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ANALYTICITY PROPERTIES OF HELICITY AMPLITUDES FOR
FIVE-BODY PROCESSES INVOLVING MASSLESS PARTICLES

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A B S T R A C T

Taking into account gauge conditions, we have extended to the massless case results previously obtained for five massive particles by using Raynal's method. This procedure allows to build in the massless case regularized helicity amplitudes the constraints of which have been explicitly derived.

1. INTRODUCTION

In a previous work ¹⁾, we have extensively studied the analytic properties of helicity amplitudes for the processes involving five massive particles of any spin. In this paper we want to obtain similar results in the case of reactions involving massless particles of arbitrary spin. Indeed a special study is necessary in order to take properly into account gauge invariance for massless particles. As we shall see later, parity conservation is needed as in the massive case and this study will apply in practice to the photon case and will give a tool to introduce spin in the models using five-point Veneziano amplitudes for instance in the photoproduction case ²⁾.

As before we have used Raynal's method ³⁾ which consists in writing helicity amplitudes as linear combinations of the spinor amplitudes $\mathcal{M}_{A_i}(p_i)$ once the gauge relations ^{4),5)} have been taken into account. ($M_{\lambda_i} = C_{\lambda_i}^{A_i} \mathcal{M}_{A_i}$.)

Whenever at least one particle is massive, the results for the spinor amplitudes apply; namely, the $\mathcal{M}_{A_i}(p_i)$ are analytic functions in the set E of the four-momentum components restricted to the mass shell and obey the covariance property. Their only singularities originate in dynamics. For a given singularity of the $C_{\lambda_i}^{A_i}$'s it is always possible, apart from the boundary of the physical region ^{*}), to choose a reference frame such that the $\mathcal{M}_{A_i}$ are analytic functions of the five independent invariants in the vicinity of this singularity. Then all the kinematical singularities are those of the $C_{\lambda_i}^{A_i}$'s.

In Section 2, we recall briefly the definition and gauge properties of spinor states for massless particles. Section 3 is devoted to only the singularities which are different from the massive case and which require a special study. The kinematical constraints are also derived whenever necessary.

As has been shown for the four-body processes ^{5),6)}, the crossing matrix for the massless case is obtained from that of the massive case by setting one or several masses equal to zero. In other words the helicity is unchanged if the massless particle is uncrossed; it is changed into its opposite if the particle is crossed. Thus it is sufficient to apply the results of Capella ⁷⁾ and of Chih Kwan Chen and Kuo Hsian Wang ⁸⁾ and the recipe for the massless case to get the corresponding crossing matrix. In the Appendix we give some technical details.

^{*}) It is the reason why we have assumed that parity conservation holds, which eliminates the neutrino case.

2. NOTATIONS AND CONVENTIONS

A. Spinor amplitudes

We will use throughout this work the notations of Ref. 5) for the massless particle formalism.

Here we recall briefly the main results.

For a massless particle of helicity $+|\lambda|$ (resp. $-|\lambda|$) and of four-momentum p , we define the spinor state $|p, +|\lambda|, A\rangle$ (resp. $|p, -|\lambda|, \tilde{A}\rangle$). By definition these states transform under a Lorentz transformation Λ by

$$\begin{aligned} U(\Lambda) |p, +|\lambda|, A\rangle &= D^{|\lambda|}(\Lambda)^{A'}_A | \Lambda p, +|\lambda|, A'\rangle \\ \text{(resp. } U(\Lambda) |p, -|\lambda|, \tilde{A}\rangle &= D^{|\lambda|}(\Lambda^{\dagger-1})^{A'}_A | \Lambda p, -|\lambda|, \tilde{A}'\rangle \end{aligned}$$

where $\Lambda^{\dagger-1}$ is the inverse of the 2×2 hermitian conjugate matrix of Λ).

Furthermore, these states obey the following gauge conditions

$$|p, +|\lambda|, A\rangle = D^{|\lambda|} \left(\frac{\sigma_e \cdot \tilde{\sigma} p}{e} \right)^{A'}_A |p, +|\lambda|, A'\rangle$$

$$|p, -|\lambda|, \tilde{A}\rangle = D^{|\lambda|} \left(\frac{\tilde{\sigma} e \cdot \sigma p}{e} \right)^{A'}_A |p, -|\lambda|, \tilde{A}'\rangle$$

(2.1)

where e is any four-vector satisfying the conditions $e \cdot p = 1$, $e^2 \neq 0$. From these states, we want to define the spinor amplitudes which obey the covariance formula

$$m_{A_i}(p_i) = m_{A'_i}(\Lambda p_i) \prod_i D^{s_i}(\Lambda)^{A'_i}_{A_i} \quad (2.2)$$

Let us treat for instance the case where only one incoming particle, say 1, is massless.

$$a) \underline{\lambda_1 \geq 0}$$

$$m_{A_3 A_4 A_5, A_1 A_2}^+ (p_3 p_4 p_5; p_1 p_2) = \langle p_3 s_3 \dot{A}_3, p_4 s_4 \dot{A}_4, p_5 s_5 \dot{A}_5 | T | p_1 +|\lambda_1| A_1, p_2 s_2 A_2 \rangle$$

satisfies the covariance property (2.2)

b) $\underline{\lambda_1 < 0}$

In this case we first define

$$m_{A_3 A_4 A_5; \tilde{A}_1 A_2} (p_3 p_4 p_5; p_1 p_2) = \langle p_3 s_3 \dot{A}_3, p_4 s_4 \dot{A}_4, p_5 s_5 \dot{A}_5 | T | p_1 -|\lambda_1| \tilde{A}_1, p_2 s_2 A_2 \rangle$$

which does not obey the covariance property (2.2). But the amplitude

$$m_{A_3 A_4 A_5; A_1 A_2}^- (p_3 p_4 p_5; p_1 p_2) = \mathcal{D}^{|\lambda_1|} \left(\frac{\tilde{\sigma} p_3}{m_3} \right)_{A_1}^{A'_1} m_{A_3 A_4 A_5; \tilde{A}'_1 A'_2} (p_3 p_4 p_5; p_1 p_2)$$

does obey the covariance property ^{*} (2.2).

