of H can be used here. It follows indeed that for the truncated Hamiltonian the function $F(g^2, J, E) = \text{Det}(H_T - E)$ is a polynomial in E , where each coefficient is a polynomial in g^2 and a rational function of J , with poles at $J = -1$ and $J = -\frac{1}{2}$. The energy is therefore again an algebraic function of g^2 and probably also of J . The full dependence on complex J will not be studied in this paper and therefore the matrix form of H is not given explicitly. For $J = 0$ the dependence of E on g^2 will be studied in somewhat more detail. In this case, $P_n = 0$ is a solution of (2.7). The remaining equations for Q_n and R_n can again be written in the form (2.9) with H a real symmetric band matrix of width three. Removal of the square roots then leads to the following form of H

$$H = \begin{vmatrix} 0 & 1 & & & & \\ 3g^2 & 1+2m & 1 & & & \\ & 2g^2 & 2 & & & \\ & & 5g^2 & 1 & & \\ & & & 3+2m & 1 & \\ & & & 4g^2 & 4 & \\ & & & & 7g^2 & 5+2m \\ & & & & & \ddots & \ddots \end{vmatrix} \quad (2.11)$$

Observe that the two models become identical if in (2.5) the variable J is set equal to one and if in (2.11) the fermion mass is taken to be zero. It came as a surprise to Bender and Wu ¹⁾ that for the anharmonic oscillator, when the energy was fixed at any complex value, the resolvent possessed an infinite series of poles in the plane of the complex coupling constant. This would mean that for any complex E there exists an infinity of numbers $g_n(E)$, such that

$$(H_0 + g_n(E) V) \phi_n = E \phi_n. \quad (2.12)$$

If the Hamiltonians for the two models are again truncated this problem can be formulated as: find the solutions $g_n^2(E)$ of the equation $F(g^2, E) = 0$ (the variable J is suppressed), where F is a polynomial in g^2 and E . The solution is in general a many sheeted algebraic function $g^2(E)$. For the actual calculation of the poles of the resolvent, it is observed that (2.12) can also be written in the form of an eigenvalue problem

$$\frac{1}{H_0 - E} V \phi_n = - \frac{1}{g_n} \phi_n. \quad (2.13)$$

For the second model with $J = 0$ the operator on the left-hand side takes the form

$$\begin{vmatrix} 0 & -\frac{\sqrt{3}}{E} & & & \\ \frac{\sqrt{3}}{1+2m-E} & 0 & \frac{\sqrt{2}}{1+2m-E} & & \\ & \frac{\sqrt{2}}{2-E} & 0 & \frac{\sqrt{5}}{2-E} & \\ & & \frac{\sqrt{5}}{3+2m-E} & 0 & \frac{\sqrt{4}}{3+2m-E} \\ & & & & \ddots \end{vmatrix} \quad (2.14)$$

By using a recurrence relation one can prove that if λ_n is an eigenvalue of the truncated version of (2.14), so is $-\lambda_n$. It makes therefore sense to calculate $g_n^2(E)$. By numerical calculation it appeared that when E takes values along the negative and positive parts of the real axis all functions $g_n^2(E)$ are real and positive. After the fact this is no surprise at all since any real value of the energy can be obtained by taking g^2 as the independent variable and giving it some positive value. For the graphical proof, see Fig. 2.

3. - FORMULATION WITH DIFFERENTIAL EQUATIONS

Although the numerical calculations are performed using the matrix formulations (2.5) and (2.11), a few remarks will be made concerning the differential equations, which describe the same problems.

In order to give the analytic continuation of the energy as a function of the coupling constant, Bender and Wu ¹⁾ used the equations for the anharmonic oscillator, i.e.,

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4} x^2 + \frac{1}{4} \lambda x^4 \right) \phi(x) = E(\lambda) \phi(x) \quad (3.1)$$

and

$$\lim_{|x| \rightarrow \infty} \phi(x) = 0,$$

in a certain region of the complex x plane. It was then shown that the branch points of $E(\lambda)$ were those values of λ for which

$$\oint_C \phi^2(x, \lambda) dx = 0, \quad (3.2)$$

where C is an appropriate path in the complex x plane. As Simon ²⁾ pointed out, this method may be hazardous to the mathematical rigour. The statement, for instance, that when λ approaches a singular point, two originally distinct eigenfunctions become equal, may be wrong. This can easily be seen in the following very simple example. Consider the system which is described by the Hamiltonian (also mentioned by Simon)

