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HARMONIC OSCILLATOR PATTERN ARISING FROM AN ALGEBRAIC APPROACH
TO CHIRAL SYMMETRY

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A B S T R A C T

In a previous work we proposed a saturation scheme for the $SU(3) \otimes SU(3)$ chiral algebra within an infinite set of $SU(6) \otimes O(3)$ states. A mixing operator was introduced which transforms the axial charges of the $SU(6)$ solution into the physical ones, Q_{i5} . Here we saturate the Weinberg equation for the $(\text{mass})^2$ operator $[Q_5^+, [Q_5^+, m^2]] = 0$, between meson states, in a perturbative approach. The generator Z of the mixing operators for which formerly only the transformation properties under $SU(6) \times O(3)$ were given, is now completely established as $Z = (\vec{W} \times \vec{M})_z$, where $\vec{W}$ is the W spin operator and $\vec{M}$ is the co-ordinate of the three-dimensional harmonic oscillator. In a perturbative expansion of the $(\text{mass})^2$ operator, the lowest term consists of two pieces, the harmonic oscillator energy and a spin-orbit coupling of the form $(-1)^{L+1} (\vec{L} \cdot \vec{S} + \frac{1}{2})$. The resulting $(\text{mass})^2$ consists of families of equispaced linearly rising trajectories.

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I. INTRODUCTION

The experimental fact that the known hadron resonances can be classified into representations of $SU(6)_S \otimes O(3)_L$ ¹⁾ suggests that the operators relevant to the strong interacting particles should also be constructed so as to have simple transformation properties under that group.

In particular, the identification of the vector (Q_i) and axial (Q_{i5}) charges with the proper $SU(6)$ generators $A(\lambda_i)$ and $A(\lambda_i \sigma_z)$ ²⁾ (in the frame of reference $P_z = \infty$) gives the commutation relations of the chiral algebra :

$$\begin{aligned} [A(\lambda_i), A(\lambda_j)] &= i f_{ijk} A(\lambda_k) \\ [A(\lambda_i), A(\lambda_j \sigma_z)] &= i f_{ijk} A(\lambda_k \sigma_z) \\ [A(\lambda_i \sigma_z), A(\lambda_j \sigma_z)] &= i f_{ijk} A(\lambda_k) \end{aligned} \quad (1)$$

Moreover, the angular conditions required by Lorentz invariance are fulfilled ; however, within the $SU(6)$ multiplets, the predicted axial couplings are consistently larger than the experimental values.

These facts suggest that the axial charges should be obtained from $A(\lambda_i \sigma_z)$ by a suitable mixing procedure :

$$Q_{i5} = U(\theta z) A(\lambda_i \sigma_z) U^\dagger(\theta z) \quad (2)$$

where Θ is a real number (mixing angle), Z is an Hermitian $SU(3)$ singlet and U is a unitary function of ΘZ ; the axial charges defined in (2) obey the commutation relations of the chiral algebra together with $A(\lambda_i)$.

With the assumption that Z transforms like a member of a $\underline{35}$ under $SU(6)$ and a vector under $O(3)$ one has ³⁾ :

$$Z = (\vec{W} \times \vec{M})_3 \quad (3)$$

where $\vec{W}$ is the $SU(3)$ singlet spin one of the $\underline{35}$ and connects only $(\underline{35} \oplus \underline{1}) \rightleftharpoons (\underline{35} \oplus \underline{1})$ for mesons and $\underline{56} \rightleftharpoons \underline{70}$ for baryons ; the operator $\vec{M}$ is a vector under $O(3)$ and generates the orbital excitation. It gives rise to a new quantum number n , which counts the number of times one has to apply $\vec{M}$ to reach a state starting from the ground state ($L = 0$). The property $\Delta L = 1$ of M gives rise to the (n, L) lattice plotted in the Figure.

Further degeneracy might be present since different paths can be taken to arrive at the same point in (n, L) plane. On the basis of Eqs. (2) and (3), the axial transitions $\underline{70} \ L = 1 \rightarrow \underline{56} \ L = 0$ and $\underline{35} \ L = 1 \rightarrow \underline{35} \ L = 0$ have been computed to first order in the mixing angle Θ : only one parameter is needed in each case and the agreement with experiment is satisfactory for baryons and very good for mesons ³⁾.

Here, we investigate whether the model presented in Ref. 3) for the axial charge is consistent with the Weinberg equation ⁴⁾ :

$$[Q_{5a}, [Q_{5b}, m^2]] \propto \delta_{ab} \quad (4)$$

We saturate Eq. (4) with the non-strange meson states and find further information on $\vec{W}$ and $\vec{M}$, for which only the transformation properties under $SU(6) \otimes O(3)$ have been given up to now.