In the case of one outgoing massless particle, say particle 4, it is easy to define also two spinor amplitudes $m_{\{A_i\}}^\pm$ satisfying the covariance property (2.2).

a) $\underline{\lambda_4 < 0}$

$$m_{A_3 A_4 A_5; A_1 A_2}^- (p_3 p_4 p_5; p_1 p_2) = \langle p_3 s_3 \dot{A}_3, p_4 -|\lambda_4| \tilde{A}'_4, p_5 s_5 \dot{A}_5 | T | p_1 s_1 A_1, p_2 s_2 A_2 \rangle \mathcal{D}^{|\lambda_4|} (\epsilon^{-1})_{A_4}^{A'_4}$$

b) $\underline{\lambda_4 > 0}$

$$m_{A_3 A_4 A_5; A_1 A_2}^+ (p_3 p_4 p_5; p_1 p_2) = \langle p_3 s_3 \dot{A}_3, p_4 +|\lambda_4| A'_4, p_5 s_5 \dot{A}_5 | T | p_1 s_1 A_1, p_2 s_2 A_2 \rangle \mathcal{D}^{|\lambda_4|} (\epsilon \frac{\tilde{\sigma} p_3}{m_3})_{A_4}^{A'_4}$$

Now these spinor amplitudes are subject to the gauge condition (2.1). For instance, if only particle 1 is massless, we can choose $e = p_3 / (p_3 \cdot p_1)$ ($e \cdot p_1 = 1$). Then,

$$m_{\{A_i\}, A_1}^+ (p_i) = \mathcal{D}^{|\lambda_1|} \left(\frac{\sigma p_3 \cdot \tilde{\sigma} p_1}{2(p_3 \cdot p_1)} \right)_{A_1}^{A'_1} m_{\{A_i\}, A'_1}^+ (p_i)$$

$$m_{\{A_i\}, A_1}^- (p_i) = \mathcal{D}^{|\lambda_1|} \left(\frac{\sigma p_1 \cdot \tilde{\sigma} p_3}{2(p_3 \cdot p_1)} \right)_{A_1}^{A'_1} m_{\{A_i\}, A'_1}^- (p_i) \quad (2.3)$$

^{*}) The four-vector p_3 may be replaced by any four-vector $K (K^2 \neq 0)$ where K is a linear combination of the p_i 's.

Analogous results hold if only particle 4 is massless, namely

$$\begin{aligned} m_{\{A_i\}, A_4}^-(p_i) &= D^{|\lambda_4|} \left(\frac{\sigma p_3 \cdot \tilde{\sigma} p_4}{2(p_3 \cdot p_4)} \right)^{A'_4}_{A_4} m_{\{A_i\}, A'_4}^-(p_i) \\ m_{\{A_i\}, A_4}^+(p_i) &= D^{|\lambda_4|} \left(\frac{\sigma p_4 \cdot \tilde{\sigma} p_3}{2(p_3 \cdot p_4)} \right)^{A'_4}_{A_4} m_{\{A_i\}, A'_4}^+(p_i) \end{aligned} \quad (2.4)$$

These spinor amplitudes which obey the covariance property and the gauge conditions, possess the same analyticity properties as in the massive case provided that at least one mass is different from zero.

B. Helicity amplitudes in the total centre-of-mass

a) Helicity states for a mass particle in the c.m. system

We recall briefly the definition

$$\begin{aligned} \text{for } \lambda_i > 0 \quad |p_i, L(i), \lambda_i\rangle &= D^{|\lambda_i|} (L(i))^{A_i}_{\lambda_i} |p_i, +|\lambda_i|, A_i\rangle \\ \text{for } \lambda_i < 0 \quad |p_i, L(i), \lambda_i\rangle &= D^{|\lambda_i|} (L(i))^{A_i}_{\lambda_i} |p_i, -|\lambda_i|, A_i\rangle \end{aligned} \quad (2.5)$$

where $L(i)$ is the Lorentz transformation which takes the standard triade $(t+n_3, n_1, n_2)$ onto $(p_i, n_1(i), n_2(i))$.

As in our previous work ¹⁾ particle 3 is chosen such that $s_1 + s_2 + s_3$ is an integer. Thus we have defined

$$\eta_i^\mu(i) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_i) p_\rho(p_3) \sigma}{\sqrt{\Delta_3(p, p_1, p_3)}} \quad i = 1, 2, 3 \quad (2.6)$$

and

$$\eta_i^\mu(i) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_4) p_\rho(p_5) \sigma}{\sqrt{\Delta_3(p, p_4, p_5)}} \quad i = 4, 5 \quad (2.7)$$

where $P = p_1 + p_2$ and $\Delta_n(p_1, \dots, p_n)$ is the Gram determinant $\begin{vmatrix} p_1 p_2 \dots p_n \\ p_1 p_2 \dots p_n \end{vmatrix}$ up to a sign $(-1)^n$.

b) Helicity amplitudes in the total centre-of-mass

Helicity amplitudes can be expressed as functions of spinor amplitudes $m_{A_i}^{\pm}(p_i)$ using definition (2.5) and taking into account the gauge invariance conditions (2.3), (2.4).