$$H = \begin{pmatrix} -1 & g \\ g & 1 \end{pmatrix}. \quad (3.3)$$

The eigenvalues are $E_1 = \sqrt{1+g^2}$ and $E_2 = -\sqrt{1+g^2}$, with the corresponding eigenvectors

$$\varphi_1 = \begin{pmatrix} \sqrt{\frac{1}{2} - \frac{1}{2E_1}} \\ \sqrt{\frac{1}{2} + \frac{1}{2E_1}} \end{pmatrix} \text{ and } \varphi_2 = \begin{pmatrix} \sqrt{\frac{1}{2} + \frac{1}{2E_1}} \\ -\sqrt{\frac{1}{2} - \frac{1}{2E_1}} \end{pmatrix}, \quad (3.4)$$

which are orthonormal in the sense $\varphi_i \cdot \varphi_j = \delta_{ij}$. In the g^2 plane $E(g^2)$ has a branch point at $g^2 = -1$. At this point the two eigenvalues are equal to zero, but the vectors φ_1 and φ_2 no longer exist.

The method of the complex x plane may in addition become very complicated when applied to other systems. This can, e.g., be seen for the model described by the Hamiltonian (2.1). It can be shown that in the co-ordinate representation, the eigenvalue problem for this model consists of the following coupled differential equations ($x \geq 0$) :

$$\begin{aligned} \left(\frac{d^2}{dx^2} - \frac{J(J-1)}{x^2} - \frac{1}{2}x^2 + E \right) \chi_1(x) &= g\sqrt{2}x\chi_2(x) \\ \left(\frac{d^2}{dx^2} - \frac{J(J+1)}{x^2} - \frac{1}{2}x^2 + E \right) \chi_2(x) &= g\sqrt{2}x\chi_1(x). \end{aligned} \quad (3.5)$$

The difficulty in applying the method of Bender and Wu is not only that two boundary conditions and the analytic continuation of two functions in the complex x plane is required, but also that it is not clear how the analogue of Eq. (3.2) for the branch points must be formulated.

It is possible, however, to derive another equation in x space, which describes the same model and which can perhaps save the Bender-Wu method. This is important, because the analytical methods of Bender and Wu, which are not so easily applied to the matrix formulation, give a much deeper insight into the structure of the Riemann surface for the energy.

This new formulation can be derived by remarking ³⁾ that the Hamiltonian for this first model can be written as $H = H_0(g) + gH_1$, where $H_0(g) = a^*a + g(a + a^*)$ and

$$H_1 = \begin{vmatrix} 0 & \alpha_0 & & \\ \alpha_0 & 0 & & \\ & & 0 & \alpha_1 \\ & & \alpha_1 & 0 \\ & & & & 0 & \alpha_2 \\ & & & & \alpha_2 & 0 \end{vmatrix} \quad (3.6)$$

with

$$\alpha_n = \sqrt{2n+2}J+1 - \sqrt{2n+1}. \quad (3.7)$$

In the co-ordinate representation, this Hamiltonian gives rise to the following Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + x^2 + 2g\sqrt{2}x\right)\phi(x) + 2g \int_{-\infty}^{\infty} K(x, x') \phi(x') dx' = 2E \phi(x), \quad (3.8)$$

where the kernel

$$K(x, x') = \sum_{n=0}^{\infty} \alpha_n \left(h_{2n}(x) h_{2n+1}(x') + h_{2n+1}(x) h_{2n}(x') \right) \quad (3.9)$$

is expressed in terms of the Hermite functions $h_n(x)$. The eigenvalues are obtained from the single normalization condition $\int_{-\infty}^{\infty} \phi^2(x) dx = 1$. The second model, at least for $J = 0$, can be cast in the same form, with a slightly different kernel K . Since $\alpha_n \sim 1/\sqrt{n}$ for $n \rightarrow \infty$ the kernel K of (3.9) is not Hilbert-Schmidt, but it is compact. This is the main difference with the anharmonic oscillator and with the model of Peres ^{6), 7)} and explains why there the energy of the ground state has a divergent power series expansion, while here it will be shown (numerically) that all energy levels, when expanded in powers of g^2 , have a finite radius of convergence.