In this paper one looks for a perturbative solution of Eq. (4). The unitary operator $U(\Theta Z)$ is expanded in powers of its argument :

$$U(\Theta Z) = 1 + i\Theta Z - \frac{1}{2}\Theta^2 Z^2 + \sum_{r=3}^{\infty} a_r (i\Theta Z)^r \quad (5)$$

The $(\text{mass})^2$ operator is expanded in a power series of Θ^2 and Eq. (4) is imposed at each order in Θ .

From the calculations up to second order, the operator Z is fully determined to be :

$$Z = i[W_+(a_- + a_+^\dagger) - W_-(a_+ + a_-^\dagger)] \quad (6)$$

where $\vec{W}$ is just the W spin operator, which has a simple interpretation in terms of the Dirac algebra, and $a_+(a_-)$ is the creation operator of one unit of orbital angular momentum in the up (down) direction obeying the commutation relations :

$$[a_i^\dagger, a_j] = \delta_{ij} \quad (7)$$

The first term m_0^2 in the expansion of m^2 in powers of Θ^2 has the simple form :

$$m_0^2 = A_0 + \Delta n + B(-)^{L+1}(\vec{L} \cdot \vec{S} + \frac{1}{2}) \quad (8)$$

Equation (8) gives rise to families of linear equispaced trajectories ; an interesting feature is the degeneracy of π and A_1 trajectories.

With Z given by Eq. (6), the term $B(-1)^L(\vec{L} \cdot \vec{S} + \frac{1}{2})$ satisfies (4) to all orders in Θ and could be a complete solution for the spin-orbit coupling in the mass operator. Its alternating sign constitutes a crucial experimental test for the whole approach ^{*)}.

The paper is organized as follows.

In Section 2, the perturbative method for Eq. (4) is fully explained. In Sections 3 and 4, the operators m_0^2 and Z are found from the second order calculations. In Section 5, we show the consistency of these results with Eq. (4) to all orders in Θ . The mass spectrum is discussed in Section 6. The explicit second order calculations are given in two Appendices.

^{*)} The possibility of an inverted triplet for $L = 2$ states has been discussed by Maglic at the Erice Summer School 1970.

2. THE PERTURBATIVE APPROACH TO EQ. (4)

We transform Eq. (4) by applying the mixing operator $U^\dagger(\theta Z)$ to obtain :

$$\begin{aligned} & [A(\frac{\tau_a}{2}, \sigma_a), [A(\frac{\tau_b}{2}, \sigma_b), U^\dagger(\theta Z) m^2 U(\theta Z)]] = \\ & = \frac{1}{3} \delta_{ab} [A(\frac{\tau_c}{2}, \sigma_c), [A(\frac{\tau_c}{2}, \sigma_c), U^\dagger(\theta Z) m^2 U(\theta Z)]] \end{aligned} \quad (9)$$

where the $SU(3)$ indices have been restricted to isospin for simplicity. Equation (9) implies that the operator $U^\dagger(\theta Z) m^2 U(\theta Z)$ must transform as a $(0,0) \oplus (\frac{1}{2}, \frac{1}{2})_{I=0}$ representation ⁴⁾ under the $SU(2) \otimes SU(2)$ algebra generated by the $SU(4)$ vector and axial charges $A(\frac{\tau^a}{2})$ and $A(\frac{\tau^a}{2} \sigma_z)$.

We recall the $SU(2) \otimes SU(2)$ content of the states belonging to the $\underline{15} \oplus \underline{1}$ representation of $SU(4)$

<u>I = 1</u>	<u>I = 0</u>
<u>S=1, S_z=0</u>	$ \bar{t}\rangle = \frac{1}{\sqrt{2}} [(1,0) - (0,1)] \quad \bar{s}\rangle = (0,0)$
<u>S=1, S_z=±1</u>	$ \vec{t} S_z=\pm 1\rangle = (\frac{1}{2}, \frac{1}{2})_{I=1} \quad \pm \vec{s} S_z=\pm 1\rangle = \pm (\frac{1}{2}, \frac{1}{2})_{I=0}$
<u>S=0</u>	$ t\rangle = \frac{1}{\sqrt{2}} [(1,0) + (0,1)] \quad s\rangle = (0,0)$