For instance, if only one initial particle, say particle 1, is massless

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} = D^{S_3}(L(3)\varepsilon)^{A_3}_{\lambda_3} D^{S_4}(L(4)\varepsilon)^{A_4}_{\lambda_4} D^{S_5}(L(5)\varepsilon)^{A_5}_{\lambda_5} m_{A_3 A_4 A_5; A_1 A_2}^{\pm}(p_3 p_4 p_5; p_1 p_2) \times \\ \times D^{|\lambda_1|}(H_1^{\pm})^{A_1}_{\lambda_1} D^{S_2}(L(2))^{A_2}_{\lambda_2} \quad (2.8)$$

where

$$H_1^+ = \frac{\sigma p_3}{2(p_1 p_3)} L^{+-1}(1) \quad \text{if } \lambda_1 > 0 \quad *)$$

$$H_1^- = \frac{m_3}{2(p_1 p_3)} L(1) \quad \text{if } \lambda_1 < 0$$

and $L(i)$ $i=2,3,4,5$ is the usual Lorentz transformation which takes the standard tetrad (t, n_1, n_2, n_3) onto $(\frac{p_i}{m_i}, n_1(i), n_2(i), n_3(i))$. On the other hand, if only a final particle, say particle 4, is massless

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} = D^{S_3}(L(3)\varepsilon)^{A_3}_{\lambda_3} D^{|\lambda_4|}(H_4^{\pm})^{A_4}_{\lambda_4} D^{S_5}(L(5)\varepsilon)^{A_5}_{\lambda_5} m_{A_3 A_4 A_5; A_1 A_2}^{\pm}(p_3 p_4 p_5; p_1 p_2) \times \\ \times D^{S_1}(L(1))^{A_1}_{\lambda_1} D^{S_2}(L(2))^{A_2}_{\lambda_2} \quad (2.9)$$

where

$$H_4^- = \frac{\sigma p_3}{2(p_4 p_3)} L^{+-1}(4) \varepsilon \quad \text{if } \lambda_4 < 0$$

$$H_4^+ = \frac{m_3}{2(p_4 p_3)} L(4) \varepsilon^{-1} \quad \text{if } \lambda_4 > 0$$

*) We may replace p_3 by any four-vector K which is a linear combination of p_i 's ($K^2 \neq 0$).

3. KINEMATICAL SINGULARITIES AND CONSTRAINTS

A. Principle of the method

We now recall the principle of Raynal's method ³⁾ we have already used in the massive case.

Let us call π the projection

$$p_i \in E \xrightarrow{\pi} \begin{cases} s = (p_1 + p_2)^2 \\ t_1 = 2 p_1 \cdot p_3 \\ t_2 = 2 p_1 \cdot p_4 \\ t_3 = 2 p_2 \cdot p_3 \\ t_4 = 2 p_2 \cdot p_4 \end{cases}$$

The helicity amplitudes are distributions defined in the domain $\pi(E)$ of dimension 5. Let us assume that $M_{A_i}(p_j)$ is analytical in a neighbourhood of a point p_j ; if we choose a function

$$\text{space of invariants} \xrightarrow{g_j} \text{space of momenta}$$

such that $\pi \circ g_j = 1$ and $g_j \circ \pi(p_j) = p_j$. Then, Eqs. (2.8) and (2.9) can be written

$$M_{\{A_i\}}(s, t_k) = \pi \mathcal{D}^{s_i} \left(\mathcal{L}(i, g_j(s, t_k))^{A_i} \right)_{\lambda_i} m_{A_i}(g_j(s, t_k)) \quad (3.1)$$

where $\mathcal{L}$ stands for $H_i^\pm$, $L(i)$ or $L(i)E$.

If g_j is chosen analytical in a neighbourhood of $\pi(p_i)$ $m_{A_i}(g_j(s, t_k))$ is also an analytical function of (s, t_k) in a neighbourhood of $\pi(p_i)$ and singularities of a kinematical type are contained in the functions

$$\mathcal{D}^{s_i} \left[\mathcal{L}(i, g_j(s, t_k))^{A_i} \right]_{\lambda_i}$$

namely

$$\begin{aligned} \Delta_4(p, p_1, p_3, p_4) &= 0 \\ \Delta_3(p, p_1, p_3) &= 0, \quad \Delta_3(p, p_4, p_5) = 0 \\ \Delta_2(p, p_i) &= 0 \quad i = 1, 2, 3, 4, 5. \end{aligned}$$

Furthermore, we can reverse relation (3.1) and get

$$M_{\{A_i\}}(g_j(s, t_k)) = \pi \mathcal{D}^{s_i}(\mathcal{L}^{-1}(\lambda, g_j(s, t_k))^{\lambda_i} M_{\{A_i\}}(s, t_k) \quad (3.2)$$

These relations (3.2) are nothing but constraints on the right-hand side whenever the M_{A_i} 's are analytical functions of (s, t_k) in a neighbourhood of the singularities of the $\mathcal{D}(\mathcal{L}(i))$.

In a previous work ¹⁾, we have shown that apart from the boundary of the physical region ($\Delta_4(P, p_1, p_3, p_4) = 0$) it is always possible to find a reference frame such that each component of the four independent momenta is an analytic function of the five invariants in the neighbourhood of a singularity $\Delta_n(P, p_1, \dots, p_n) = 0$ ($n=2,3$). Then, to study these singularities, it is sufficient to choose a reference frame the four vectors of which are not singular.

B. Kinematical singularities at $\Delta_4(P, p_1, p_3, p_4) = 0$

There does not exist any $g_j(s, t_k)$ which is analytic on the boundary of the physical region. So we need as in the massive case parity conservation which, for massless particles, reads

$$U(\pi) |p, L(p), \lambda\rangle = \eta^{|\lambda|+2} | \tilde{p}, L(\tilde{p}), -\lambda \rangle \quad (\tilde{p} = (p^0, -\vec{p}))$$

If parity conservation does not hold (the neutrino case, for instance), then it is impossible to get rid of this singularity which appears even in the spinless case.

With parity conservation (the photon case, for instance) the regularization is the same as in the massive case.

C. Kinematical singularities at $\Delta_3(P, p_1, p_3) = 0$ and $\Delta_3(P, p_4, p_5) = 0$

Only the 2-axis of the helicity tetrad or triad are singular at $\Delta_3 = 0$. In any case, the Lorentz transformation $L(i)$ can be written as $L(i) = R_i B(i)$ where $B(i)$ is a boost and R_i a rotation which does not depend on the mass of the particle i . Furthermore, $R_i^{\dagger-1} = R_i$. Then, provided that at least one particle is massive, the regularization for the massive case is still valid.