For $J = 0$ the second (pair) model can also be written as two coupled differential equations

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + x^2 - 2E\right) \chi_1(x) &= -2\sqrt{2}g x \chi_2(x) \\ \left(-\frac{d^2}{dx^2} + x^2 + \frac{2}{x^2} + 2m - 2E\right) \chi_2(x) &= -2\sqrt{2}g x \chi_1(x), \end{aligned} \quad (3.10)$$

with only one normalization condition

$$\int_0^{\infty} (\chi_1^2(x) + \chi_2^2(x)) dx = 1, \quad (3.11)$$

which again determines the spectrum. From Eq. (3.10) it can be seen that for large x the coupling terms never dominate the harmonic potential and this is again the reason why $g = 0$ is not a singular point. This difference in the two types of models manifests itself in the matrix formulation in that for the models without singularity in $g = 0$, e.g., (2.3), the off-diagonal elements do not grow as fast as the diagonal elements, whereas this is not the case for the anharmonic oscillator. Also in other models^{13),14)} dominance of the interaction has been proven to lead to a singularity in $g = 0$. If this dominance, which was also essential in Dyson's¹⁵⁾ heuristic argument, is absent, no such singularity is expected to occur. This is shown, by way of example, for the models discussed in the present paper.

For field theories with zero radius of convergence in the coupling constant, the question arises what meaning should be given to the formal power series. In the case of the anharmonic oscillator with a gx^4 coupling, it has been shown¹⁶⁾ that the series for the energy level E_n is of the Stieltjes type and that the diagonal Padé approximants approach E_n . With a gx^8 coupling the series for E_n is again of the Stieltjes type¹⁴⁾, but convergence to E_n has only been obtained with a generalized Borel summation.

For a theory with finite radius of convergence in g there is no difficulty in principle, since the value of the function outside this radius can be defined by analytic continuation. In practice, however, and especially when the function is required far outside the radius of convergence, methods must be given for its numerical calculation.

Since the two models considered in this article give rise to power series, which are not of the Stieltjes type, the general method¹⁶⁾ for proving the convergence of the Padé approximants does not apply. The numerical calculations show, however, that there is convergence for almost all positive g^2 , except for those values where neighbouring energy levels try to cross.

4. - RESULTS OF THE COMPUTATIONS

1. - The numerical method

The problem consists in finding the eigenvalues of the truncated forms of the complex tri-diagonal matrices (2.5) and (2.11), which describe the first and the second model respectively. In order to solve this problem, a standard¹⁷⁾ LR method was used, with a properly chosen shift at each step and deflation of the dimension N of the matrix.

Although for unsymmetric complex band matrices no proof exists for the convergence of the LR method a rapid determination of all N eigenvalues was in fact obtained. The energy levels were ordered according to increasing real parts, or, when two of these differed by less than 10^{-8} , according to increasing imaginary parts. For $N = 40$ the CDC-6600 computer gave all eigenvalues in 0.1 second. For $N = 60$ this number became 0.2 seconds, depending a little bit on the value of g^2 .

It was found that in the region $-7.5 \leq g^2 \leq 7.5$ the first five significant figures of the real and the imaginary parts of the first ten eigenvalues did not change when the dimension N was changed from 40 to 60. It can therefore be safely assumed that the first ten eigenvalues as calculated from the truncated matrices very well approximate the first ten exact eigenvalues.

2. - The energy levels $E_n(g^2)$

The real parts of E_n ($n=1, \dots, 16$) are calculated and plotted in Figs. 1 for model I, and 2, 3 and 4 for model II, with g^2 varying along the negative and positive real axis. The mass occurring in model II was arbitrarily given the value $m = 2.3$. For positive g^2 all energies are of course real, because the Hamiltonian is hermitian in that case. No degeneracy occurs, although two levels may come very close. This repulsion of levels is well known for systems which have no extra symmetry.

When a line carries two levels, the two energies are complex conjugate to each other. This can only happen for negative g^2 . If a line carries only one level the energy is real, also for negative g^2 . For the first model, the energies have been calculated down to $g^2 = -30$ and no further collapse or crossing was observed. The crossing of E_1 with the pair (E_2, E_3) at $g^2 = g_0^2 = -26.4$ in Fig. 3, does not occur at a branch point, since $\text{Im } E_2 \neq 0$ at that point, whereas E_1 is real for $g^2 \geq g_0^2$. For $g^2 < g_0^2$ the energies E_1 and E_2 are complex conjugate, but now E_3 is real. The curve which is the smooth continuation of E_1 therefore carries a real energy. This is also true for the smooth continuation of E_2 (see Fig. 4), which winds up, oscillating in the g^2 variable and eventually approaches the $g^2 = 0$ axis. The energy along this curve is real and the forks on this curve are branch points, since the levels which meet at these points have vanishing imaginary parts. This is shown in Fig. 5. The branch points (3.4) and (4.5) are so close that on the scale of Fig. 5 they coalesce. Other curves carrying real energies are formed by the pairs (3,4), (5,6), (7,8), etc. (see Fig. 4). The forks of these curves are again branch points. It will now be clear that Fig. 4 shows a cross-section along the negative real g^2 axis of the sheets of one Riemann surface. The pieces of the curves where E_1 and its continuation, E_2 and its continuation, and the pairs (E_3, E_4) , (E_5, E_6) , etc., are real, represent the different sheets. The short horizontal lines connecting E_2 with the pairs (E_3, E_4) , etc., give the cuts along the negative real axis. The horizontal parts extending to the left, except E_1 , are also cuts along the negative real axis, but it is not known whether they are finite or not.