It is easy to see that the only matrix elements one can build within the above-defined set of states so that they do not behave according to (9) under $A \left[\frac{\tau_a}{2} \frac{(1 \pm \sigma_z)}{2} \right]$, are :

$$\langle (1,0) | (1,1)_{I=0} | (0,1) \rangle \text{ and } \langle (\frac{1}{2}, \frac{1}{2}) | (1,1)_{I=0} | (\frac{1}{2}, \frac{1}{2}) \rangle$$

The absence of $(1,1)_0$ in $U^\dagger(\Theta Z) m^2 U(\Theta Z)$ requires that :

$$\begin{aligned} \langle t n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | t n L L_2 \rangle = \\ = \langle \bar{t} n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | \bar{t} n L L_2 \rangle \end{aligned} \quad (10)$$

$$\begin{aligned} \langle \vec{v} S'_2 n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | \vec{v} S_2 n L L_2 \rangle = \\ = \langle v_4 S'_2 n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | v_4 S_2 n L L_2 \rangle \end{aligned} \quad (11)$$

$$\begin{aligned} \langle t n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | \bar{t} n L L_2 \rangle = \\ = \langle \bar{t} n' L' L_2 | U^\dagger(\Theta Z) m^2 U(\Theta Z) | t n L L_2 \rangle \end{aligned} \quad (12)$$

The $(\text{mass})^2$ operator is developed in a power series of Θ^2 :

$$m^2 = m_0^2 + \Theta^2 m_2^2 + \Theta^4 m_4^2 + \dots \quad (13)$$

Then, with the expansion (5) for U , one searches for m_i^2 operators, which solve (10), (11) and (12) order by order in the mixing angle Θ , assuming them to be diagonal in the base (InLSJ) and independent of the helicity $h = L_z + S_z$ and I_3 .

3. DETERMINATION OF m_0^2

Equations (10), (11) and (12) are considered at the order in given by $\max(|n'-n|, |L'-L|)$ so that only m_0^2 , the first term in the expansion (13), is present.

At 0^{th} order with $n' = n$ and $L' = L$ one finds ⁵⁾

$$m_0^2 = A(n, L) + B(n, L) (\vec{L} \cdot \vec{S} + \frac{1}{2}) \quad (14)$$

for the 15 (ω, ρ and π states) and

$$m_0^2 = \tilde{\eta}(n, L) \quad (14')$$

for the singlet ($\tilde{\eta}$ states). The ρ and ω states with the same (nLJ) quantum numbers are degenerate ; the only mass-splitting inside the $SU(4) \otimes O(3)$ multiplets arises from the $\vec{L} \cdot \vec{S}$ term, which breaks chiral symmetry ⁶⁾.

The operators A , B and $\tilde{\eta}$ are arbitrary functions of (n,L).

Equation (12), at first order in Θ with $n' = n \pm 1$ and $L' = L \pm 1$, gives no condition on $A(n, L)$ and $\tilde{\eta}(n, L)$, but relates the coefficients of the spin-orbit term :

$$B(n, L) = (-)^{n+1} B = (-)^{L+1} B \quad (15)$$

where B is a constant.

The calculations, concerning (10) and (11) at second order in Θ with $\max(|n'-n|, |L'-L|) = 2$, are performed in Appendix I and give :

$$\tilde{\eta}(n, L) = A(n, L) = A_0 + \Delta n \quad (16)$$

where A_0 and Δ are constants.

One is led to the following form for m_0^2 :

$$m_0^2 = A_0 + \Delta n + B(-)^{L+1} (\vec{L} \cdot \vec{S} + \frac{1}{2})$$

Note that there is degeneracy between $I = 0$ and $I = 1$ states.

4. DETERMINATION OF Z MATRIX ELEMENTS

The matrix elements of $W_{\pm}$ and $M_{\pm}$, the operators introduced in (3), are :

$$\underline{I=1}: W_{\pm} |t\rangle = |\vec{v} S_z = \pm 1\rangle ; W_{\pm} |\bar{t}\rangle = 0$$

$$\underline{I=0}: W_{\pm} |s\rangle = \pm \lambda |\vec{v}_4 S_z = \pm 1\rangle ; W_{\pm} |\bar{s}\rangle = 0$$

$$\begin{aligned} \langle I=1 \ n' L \ L_3 \pm 1 | M_{\pm} | I=1 \ n \ L-1 \ L_3 \rangle = \\ = \frac{1}{\sqrt{2}} \sqrt{(L \pm L_3)(L \pm L_3 + 1)} C_{n \ L-1}^{n' L} \end{aligned}$$

$$\begin{aligned} \langle I=0 \ n' L \ L_3 \pm 1 | M_{\pm} | I=0 \ n \ L-1 \ L_3 \rangle = \\ = \frac{1}{\sqrt{2}} \sqrt{(L \pm L_3)(L \pm L_3 + 1)} \tilde{C}_{n \ L-1}^{n' L} \end{aligned}$$

with λ , C and $\tilde{C}$ real numbers. Notice that different matrix elements are allowed for isospin 0 and 1.