D. Kinematical singularities at $\Delta_2(P, p_i) = 0$ ($i=1,3,4,5$)

D.1 $m_1 = 0, m_2 \neq 0$ (or $m_2 = 0, m_1 \neq 0$)

The threshold coincides with the pseudothreshold. Thus a special study is needed at $s = m_2^2$.

In the centre-of-mass frame R_2 ($\hat{p}_3$ is chosen as 3-axis and $\hat{p}_3 \times \hat{p}_4$ as 2-axis), the spinor amplitudes are analytic near $s = m_2^2$, and so are $L(i)$ $i = 3, 4, 5$. From formula (3.8) we write

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} = D^{|\lambda_1|}(H_1^\pm)_{\lambda_1}^{A_1} D^{s_2}(L(2))_{\lambda_2}^{A_2} C_{A_1 A_2}$$

where $C_{A_1 A_2}$ is analytic in the neighbourhood of $s = m_2^2$.

In the reference frame R_2

$$L(1) = R(1) B(1) \quad (\text{see the Appendix})$$

$$L(2) = R(1) B(2)$$

where

$$B(1) = \begin{pmatrix} \frac{1}{\sqrt{|\vec{p}_1|}} & 0 \\ 0 & \sqrt{|\vec{p}_1|} \end{pmatrix}, \quad B(2) = \begin{pmatrix} \sqrt{\frac{m_2}{s}} & 0 \\ 0 & \sqrt{\frac{s}{m_2}} \end{pmatrix}, \quad |\vec{p}_1| = \frac{s - m_1^2}{2\sqrt{s}}$$

and $R(1)$ is the rotation of Euler angle $(\varphi_1 + \pi, -\theta_{13}, 0)$ where θ_{13} is the angle of the pure rotation around $\vec{p}_1 \times \vec{p}_3$ which takes $\hat{p}_1$ onto $\hat{p}_3$ ($\sin \theta_{13} \approx \cos \theta_{13} \approx 1/(s - m_2^2)$) and φ_1 is the azimuthal angle of particle 1 and is regular at $s = m_2^2$.

Then, for $\lambda_1 > 0$

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} = (-1)^{s_2 - \lambda_2} D^{|\lambda_1|}(B_{(1)}^\dagger)_{\lambda_1}^{A_1} \sum_{\{A_1, A_2\}} d^{|\lambda_1|}(-\theta_{13})_{\lambda_1}^{A_1} d^{s_2}(-\theta_{13})_{-\lambda_2}^{A_2} C'_{A_1 A_2}$$

where $C'_{A_1 A_2}$ is analytic near $s = m_2^2$. [For $\lambda_1 < 0$ we have to replace $B_{(1)}^{\dagger -1}(1)$ by $B(1)$]. Turning around the point $s = m_2^2$ induces the following changes

$$\begin{aligned} \sqrt{|\vec{p}_1|} &\rightarrow -\sqrt{|\vec{p}_1|} \\ \frac{\theta_{13}}{2} &\rightarrow \frac{\theta_{13}}{2} \pm \pi \end{aligned}$$

Then

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2}(-\sqrt{s - m_1^2}) = (-1)^{2\lambda_1} M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2}(\sqrt{s - m_1^2}) \quad (3.3)$$

It is now easy to find the kinematical poles at $s = m_2^2$,

$$\text{for } \lambda_1 < 0 \quad D^{|\lambda_1|} (B(1)) \lambda_1 \lambda_2 \approx \sqrt{|\vec{p}_1|} |\lambda_1|$$

$$\text{for } \lambda_1 > 0 \quad D^{|\lambda_1|} (B^{\dagger}(1)) \lambda_1 \lambda_2 \approx \sqrt{|\vec{p}_1|} |\lambda_1|$$

$$d^{|\lambda_1|} (-\theta_{13}) \lambda_1 \lambda_2 \approx \sqrt{|\vec{p}_1|}^{-2|\lambda_1|} \text{ and } d^{s_2} (-\theta_{13}) \lambda_1 \lambda_2 \approx \sqrt{|\vec{p}_1|}^{-2s_2}$$

Then we deduce

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} \approx (\sqrt{s - m_2^2})^{-2s_2} \quad (*) \quad (3.4)$$

In order to regularize now, we consider the $H_{\{\lambda_i\}}^1$ functions which are analytic at $\Delta_4(P, p_1, p_2, p_3) = 0$, $\Delta_3(P, p_1, p_3) = 0$ and $\Delta_3(P, p_4, p_5) = 0$, namely [see formula (2.30) of Ref. 1)]

$$H_{\{\lambda_i\}}^1 = \sin \frac{\theta_{13}}{2} \frac{|\lambda_3 - (\lambda_1 - \lambda_2)|}{2} \cos \frac{\theta_{13}}{2} \frac{|\lambda_3 + (\lambda_1 - \lambda_2)|}{2} \sin \frac{\theta_{45}}{2} \frac{|\lambda_4 + \lambda_5|}{2} \cos \frac{\theta_{45}}{2} \frac{|\lambda_4 - \lambda_5|}{2} F_{\{\lambda_i\}}^1 \quad (3.5)$$

where θ_{45} is the angle of the pure rotation around $\vec{p}_4 \times \vec{p}_5$ which takes $\hat{p}_4$ onto $\hat{p}_5$ in the centre-of-mass frame and $F_{\{\lambda_i\}}^1$ are analytic at $\Delta_4(P, p_1, p_3, p_4) = 0$ (**)

$$\begin{aligned} F_{\{\lambda_i\}}^1 &= M_{\{\lambda_i\}} + \eta_{\{\lambda_i\}} M_{\{\lambda_i\} - \lambda_i} \\ F_{\{\lambda_i\}}^2 &= \frac{M_{\{\lambda_i\}} - \eta_{\{\lambda_i\}} M_{\{\lambda_i\} - \lambda_i}}{\sqrt{\Delta_4(P, p_1, p_3, p_4)}} \\ \eta_{\{\lambda_i\}} &= \prod_{i=1}^5 \eta_i (-1)^{s_i + \lambda_i} \end{aligned} \quad (3.6)$$

η_i is the intrinsic parity of particle i (***).