For positive g^2 the levels E_n with $n \geq 4$ collide $n-3$ times with their neighbouring levels (model II). Since they do not, however, become quite degenerate, this means that the n th sheet will now have n conjugate pairs of short cuts very close to the real axis. They will be connected in a similar way as shown in Fig. 4, but this has not been studied in detail. The singularities in the first three sheets will be discussed under point 5 of this section.

Another way of getting some insight into the structure of the Riemann surface is by letting g^2 traverse a circle, $g^2 = |g^2|e^{i\varphi}$, $\varphi = 0 \rightarrow 2\pi$ and follow the paths of the energy levels in the complex E plane. The results of the calculations are shown in Figs. 6, 7 and 8 for model I and in Fig. 9 for model II. For model I it is indicated where for a given energy level the path, which is always traversed counter clockwise, jumps to another sheet. Since the cuts are defined by $\text{Re}(E_n) = \text{Re}(E_{n+1})$, the real part of the energy does not change, i.e., the jump is in the vertical direction. Figure 8 shows how the levels E_{10} , E_{11} , E_{12} and E_{13} wander through the E plane when φ changes from 0 to 2π . It is as follows :

$n = 10 : 1, 2, 3, 4, 5, 6, 7, 1$
 $n = 11 : 8, 3, 2, 9, 10, 7, 6, 8$
 $n = 12 : 11, 12, 13, 10, 9, 14, 15, 11$
 $n = 13 : 16, 17, 13, 12, 18, 19, 15, 14, 17, 16$

3. - Power series

For both models the first 100 coefficients were calculated in the series expansions $E_n = \sum_{j=0} c_j^{(n)} g^{2j}$ ($n=1, \dots, 10$). The radii of convergence r_n , obtained from these series, are plotted in Figs. 10 and 11 for the first and the second model respectively. For the first model $r_{2n} = r_{2n-1}$ ($n=1, 2, \dots$). This means that the branch points on the negative real g^2 axis, which can be recognized in Fig. 1, are the branch points closest to the origin. The numerical values of these branch points b_n are given in Table I. The first ten r_n , as obtained from the power series, have indeed reached values which exceed the corresponding $|b_n|$ by not more than 10%.

For the second model it is found that $r_{2n+1} = r_{2n}$ ($n=2, 3, \dots$). The corresponding branch points lie close to the positive real axis and are represented in Fig. 2 by the first points of near collision of E_{2n} and E_{2n+1} . Since for both models r_n tends to zero with n going to infinity the origin in the g^2 plane is a limit point of singularities. This behaviour was also found ¹⁾ for the anharmonic oscillator. The difference with the present cases is, however, that there already in one sheet the branch points accumulated near the origin, whereas here all branches are involved. Other peculiarities of the power series are :

- for model I : - all series with n odd are alternating ;
 - starting with the second term also the series with n even are alternating ;
 - with increasing j the ratios of $c_j^{(2n-1)}$ and $c_j^{(2n)}$ ($n=1,2,\dots$) very rapidly approach -1 ;
 - the first five coefficients for each of the series $E_1, \dots, E_5$ are given in Table II ;
- for model II : - for E_1 the series is alternating in groups of three or four terms ;
 - the series for E_2 and E_3 are alternating ;
 - for $j > 10$ the coefficients $c_j^{(2n)}$ are negative and the coefficients $c_j^{(2n+1)}$ are positive ($n=2,3,\dots$), with their ratio approaching -1 ;
 - the first five coefficients for each of the series $E_1, \dots, E_5$ are given in Table III.