In the second order calculation of Appendix I, one finds also some conditions on the matrix elements of the operator Z . Firstly, there is "exchange degeneracy", i.e. $\langle \rho | Z | \pi \rangle = \langle \omega | Z | \tilde{\eta} \rangle$, which implies the same internal excitation of isospin 0 and 1 particles. One obtains also :

$$\lambda \tilde{C}_{n-1 \ L-1}^{n \ L} = C_{n-1 \ L-1}^{n \ L} = \frac{f_+(n+L)}{\sqrt{4L^2-1}} \quad (17)$$

$$\lambda \tilde{C}_n^{n-1 \ L} = C_n^{n-1 \ L} = \frac{f_-(n-L)}{\sqrt{4L^2-1}}$$

Up to now, perturbative conditions have been considered, where only m_0^2 is present. When both m_0^2 and m_2^2 appear, the former being surrounded by two more powers in Z than the latter, some inconsistency might appear in the helicity dependence between the m_0^2 and the m_2^2 sides of the equation. This problem is investigated for the case $n' = n$, $L' = L$ at second order in θ and is presented in Appendix II. As a result, one obtains :

$$f_{\pm}(n \pm L) = \sqrt{n \pm L + 1} \quad (18)$$

Further, an $(\vec{L} \cdot \vec{S})^2$ term is incompatible with our perturbative approach to Eq. (4).

It can now be easily checked that, with (17) and (18), the extra $\hbar$ dependence is cancelled out in all the equations concerning only m_0^2 and m_2^2 at higher orders.

From Eq. (8) one can derive the commutation relation :

$$[M_+, M_-] = 0 \quad (19)$$

The result found for $\vec{M}$ can be put in a simpler form by going from the base $|nLL_z\rangle$ to another one $|n_+, n_0, n_-\rangle$. The n 's denote the number of orbital angular momentum excitations in each direction (up, collinear and down) ⁷⁾.

The quantum numbers n and L_z are diagonal in the new base and given by :

$$n = n_+ + n_0 + n_-$$

$$L_z = n_+ - n_-$$

while :

$$\begin{aligned} L_+ |n_+ n_0 n_- \rangle &= \sqrt{(n_++1)n_0} |n_++1 n_0-1 n_- \rangle + \\ &+ \sqrt{(n_0+1)n_-} |n_+ n_0+1 n_--1 \rangle. \end{aligned}$$

In this base M_+ is given by :

$$M_+ |n_+ n_0 n_- \rangle = \sqrt{(n_+ + 1)} |n_+ + 1 n_0 n_- \rangle + \sqrt{n_-} |n_+ n_0 n_- - 1 \rangle$$

and may be written in the operator form :

$$M_+ = a_+ + a_-^\dagger \quad (20)$$

with the commutation relations of the harmonic oscillator for the a 's :

$$[a_i^\dagger, a_j] = \delta_{ij} \quad (7)$$

5. HIGHER ORDER CONSISTENCY CONDITIONS

The calculations up to second order in Θ have given :

$$Z = i[W_+(a_- + a_+^\dagger) - W_-(a_+ + a_-^\dagger)] \quad (6)$$

$$m_0^2 = A_0 + \Delta n + B(-)^{L+1}(\vec{L} \cdot \vec{S} + \frac{1}{2}) \quad (8)$$

At higher orders, whenever the order in Θ considered is $N = \max(|n' - n|, |L' - L|)$, only m_0^2 appears and there is no freedom left, since Z is given, to fulfil (10), (11) and (12). Indeed these equations are satisfied, as we shall see in the following.