*) As in the four-body case ⁵⁾, the exponent of $s - m_2^2$ does not depend on the value of the helicity of the massless particle. Furthermore we do not have to associate $M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2}$ with $M_{\lambda_3 \lambda_4 \lambda_5; -\lambda_1 - \lambda_2}$ to regularize.

**) We recall that we have assumed parity conservation.

***) If particle i is massless $(-1)^{s_i + \lambda_i}$ has to be replaced by $(-1)^{|\lambda_i| + \lambda_i}$.

From formulae (3.3) and (3.4) we deduce

$$H_{\{\lambda_i\}}^i(-\sqrt{s-m_2^2}) = (-1)^{2s_2+2s_3} H_{\{\lambda_i\}}^i(\sqrt{s-m_2^2})$$

and

$$H_{\{\lambda_i\}}^i \simeq \sqrt{s-m_2^2}^{-2[s_2 + \text{Max}(|\lambda_3|, |\lambda_1 - \lambda_2|)]} \quad (3.7)$$

To summarize the functions

$$\sqrt{s-m_2^2}^{2m_{12}} H_{\{\lambda_i\}}^i \quad \text{where } m_{12} = s_2 + \text{Max}(|\lambda_3|, |\lambda_1 - \lambda_2|) \quad (3.8)$$

are regular at $s=m_2^2$.

As far as the constraints are concerned, we define as in Ref. 5) the following "transversity" amplitudes

$$T_{\{\tau_i\}; \lambda_1 \tau_2} = D^{s_1}(R)^{\lambda_1}_{\tau_1} D^{s_2}(R^*)^{\lambda_2}_{\tau_2} D^{s_3}(R^*)^{\lambda_3}_{\tau_3} D^{s_4}(R^*)^{\lambda_4}_{\tau_4} D^{s_5}(R^*)^{\lambda_5}_{\tau_5} M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2} \quad (3.9)$$

which behaves as $e^{i\theta_{13}\tau_2}$ near $s=m_2^2$. Thus

$$T_{\{\tau_i\}; \lambda_1 \tau_2} \simeq (s-m_2^2)^{\pm \tau_2}$$

if

$$\text{tg } \theta_{13} = \mp i$$

near $s=m_2^2$ and the constraints read

$$(s-m_2^2)^{s_2} T_{\{\tau_i\}; \lambda_1 \tau_2} \simeq (s-m_2^2)^{s_2 \pm \tau_2}$$

$$\text{if } \text{tg } \theta_{13} = \mp i \quad \text{near } s=m_2^2 \quad *) \quad (3.10)$$

D.2 $m_i=0, i=3,4,5$

We need a special study at $P.p_4=0$ since, if $m_4=0$

$$D_2^+(p, p_4) = D_2^-(p, p_4) = P.p_4 \quad (\Delta_2(p, p_4) = D_2^+(p, p_4) D_2^-(p, p_4))$$

[see Appendix A of Ref. 1)].

*) Actually, as shown in Ref. 5) we have to regularize $T_{\{\tau_i\}; \tau_2 \lambda_1}$ at $\Delta_4(P, p_1, p_3, p_4)=0$, $\Delta_3(P, p_4, p_5)=0$, $\Delta_2(P, p_4)=0$ and $\Delta_2(P, p_5)=0$ in order to have an analytic form for these constraint equations.

In the centre-of-mass frame R_2 the spinor amplitudes $m_{\{Ai\}}^{\pm}$ are regular at $P \cdot p_4 = 0$. So are the $L(i)$'s $i=1,2,3,5$. From formula (2.9) we deduce

$$M_{\{2i\}} = D^{|\lambda_4|} (H_4^{\pm})_{\lambda_4} C_{A_4}^{\pm}$$

where $C_{A_4}^{\pm}$ is regular at $P \cdot p_4 = 0$.

$$H_4^{+} = \frac{m_3}{2(p_3 \cdot p_4)} L(4) \varepsilon^{-1} \quad \text{if } \lambda_4 > 0$$

$$H_4^{-} = \frac{\sigma p_3}{2(p_3 \cdot p_4)} L^{\dagger}(4) \varepsilon \quad \text{if } \lambda_4 < 0$$

In the R_2 frame, $L(4) = R(4)B(4)$ where

$$B(4) = \begin{pmatrix} \frac{1}{\sqrt{|p_4|}} & 0 \\ 0 & \sqrt{|p_4|} \end{pmatrix} \quad |\vec{p}_4| = \frac{P \cdot p_4}{\sqrt{s}}$$

and $R(4)$ is the rotation of Euler angle $(-\pi, \theta_4, \pi)$ where θ_4 is the angle of the pure rotation around $\vec{p}_3 \times \vec{p}_4$ which takes $\hat{p}_3$ onto $\hat{p}_4$ ($\sin \theta_4$ and $\cos \theta_4 \approx 1/(P \cdot p_4)$).

It is easy to check that $L(4)$ is regular at $P \cdot p_4 = 0$. Thus $M_{\{2i\}}$ is regular at $P \cdot p_4 = 0$. Now we have to regularize the $H_{\{2i\}}^{\pm}$ as defined in formulae (3.5) and (3.6). Noticing that $\sin \theta_{45}$ and $\cos \theta_{45} \approx (P \cdot p_4)^{-1}$ near $P \cdot p_4 = 0$. We deduce at once

$$H_{\{2i\}}^{\pm} (-\sqrt{P \cdot p_4}) = (-1)^{2S_5} H_{\{2i\}}^{\pm} (\sqrt{P \cdot p_4})$$

$$H_{\{2i\}}^{\pm} \approx (P \cdot p_4)^{-\text{Max}(|\lambda_4|, |\lambda_5|)} \quad \text{near } P \cdot p_4 = 0 \quad (3.11)$$

and

$$(P \cdot p_4)^{\text{Max}(|\lambda_4|, |\lambda_5|)} H_{\{2i\}}^{\pm} \text{ is regular at } P \cdot p_4 = 0$$