If $f_j = (-1)^j c_j$ and

$$D(m, n) = \begin{vmatrix} f_m & f_{m+1} & \dots & f_{m+n} \\ f_{m+1} & f_{m+2} & \dots & f_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n} & f_{m+n+1} & \dots & f_{m+2n} \end{vmatrix},$$

it is found with the coefficients of Tables II and III that for a given energy not all $D(0, n)$ and $D(1, n)$ are positive. They are therefore not series of Stieltjes and it is not to be expected that the functions $E_n(g^2)$ are analytic in the plane cut along the negative real axis¹⁸⁾. Other branch points not on the real axis have indeed been found.

4. - Padé approximants

The exact $E_n(g^2)$ have been compared with the $[6, 15]$ Padé approximant. For negative g^2 the $[6, 15]$ shows a very wild behaviour and is not shown here. The results for positive g^2 are given in Figs. 12 (model I) and 13 (model II). The full curves represent the exact eigenvalues, whereas the Padé approximations are given by dashed curves. If the two overlap the curve is also dashed. For not too large values of g^2 there is good agreement and for most positive g^2 convergence of the Padé approximants can be expected. When, however, two levels come very close together the agreement on a finer scale becomes worse. Figures 14 and 15 show that the higher Padé approximants must describe functions which make a jump from one level to another. It is not clear whether at this point of discontinuity convergence can be obtained.

5. - Branch points

The branch points on the negative real axis have been mentioned under points 2 and 3 of this section. For model II the branch points on the interval $-30 \leq g^2 \leq 0$ are given in Table IV. They can be recognized in Fig. 4. The branch points near the positive real axis will not be studied, so that the remaining question is whether the rest of the complex g^2 plane contains branch points. It is immediately clear that if g^2 is a branch point its complex conjugate is one too. Therefore only the upper half g^2 plane is considered. Now, except for the singularities on the negative real axis, no further branch points were found - in neither of the two models - in the functions $E_1(g^2)$, $E_2(g^2)$ and $E_3(g^2)$ and with g^2 restricted by $-15 \leq \text{Re } g^2 \leq 0$ and $0 \leq \text{Im } g^2 \leq 30$. In the upper-right quadrant of the g^2 plane these three functions did have branch points. They are drawn in Figs. 16 and 17 for model I and model II respectively. In considering these figures it must be borne in mind that, although a branch point in the n th sheet was defined as a point g^2 for which $E_n(g^2) = E_{n+1}(g^2)$, such a point is of course also a singular point in the $(n+1)$ st sheet.

The most striking features of these figures are :

- a) in both models and for each of the three energies, the line of singularities, which originally is parallel to the imaginary axis, bends over to a horizontal direction ;
- b) there seem to be no branch points with $\text{Im } g^2 > 15$;
- c) only for the second model and for the smallest branch point of E_1 (and perhaps E_2) is there a clear indication that the poles of the $[25,24]$ Padé approximant simulates a branch line.

Just to illustrate what an analytic function near a square root branch point looks like, perspective drawings have been made (CERN program SUR F 3 D) of the real and imaginary part of E_1 (model II) near its smallest branch point $g^2 = (1.897, 2.426)$. The crest in the real part and the cliff in the imaginary part both end at the branch point.

6. - Isospin dependence of the energy

In order to calculate E_n as a function of J the Hamiltonian for the first model (2.5) was diagonalized with $g^2 = 3$ and J varying between -7.5 and $+7.5$. For model II a band matrix of width five had to be considered. Using the same LR method the eigenvalues could again be rapidly calculated. The results are shown in Fig. 20 for model I and in Figs. 21 and 22 for the other model. On the full curves the energy is again real, on the dashed parts two complex conjugated energies are situated. The independent variable J has been plotted in vertical direction in order to get a closer analogy with the usual Regge trajectories. As long as the energy is real and decreases with decreasing J this is a correct analogy, otherwise it is misleading. Another way of looking at these curves is to consider them as a cross-section along the real J axis

through the sheets of the Riemann surface on which the algebraic function $E(J)$ is defined. Forks again correspond to branch points on the real axis. Repulsion and attraction of Regge trajectories as noticed in refs. 8) and 9) here too are observed. In Fig. 22, a specially complicated crossing is shown. It is not known whether the three lines ought to pass through a single point.