The "exchange degeneracy" for $I = 0$ and $I = 1$ makes (11) automatically satisfied when $S_z = S'_z$, while when $S_z + S'_z = 0$ it takes the form :

$$\langle v - S_z n' L' L'_z | U^\dagger(\theta z) m^2 U(\theta z) | v S_z n L L_z \rangle = 0 \quad (11')$$

First we show that the term $B(-1)^{L+1}(\vec{L} \cdot \vec{S} + \frac{1}{2})$ obeys (10), (11') and (12) at any order in θ . To this extent, one has to prove, since $Z|\bar{t}\rangle = 0$, that :

$$\begin{aligned} \langle t n' L' | U^\dagger(\theta z) (-)^L (\vec{L} \cdot \vec{S} + \frac{1}{2}) U(\theta z) | t n L \rangle = \\ = \frac{1}{2} (-)^L \delta_{L L'} \delta_{n n'} \end{aligned} \quad (21)$$

$$\langle v - S_z n' L' h + S_z | U^\dagger(\theta z) (-)^L (\vec{L} \cdot \vec{S} + \frac{1}{2}) U(\theta z) | v S_z n L h - S_z \rangle = 0$$

$$\begin{aligned} \langle \bar{t} n' L' | (-)^L (\vec{L} \cdot \vec{S} + \frac{1}{2}) U(\theta z) | t n L \rangle = \\ = \langle t n' L' | U^\dagger(\theta z) (-)^L (\vec{L} \cdot \vec{S} + \frac{1}{2}) | \bar{t} n L \rangle \end{aligned} \quad (22)$$

The selection rule $\Delta(L+L_z) = \text{even}$, which holds for Z , implies that $L_+ S_- + L_- S_+$ does not contribute to the left-hand side of (21). From the identities :

$$\begin{aligned} L_z S_z &= h S_z - S_z^2 \\ Z S_z Z &= 0 \end{aligned}$$

$$[z, (-)^L (S_z^2 - \frac{1}{2})] = 0$$

$$[z, h] = 0$$

one has :

$$\begin{aligned} U^\dagger(\theta z) (-)^L (L_z S_z + \frac{1}{2}) U(\theta z) &= \\ &= U^\dagger(\theta z) (-)^L [h S_z - S_z^2 + \frac{1}{2}] U(\theta z) = \\ &= h (-)^L [U^\dagger(\theta z) S_z + S_z U(\theta z)] + \frac{1}{2} (-)^L + S_z^2 \end{aligned}$$

from which (21) follows.

As far as (22) is concerned, it takes the form :

$$\langle t n' L' | [M_z, \frac{U(\theta z) - 1}{i z}] | t n L \rangle = 0$$

which is a consequence of (19).

The term Δn trivially obeys Eq. (12) since $z|\bar{t} \rangle = 0$ and n does not act on the $SU(4)$ space. Now we show that (10) and (11') are satisfied to the order in θ given by $N = \max(|n' - n|, |L' - L|)$. To prove this, we exploit the unitarity of U , which implies :

$$\sum_{r=0}^N (-)^r a_r a_{N-r} = \delta_{N0} \quad (23)$$

with the a_i defined in (5).

When $N = n' - n$ the left-hand side of (10) and (11') reads :

$$\begin{aligned}
 \langle n+N | \sum_{r=0}^N (-)^r a_r a_{N-r} z^{N-r} n z^r | n \rangle &= \\
 = \sum_{r=0}^N (-)^r (n+r) a_r a_{N-r} \langle n+N | z^N | n \rangle &= \\
 = n \delta_{N0} \langle n | n \rangle &
 \end{aligned}$$

which is just the right-hand side of the same equations. When $N = L' - L > |n' - n|$, there are several paths in the (n, L) lattice (see the Figure) which go in N steps from (n, L) to $(n', L+N)$. An interesting property is, however, that the different paths give the same contribution to

$$\langle n' L+N | z^N | n L \rangle$$

It is easily seen that an equal number of paths pass through the points $(n+h, L+\frac{N}{2}-K)$ and $(n'-h, L+\frac{N}{2}+K)$. Therefore,

$$\begin{aligned}
 \sum_{r=0}^N (-)^r a_r a_{N-r} \langle n' L+N | z^{N-r} n z^r | n L \rangle &= \\
 = \frac{n+n'}{2} \sum_{r=0}^N (-)^r a_r a_{N-r} \langle n' L+N | z^N | n L \rangle &= \\
 = n \delta_{N0} &
 \end{aligned}$$

which proves our assertion.

6. DISCUSSION OF THE RESULTS

The motivation for this work was to investigate whether the simple behaviour under $SU(6) \otimes O(3)$ assumed for Z , the generator of the mixing operator transforming the $SU(6)$ charges into the physical ones, was consistent with the Weinberg equation (4). The mathematical approach to this problem is the perturbative method described in Section 2. The computations up to second order result in the complete determination of Z and give an expression for the first term in the expansion of m^2 in powers of Θ^2 , m_0^2 , in terms of three parameters. It is a remarkable fact that the form found for Z and m_0^2 is such that, at any order in Θ , the equations concerning only m_0^2 are obeyed.