Obviously the same kind of proof holds if particle 5 is massless and particle 4 massive or massless *)

$$H_{\{2i\}}^{\pm} (-\sqrt{P \cdot p_5}) = (-1)^{2S_4} H_{\{2i\}}^{\pm} (\sqrt{P \cdot p_5})$$

$$H_{\{2i\}}^{\pm} \approx (P \cdot p_5)^{-\text{Max}(|\lambda_4|, |\lambda_5|)} \quad \text{near } P \cdot p_5 = 0$$

(3.12)

*) The same result is valid if two other particles are massless.

and

$$(P, p_5)^{\text{Max}(|\lambda_4|, |\lambda_5|)} H_{\{\lambda_i\}}^i$$

is regular at $P \cdot p_5 = 0$. The same result applies for the singularity at $P \cdot p_3 = 0$ if $m_3 = 0$.

$$\begin{aligned} H_{\{\lambda_i\}}^i(-\sqrt{P \cdot p_3}) &= (-1)^{2J_1 + 2J_2} H_{\{\lambda_i\}}^i(\sqrt{P \cdot p_3}) \\ H_{\{\lambda_i\}}^i &\approx (P \cdot p_3)^{-\text{Max}\{|\lambda_3|, |\lambda_1 - \lambda_2|\}} \quad \text{near } P \cdot p_3 = 0 \end{aligned} \quad (3.13)$$

Since the $M_{\{\lambda_i\}}$ are regular at $P \cdot p_i = 0$ $i=4,5,3$, there is no kinematical constraint.

$$D.3 \quad m_1 = m_2 = 0, \quad m_3 \neq 0$$

In this case both the threshold and the pseudothreshold coincide with $s=0$. We choose a new reference frame R_3 which is regular at $s=0$ ($m_3 \neq 0$)

$$t = \frac{p_3}{m_3}, \quad n_2^\mu = \frac{\varepsilon^{\mu\nu\sigma} (p_3)_\nu (p_4)_\sigma (p_5)_\sigma}{\sqrt{\Delta_3(p_3, p_4, p_5)}}, \quad n_3^\mu = \frac{\begin{vmatrix} m_3^2 & p_3 \cdot p_4 \\ p_3^\mu & p_4^\mu \end{vmatrix}}{m_3 \sqrt{\Delta_2(p_3, p_4)}} \quad (3.14)$$

Then the spinor amplitudes are regular at $s=0$ and so are $L(4)$ and $L(5)$. In the Appendix we show that $D^{|\lambda_i|}_{(L(i))} \lambda_i$ behaves like $\sqrt{s} |\lambda_i|$ in the vicinity of $s=0$ ($i=1,2$).

Furthermore, the laboratory frame of particle 3, $L(3)$ reduces to a rotation $R(0,0,\varphi)$ where $\cos \varphi$ and $\sin \varphi$ behaves as $1/\sqrt{s}$ near $s=0$ ^{*)}. Then

$$D^{s_3}(L(3)\varepsilon) A_{\lambda_3}^3 \approx e^{i\lambda_3\varphi}$$

and

$$M_{\{\lambda_i\}} \approx \sqrt{s} (|\lambda_1| + |\lambda_2| + \varepsilon \lambda_3) \quad (\varepsilon = \pm 1)$$

$$\text{if } \text{tg } \varphi = \pm i \quad \text{near } s=0$$

(3.15)

Now we have to regularize at $s=0$ the $H^i_{\{\lambda_i\}}$. For this configuration of mass ($m_1 = m_2 = 0$, $m_i \neq 0$ $i=3,4,5$) the angle θ_{13} is regular at $s=0$. But $\sin \theta_{45}$ exhibits a $\sqrt{s}$ singularity which means that either $\sin \theta_{45}/2$ or $\cos \theta_{45}/2$ has a $\sqrt{s}$ singularity.

*) The explicit form of θ and φ is given in the Appendix.

Then, $H^i_{\{\lambda_i\}}$ which is a linear combination of $M_{\{\lambda_i\}}$ and $M_{\{\lambda_i\}}$ behaves as

$$\sqrt{s} \left| \lambda_1 + |\lambda_2| - |\lambda_3| + |\lambda_4 + \varepsilon' \lambda_5| \right|$$

near $s=0$ where $\varepsilon' = +1$ if $\sin \theta_{45}/2 \simeq \sqrt{s}$, $\cos \theta_{45} = +1$ at $s=0$, $\varepsilon' = -1$ if $\cos \theta_{45}/2 \simeq \sqrt{s}$, $\cos \theta_{45} = -1$ at $s=0$. We get rid of this ε' ambiguity once we have regularized $H^i_{\{\lambda_i\}}$ both at $\Delta_2(P, p_4) = 0$ and $\Delta_2(P, p_5) = 0$.

The $F^i_{\{\lambda_i\}}$ which are regularized at all the other singularities but $s=0$ behaves near $s=0$ as

$$F^i_{\{\lambda_i\}} \simeq \sqrt{s} \left| \lambda_1 + |\lambda_2| - |\lambda_3| + \min \{ |\lambda_4 - \lambda_5|, |\lambda_4 + \lambda_5| \} \right| \alpha^i_{\{\lambda_i\}} \quad (3.16)$$

where $\alpha^i_{\{\lambda_i\}}$ is analytic at $s=0$ if both particles 3 and 4 are bosons. A $\sqrt{s}$ singularity cannot be avoided in the other cases.

D.4 Case $m_1 = m_2 = m_3 = 0$

For the sake of completeness we will treat this case.

In the laboratory frame of particle 4 (or 5 if $m_4 = 0$), the spinor amplitudes are regular at $s=0$ and so are the $L(i)$ $i=4,5$. A short calculation in the Appendix shows that as before $L(1)$ and $L(2)$ behave as $\sqrt{s}$ whereas $L(3)$ behaves as $\sqrt{s}^{-1}$.