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n	b_n	n	b_n	n	b_n
1	-0.106	11	-0.0536	21	-0.0393
3	-0.0855	13	-0.0496	23	-0.0374
5	-0.0731	15	-0.0464	25	-0.0359
7	-0.0647	17	-0.0436	27	-0.0344
9	-0.0584	19	-0.0413	29	-0.0331

TABLE I : Branch points on the negative real g^2 axis for the first model.

	c_0	c_1	c_2	c_3	c_4
E_1	0	-3	6	-28	$163\frac{1}{3}$
E_2	1	1	-10	$22\frac{2}{3}$	$-147\frac{7}{9}$
E_3	2	-3	14	$-54\frac{2}{3}$	$412\frac{2}{9}$
E_4	3	1	-18	60	-322
E_5	4	-3	22	$-102\frac{2}{3}$	$735\frac{7}{9}$

TABLE II : The first five expansion coefficients of the power series for $E_1, \dots, E_5$ (model I).

	c_0	c_1	c_2	c_3	c_4
E_1	0	-0.536	-0.0444	-0.00528	-0.000220
E_2	2	-1.448	0.367	-0.245	0.202
E_3	4	-2.361	1.358	-1.942	3.746
E_4	5.6	1.091	-0.659	1.010	-2.498
E_5	6	-3.274	2.928	-6.448	19.775

TABLE III : The first five expansion coefficients of the power series for $E_1, \dots, E_5$ (model II with $m = 2.3$).

2	3	4	5	6	7	8	9	10
-0.7694	-0.4683	-0.4731	-0.3271	-0.3363	-0.2491	-0.2604	-0.2005	-0.2124
	-0.2869		-0.1864		-0.1382		-0.1099	

TABLE IV : Branch points on the negative real g^2 axis for the second model.

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FIGURE CAPTIONS

- Figure__1 $\text{Re}(E_n)$ vs. g^2 for model I. $n = 1, \dots, 16$.
- Figure__2 $\text{Re}(E_n)$ vs. g^2 for model II. $n = 1, \dots, 16$.
- Figure__3 $\text{Re}(E_n)$ vs. g^2 for model II. Large negative g^2 .
- Figure__4 $\text{Re}(E_n)$ vs. g^2 for model II. $-1 \leq g^2 \leq 0$.
- Figure__5 $\text{Im}(E_n)$ vs. g^2 for model II.
- Figure__6 $\text{Im}(E_n)$ vs. $\text{Re}(E_n)$ for model I. $n = 1, \dots, 15$. $|g^2| = 0.5$.
- Figure__7 $\text{Im}(E_n)$ vs. $\text{Re}(E_n)$ for model I. $n = 1, \dots, 15$. $|g^2| = 0.1$.
- Figure__8 $\text{Im}(E_n)$ vs. $\text{Re}(E_n)$ for model I. $n = 10, 11, 12, 13$. $|g^2| = 0.1$.
- Figure__9 $\text{Im}(E_n)$ vs. $\text{Re}(E_n)$ for model II. $n = 1, \dots, 15$. $|g^2| = 1$.
- Figure_10 Radius of convergence r_n vs. n for model I.
- Figure_11 Radius of convergence r_n vs. n for model II.
- Figure_12 Exact E_n (full curves) and their $[16,15]$ Padé approximants (dashed curves) for model I.
- Figure_13 Exact E_n (full curves) and their $[16,15]$ Padé approximants (dashed curves) for model II.
- Figure_14 Detail of Fig. 13 near $g^2 = 0.7$.
- Figure_15 Detail of Fig. 13 near $g^2 = 1.3$.
- Figure_16 Branch points of E_1 (black squares), E_2 (black triangles) and E_3 (black circles) for model I. $\text{Re}(g^2)$ extends from 0 to 15, $\text{Im}(g^2)$ from 0 to 30.
- Figure_17 Branch points of E_1 (black squares), E_2 (black triangles) and E_3 (black circles) and their $[25,24]$ Padé approximants (open squares, triangles and circles respectively) for model II. $\text{Re}(g^2)$ extends from 0 to 15, $\text{Im}(g^2)$ from 0 to 30.
- Figure_18 Perspective plot of $\text{Re}(E_1)$ for model II near the absolute smallest branch point.

Figure_19 Perspective plot of $\text{Im}(E_1)$ for model II near the absolute smallest branch point.

Figure_20 J vs. $\text{Re}(E_n)$ for model I. $n = 1, \dots, 10$. $g^2 = 3$.

Figure_21 J vs. $\text{Re}(E_n)$ for model II. $n = 1, \dots, 10$. $g^2 = 3$.

Figure_22 Detail of Fig. 21.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 11

Fig. 10

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 22