The expressions found for Z and m_0^2 are simple and seem to come from the same source. In fact, the operator $\vec{M}$, responsible for the excitation in the orbital angular momentum $\vec{L}$, is just the co-ordinate operator of the three-dimensional harmonic oscillator while the term Δn in m_0^2 is the energy of the same quantum system.

The spectrum described by m_0^2 consists of six families of equispaced linear trajectories :

$m_0^2(J)$	Spin J	C	Normality
$A - \frac{B}{2} + \Delta J + 2\Delta K$	all	+	-
$A + \frac{B}{2} + \Delta J + 2\Delta K$	all except 0	-	-
$A - \frac{B}{2} - \Delta + (\Delta + B)J + 2\Delta K$	all even except 0	+	+
$A - \frac{3}{2}B + \Delta + (\Delta - B)J + 2\Delta K$	all even	+	+
$A + \frac{B}{2} - \Delta + (\Delta - B)J + 2\Delta K$	all odd	-	+
$A + \frac{3}{2}B + \Delta + (\Delta + B)J + 2\Delta K$	all odd	-	+

where K is a positive integer giving the order of the daughter.

A complete degeneracy shows up between $I = 0$ and $I = 1$ states ; the π and A_1 trajectories coincide.

With $B = 0$ one has the typical spectrum of dual two-body amplitudes ⁸⁾, i.e., an universal slope $\frac{1}{\Delta}$ for the trajectories and several values of J for the same mass ; in such a case ρ and A_2 trajectories are "exchange degenerate", which is also a common result of dual theories ⁹⁾. However, the spin orbit coupling seems to be a peculiar feature of our approach since $(\vec{L} \cdot \vec{S} + \frac{1}{2})(-1)^{L+1}$ satisfies the Weinberg equation to all orders in Θ with the present form of Z .

The next term in the expansion of m^2 , m_2^2 splits the $I = 1$, $S = 0$ and $S = 1$ states by an amount independent of both n and L :

$$\Theta^2 [m_2^2(I=1, S=1) - m_2^2(I=1, S=0)] = 2\Theta^2 \Delta$$

so that, up to the order Θ^2 , one has :

$$m^2(\rho) - m^2(\pi) = (1 - G_{\rho\pi}^2) [m^2(A_1) - m^2(\pi)]$$

which gives the right sign and order of magnitude for the $\rho - \pi$ mass difference. However, this new term spoils the nice equispacing of the $I = 1$, $L = 1$ states, since the B meson $(\text{mass})^2$ is depressed by the same amount as the pion $(\text{mass})^2$. One should, therefore, rather wait until one has the exact expression for m^2 before drawing a precise conclusion. It is worth stressing that the daughter trajectories ($K \neq 0$) play an important role, as it can be seen from the C 's in Eqs. (17) and (18).

This is in agreement with the current attempts to build t channel Regge poles with s channel resonances ¹⁰⁾.

7. OUTLOOK

In this work, we have been concerned mainly with the first term, m_0^2 , of the $(\text{mass})^2$ expansion in powers of Θ^2 ; the consideration of the next term, m_2^2 , should result in the determination of $U(\Theta Z)$. We could then write all the matrix elements of the axial charge as a function of Θ through Eqs. (2) and (6). It is an open problem whether or not Eq. (9) can be solved by our perturbative method.

Finally, since both the term Δn in m_0^2 and the form of $\vec{M}$, suggest an underlying three-dimensional harmonic oscillator structure, it should be interesting to look for a simple dynamical model giving rise to the $(\text{mass})^2$ spectrum and the axial charge of this approach. Some symmetric quark models with harmonic interaction have been proposed on heuristic bases ¹¹⁾. We consider that the simple pattern emerging from this work is a good encouragement for further research in this direction.

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APPENDIX I

We shall solve the perturbative equations of order θ^2 which are imposed on the operator m_0^2 . The conditions to be satisfied are :

$$\langle t n' L' | [Z, [Z, m_0^2]] | t n L \rangle = 0 \quad (\text{I.1})$$

and

$$\begin{aligned} \langle \vec{v} S'_z n' L' | [Z, [Z, m_0^2]] | \vec{v} S_z n L \rangle = \\ = \langle v_4 S'_z n' L' | [Z, [Z, m_0^2]] | v_4 S_z n L \rangle \end{aligned} \quad (\text{I.2})$$

with $\text{Max}(|n' - n|, |L' - L|) = 2$.

We take m_0^2 from (14) and (14') ; previously established results are eventually used to simplify the new equations.

i) $n' = n \pm 2, L' = L + 2$.