Then,

$$M_{\{\lambda_i\}} \simeq \sqrt{s} \left| \lambda_1 + |\lambda_2| - |\lambda_3| \right| \quad (3.17)$$

The same procedure as before leads to the behaviour near $s=0$ of the $F^i_{\{\lambda_i\}}$ [see formula (3.15)]. Now $\alpha^i_{\{\lambda_i\}}$ is analytic if particle 4 is a boson.

4. SUMMARY AND CONCLUSIONS

Using Raynal's method and taking into account gauge conditions, we have extended to the massless case results previously obtained for massive particles ⁵⁾; namely, we have given an explicit way to free the helicity amplitudes of kinematical singularities and to give the kinematical constraints. However, we want to recall i) that all these results are valid if we assume parity conservation in order to get rid of the singularity on the boundary of the physical region. The neutrino case, for instance, is not treated, but we have given the correct tool to introduce spin complications in the photon case; ii) the method does not apply to the very academic case

of 5 massless particles since the spinor amplitudes have no longer the correct analytic properties; iii) in some cases, due to our regularization, an extra $\sqrt{s}$ singularity cannot be removed.

It may be noticed that, for the massless particles, the singularity at $\Delta_2(P, p_i) = 0$ factorizes out explicitly (we do not have to combine $M \{\lambda_k\}; \lambda_i$ and $M \{\lambda_k\}, -\lambda_i$). Thus, the number of amplitudes we have to combine is less than in the massive case.

Applications of this work are quite the same as in the massive particles case. It gives a correct tool to introduce spin in phenomenological models, multi-Regge, Veneziano B_5 function of multiparticle processes involving photons.

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A P P E N D I X

In this Appendix we show that in the case $m_1 = m_2 = 0$

$$i) \quad D^{|\lambda_i|} (L(i))^{L_i} \lambda_i \approx \sqrt{s}^{|\lambda_i|} \quad \text{near } s=0 \quad i=1,2$$

$$ii) \quad D^{s_3} (L(3))^{A_3} \lambda_3 \approx \sqrt{s}^{\pm \lambda_3} \left(\text{resp } \sqrt{s}^{-|\lambda_3|} \right) \text{ if } m_3 \neq 0 \text{ (resp } m_3=0)$$

A. Behaviour of $D^{|\lambda_i|} (L(i))^{A_i} \lambda_i$ near $s=0$

Moussa and Stora ⁹⁾ have shown that it is possible to write the Lorentz transformation $L(i)$ in the following form

$$m_1=0 \quad L(1) = \Omega(q_{12} \leftarrow n_3(P)) [\omega_1 \hat{P} + |q_{12}| n_3(P) \leftarrow \hat{P} + n_3(P)] [P]$$

$$m_2=0 \quad L(2) = \Omega(q_{12} \leftarrow n_3(P)) [\omega_2 \hat{P} - |q_{12}| n_3(P) \leftarrow \hat{P} - n_3(P)] [P] \gamma$$

(A.1)

where $[P]$ is an arbitrary Lorentz transformation which takes t onto $P/\sqrt{s}$ ($P=p_1+p_2$) ω_1 is the massless particle barycentric energy, q_{12} is the four-vector in the two plane, p_1, p_2 orthogonal to P ($q_{12} \cdot P=0$ $q_{12}^2=-1$) $[\omega_1 \hat{P} \pm |q_{12}| n_3(P) \leftarrow \hat{P} \pm n_3(P)]$ is the pure Lorentz transformation acting in the two-plane ($\hat{P}, n_3(P) = [P] n_3$) and transforming $\hat{P} \pm n_3(P)$ (lightlike) into $\omega_1 \hat{P} \pm |q_{12}| n_3(P)$. $\Omega(q_{12} \leftarrow n_3(P))$ is a Lorentz transformation taking $n_3(P)$ onto q_{12} in the two-plane ($n_3(P), q_{12}$). is a rotation through an angle π around the n_2 axis.

From (A.1) we deduce

$$L(1) \begin{bmatrix} t \\ n_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} ch \chi \hat{P} + sh \chi \hat{q}_{12} \\ sh \chi \hat{P} + ch \chi \hat{q}_{12} \end{bmatrix}$$

where

$$\omega = \frac{\sqrt{s}}{2} = c \chi \quad \hat{q}_{12} = \frac{p_1 - p_2}{2\omega} \quad \hat{P} = \frac{p_1 + p_2}{2\omega}$$

Then

$$L(1) \begin{bmatrix} t \\ n_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} p_1 + \frac{p_2}{\omega^2} \\ p_1 - \frac{p_2}{\omega^2} \end{bmatrix} = \begin{bmatrix} n_0(1) \\ n_3(1) \end{bmatrix}$$

and $L(1)$ can be written down ¹⁰⁾

$$N L(1) = \sigma n_0(1) \tilde{\sigma} t - \sigma n_3(1) \tilde{\sigma} n_3 - \sigma n_2(1) \tilde{\sigma} n_2 - \sigma n_1(1) \tilde{\sigma} n_1$$

where N is calculated in such a way that $\det L = +1$. In this case $n_0(1)$ and $n_3(1)$ behave like $1/\omega^2$ whereas $n_2(1)$ and $n_1(1)$ behave like $1/\omega$ [$\Delta_3(p, p_1, p_3) \simeq \sqrt{s}$]. Hence, apparently N^2 behaves like $1/\omega^4$, but [see Appendix of Ref. 1)] the $1/\omega^4$ term actually vanishes and N^2 behaves as $1/\omega^3$.