From (I.1) we obtain

$$A(n \pm 1, L + 1) + A(n \mp 1, L - 1) = 2A(n, L) \quad (\text{I.3})$$

and from (I.2) with $S'_z = -S_z$

$$A(n, L) = \tilde{\eta}(n, L) \quad (\text{I.4})$$

Equation (I.2) with $S'_Z = S_Z$ then implies

$$\lambda^2 \tilde{C}_{n L}^{n-1 L-1} \tilde{C}_{n L}^{n+1 L+1} = C_{n L}^{n-1 L-1} C_{n L}^{n+1 L+1} \quad (I.5)$$

$$\lambda^2 \tilde{C}_{n L}^{n+1 L-1} \tilde{C}_{n L}^{n-1 L+1} = C_{n L}^{n+1 L-1} C_{n L}^{n-1 L+1}$$

ii) $n' = n \pm 2 \quad L' = L$

In this case, we get from (I.1)

$$\begin{aligned} & L(L-1) [A(n+1, L) + A(n-1, L) - 2A(n, L-1)] \times \\ & \times C_{n L-1}^{n-1 L} C_{n L-1}^{n+1 L} + \end{aligned} \quad (I.6)$$

$$\begin{aligned} & + (L+1)(L+2) [A(n+1, L) + A(n-1, L) - 2A(n, L+1)] \times \\ & \times C_{n L+1}^{n+1 L} C_{n L+1}^{n-1 L} = 0 \end{aligned}$$

and

$$\begin{aligned} & h^2 [A(n+1, L) + A(n-1, L) - 2A(n, L-1) + \\ & - 2B(-)^{L+1} (2L-1)] C_{n L-1}^{n-1 L} C_{n L-1}^{n+1 L} + \\ & + h^2 [A(n+1, L) + A(n-1, L) - 2A(n, L+1) + \\ & + 2B(-)^{L+1} (2L+3)] C_{n L+1}^{n+1 L} C_{n L+1}^{n-1 L} = \\ & = 0 \end{aligned} \quad (I.7)$$

and from (I.2) with $S'_Z = -S_Z$

$$\lambda^2 \tilde{C}_{n, L-1}^{n-1, L} \tilde{C}_{n, L-1}^{n+1, L} = C_{n, L-1}^{n-1, L} C_{n, L-1}^{n+1, L} \quad (\text{I.8})$$

$$\lambda^2 \tilde{C}_{n, L+1}^{n+1, L} \tilde{C}_{n, L+1}^{n-1, L} = C_{n, L+1}^{n+1, L} C_{n, L+1}^{n-1, L}$$

So, consider (I.2) with $S'_Z = S_Z$ to obtain,

$$\begin{aligned} & [A(n+1, L) + A(n-1, L) - 2A(n, L-1)] C_{n, L-1}^{n-1, L} C_{n, L-1}^{n+1, L} + \\ & + [A(n+1, L) + A(n-1, L) - 2A(n, L+1)] C_{n, L+1}^{n+1, L} C_{n, L+1}^{n-1, L} = \\ & = 0 \end{aligned} \quad (\text{I.9})$$

Equations (I.6), (I.7), (I.9) together give

$$A(n, L-1) = A(n, L+1) = \frac{1}{2} [A(n+1, L) + A(n-1, L)] \quad (\text{I.10})$$

and

$$\frac{C_{n, L-1}^{n-1, L} C_{n, L-1}^{n+1, L}}{C_{n, L+1}^{n+1, L} C_{n, L+1}^{n-1, L}} = \frac{2L+3}{2L-1} \quad (\text{I.11})$$

iii) $n' = n \quad L' = L \pm 2$

Equation (I.1) implies for this case

$$\begin{aligned}
 & [A(n, L+1) + A(n, L-1) - 2A(n-1, L)] C_{n, L-1}^{n-1, L} C_{n, L+1}^{n-1, L} + \\
 & + [A(n, L+1) + A(n, L-1) - 2A(n+1, L)] C_{n, L-1}^{n+1, L} C_{n, L+1}^{n+1, L} = \\
 & = 0
 \end{aligned}$$

Using (I.3) and (I.10), we get

$$C_{n, L-1}^{n-1, L} C_{n, L+1}^{n-1, L} = C_{n, L-1}^{n+1, L} C_{n, L+1}^{n+1, L} \quad (\text{I.12})$$

Analogously, for (I.2) with $S'_Z = \pm S_Z$:

$$\tilde{C}_{n, L-1}^{n-1, L} \tilde{C}_{n, L+1}^{n-1, L} = \tilde{C}_{n, L-1}^{n+1, L} \tilde{C}_{n, L+1}^{n+1, L} \quad (\text{I.13})$$

iv) Results

Solving (I.3) and (I.10), we obtain,

$$A(n, L) = A_0 + \Delta n$$

where A_0 and Δ are two constants.