As far as the numerator is concerned, we notice that $\sigma n_0(1) \tilde{\sigma} t - \sigma n_3(1) \tilde{\sigma} n_3 = \frac{1}{2} \sigma p_1 \tilde{\sigma} (t - n_3) + 1/(2\omega^2) \sigma p_2 \tilde{\sigma} (t + n_3)$ but

$$\mathcal{D}^{|\lambda_1|} (\tilde{\sigma} (t + n_3))^{A_1}_{\lambda_1} \equiv 0 \quad \text{for } \lambda_1 = -|\lambda_1|$$

This proves that

$$\mathcal{D}^{|\lambda_1|} (L(1))^{A_1}_{-\lambda_1}$$

behaves as

$$\left(\frac{\frac{1}{\omega}}{\frac{1}{\omega^{3/2}}} \right)^{2|\lambda_1|} = \omega^{|\lambda_1|} = \left(\frac{\sqrt{s}}{2} \right)^{|\lambda_1|}$$

The same result is valid if λ_1 is positive since now

$$N L^{1/2}(1) = \frac{1}{2} \tilde{\sigma} p_1 \sigma (t - n_3) + \frac{1}{2\omega^2} \tilde{\sigma} p_2 \sigma (t + n_3) + \dots$$

but

$$\mathcal{D}^{|\lambda_1|} (\sigma (t + n_3))^{A_1}_{\lambda_1} \equiv 0 \quad \text{for } \lambda_1 = |\lambda_1|$$

Then

$$\mathcal{D}^{|\lambda_1|} (L^{1/2}(1))^{A_1}_{|\lambda_1|} \text{ behaves as } \sqrt{s}^{|\lambda_1|}$$

B. Case $m_3 \neq 0$

In the laboratory frame of particle 3 $L(3)$ is a rotation since by definition

$$L(3) \begin{Bmatrix} t \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} p_3/m_3 \\ n_2(3) \\ n_3(3) \end{Bmatrix}$$

where

$$t = p_3/m_3, \quad n_2^\mu = \frac{\varepsilon^{\mu\nu\sigma} p_\nu(p_4)_\sigma(p_5)_\sigma}{\sqrt{\Delta_3(p, p_4, p_5)}} \quad n_3^\mu = \frac{\begin{vmatrix} m_3^2 & p_3 \cdot p_4 \\ p_3^\mu & p_4^\mu \end{vmatrix}}{m_3 \sqrt{\Delta_2(p_3, p_4)}}$$

and

$$\eta_2^\mu(3) = \frac{\varepsilon^{\mu\nu\sigma\tau} p_\nu(p_1) p_\sigma(p_3) \tau}{\sqrt{\Delta_3(p, p_1, p_3)}}, \quad \eta_3^\mu(3) = \frac{\begin{vmatrix} p^2 & p \cdot p_3 \\ p_3^\mu & p^\mu \end{vmatrix}}{m_3 \sqrt{\Delta_2(p, p_3)}}$$

Then $L(3)$ reduces to the rotation of Euler angle $(0, \theta, \varphi)$ where

$$\cos \theta = -\eta_3 \cdot p_3(3)$$

$$\cos \varphi = -\eta_2 \cdot \eta_3(3)$$

$$\cos \theta = \frac{\begin{vmatrix} p_3 & p_4 \\ p_3 & p \end{vmatrix}}{\sqrt{\Delta_2(p, p_3)} \sqrt{\Delta_2(p_4, p_3)}}$$

$$\sin \theta = \frac{m_3 \sqrt{\Delta_3(p, p_3, p_4)}}{\sqrt{\Delta_2(p, p_3)} \sqrt{\Delta_2(p_4, p_3)}}$$

$$\cos \varphi = \frac{\begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_3 \end{vmatrix}}{\sqrt{\Delta_3(p, p_4, p_5)} \sqrt{\Delta_3(p, p_1, p_3)}}$$

$$\sin \varphi = \frac{\varepsilon(p, p_1, p_3, p_4) \sqrt{\Delta_2(p, p_3)}}{\sqrt{\Delta_3(p, p_4, p_5)} \sqrt{\Delta_3(p, p_1, p_3)}}$$

near $s=0$.

$$\Delta_3(p, p_1, p_3) = \frac{s}{2} \left[(p \cdot p_3)^2 + (p \cdot p_3)(p_1 \cdot p_3) - 2(p_1 \cdot p_3)^2 - m_3^2 \frac{s}{2} \right] \approx s$$

Then θ is regular at $s=0$ but

$$\begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \approx \frac{1}{\sqrt{s}}$$

near $s=0$, then

$$e^{i\varphi} \approx \sqrt{s}^{\pm 1}$$

and

$$D^{s_3}(R(0, \theta, \varphi))_{- \lambda_3}^{A_3} \approx e^{i \lambda_3 \varphi} \approx \sqrt{s}^{\pm \lambda_3} \quad \text{if } \operatorname{tg} \varphi = \pm i$$

near $s=0$.

C. Case $m_3=0$ ($m_1=m_2=0$)

Following Moussa-Stora ⁹⁾, we can write

$$L(3) = \Omega(\eta_3(p) \rightarrow \hat{q}_3) [\omega_3(\hat{p} + \eta_3(p)) \leftarrow \hat{p} + \eta_3(p)] [P]$$

where

$$\hat{q}_3 = p_3/\omega_3 - \hat{p} \quad (\hat{q}_3 \cdot P = 0 \quad \hat{q}_3^2 = -1)$$

and $\omega_3 = \frac{P \cdot p_3}{\sqrt{s}}$

Then

$$L(3) \begin{Bmatrix} t \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \hat{p}/\omega_3 + \frac{1}{2} (1 - 1/\omega_3^2) p_3 \\ -\hat{p}/\omega_3 + \frac{1}{2} (1 + 1/\omega_3^2) p_3 \end{Bmatrix}$$

which means that

$$n_3(3) = -(P \cdot p_3)^{-1} P + \frac{1}{2} (1 + P^2/(P \cdot p_3)^2) p_3$$

$$n_0(3) = (P \cdot p_3)^{-1} P + \frac{1}{2} (1 - P^2/(P \cdot p_3)^2) p_3$$

Then $n_0(3)$ and $n_3(3)$ are analytic near $P^2 = s = 0$. On the other hand $n_2(3)$ behaves as $1/\sqrt{s}$ ($\Delta_3(P, p_1, p_3) \simeq s$) and so does $n_1(3)$. By exactly the same proof of Ref. 1) (see Appendix C), we deduce that $L(3) \simeq s^{-1/4}$ and $D|\lambda_3|(L(3))^{A_3} \lambda_3 \simeq \sqrt{s} - |\lambda_3|$.

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