Equations (I.11) and (I.12) have the solution

$$C_{n-1, L-1}^n = \frac{f_+(n+L)}{\sqrt{4L^2-1}} \quad C_{n-1, L-1}' = \frac{f_-(n-L)}{\sqrt{4L^2-1}}$$

where $f_{\pm}$ are arbitrary functions of $(n \pm L)$ with the "boundary condition" $f_-(-1) = 0$.

Finally, from (I.5), (I.8) and (I.13), we obtain

$$\lambda \tilde{C}_{n, L}^{n', L'} = C_{n, L}^{n', L'}$$

APPENDIX II

We discuss here the $(\text{mass})^2$ conditions, Eqs. (10) and (11), at order Θ^2 , in the case $n' = n, L' = L$, i.e., those equations are imposed on the operator $m_2^2 - \frac{1}{2}[\underline{Z}, [\underline{Z}, m_0^2]]$. Consider the general expansion in terms of the spin-operator S .

$$m_2^2 = \alpha(I n L) + \beta(I n L) [\vec{L} \cdot \vec{S} + \frac{1}{2}] + \frac{1}{2} \delta(I n L) S^2 + \gamma(I n L) (\vec{L} \cdot \vec{S})^2 \quad (\text{II.1})$$

and the expression for m_0^2 , Eq. (8). Remembering that both m_0^2 and Z are "exchange degenerate", we take (11) with $S'_Z = S_Z$ to obtain

$$m_2^2(\omega n L J) = m_2^2(\rho n L J) \quad (\text{II.2})$$

In particular,

$$\gamma(I=0 n L) = \gamma(I=1 n L) = \gamma(n L)$$

The same equation, with $S'_Z = -S_Z$, will now imply

$$\gamma(n L) = \Delta[(C_{n+1}^{n+1} L_{+1})^2 - (C_n^{n+1} L_{+1})^2 - (C_n^{n-1} L_{-1})^2 + (C_{n+1}^{n-1} L_{-1})^2] \quad (\text{II.3})$$

Finally, one considers (10) and gets

$$\begin{aligned}
 & \delta(I=1 \text{ n } L) + [L(L+1) - h^2] \gamma(nL) = \\
 & = [L(L-1) + h^2] \Delta[(C_{n+1}^{n+1} L^{-1})^2 - (C_n^{n-1} L^{-1})^2] + \\
 & + [(L+2)(L+1) + h^2] \Delta[(C_{n+1}^{n+1} L^{+1})^2 - (C_n^{n-1} L^{+1})^2] + \\
 & + h^2 B(-)^L \{ (2L+3)[(C_{n+1}^{n+1} L^{+1})^2 + (C_n^{n-1} L^{+1})^2] \\
 & \quad - (2L-1)[(C_{n+1}^{n+1} L^{-1})^2 + (C_n^{n-1} L^{-1})^2] \} \quad (\text{II.4})
 \end{aligned}$$

Consider the h^2 terms of this equation, use $\gamma(n,L)$ as given by (II.3) and the form (17) for the C's, to obtain

$$\begin{aligned}
 & (-)^{L+1} B [f_+^2(n+L+2) + f_-^2(n-L-1) + \\
 & \quad - f_+^2(n+L) - f_-^2(n-L+1)] = \\
 & = 2 \Delta \left[\frac{f_+^2(n+L+2) - f_-^2(n-L-1)}{2L+3} + \right. \\
 & \quad \left. + \frac{f_+^2(n+L) - f_-^2(n-L+1)}{2L-1} \right] \quad (\text{II.5})
 \end{aligned}$$

The solution of (II.5) that obeys the "boundary condition"
 $f_{-}(-1) = 0$ is

$$f_{+}^2(x) = f_{-}^2(x) = \text{const.}(x+1)$$

and with a consistent choice of phases and normalization, we get

$$f_{\pm}(n \pm L) = \sqrt{n \pm L + 1}$$

Equation (II.3) now gives

$$\gamma(nL) = 0$$

i.e., no $(L.S)^2$ term can be present in the m_2^2 operator.

Putting all this back to (II.4), we have no more h^2 dependence in this equation and get

$$\delta(I=1 \ nL) = 2\Delta$$

which gives a constant splitting between $S = 1$ and $S = 0$ states.

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