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CM-P00058796

KINEMATICAL SINGULARITIES AND CONSTRAINTS OF
HELICITY AMPLITUDES FOR FIVE-PARTICLE PROCESSES

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A B S T R A C T

We have extended to the five - particle processes involving massive particles of any spin, Raynal's method of getting the kinematical singularities of the helicity amplitudes. This procedure allows to build regularized helicity amplitudes. Constraints between these amplitudes have been explicitly derived.

Ref.TH.1295-CERN

16 March 1971

I. INTRODUCTION

The interest in the $2 \rightarrow n$ ($n \geq 3$) multiparticle processes has been increased by the success of the phenomenological study of multiparticle reaction amplitudes (for instance, the multi-Regge formalism ¹⁾, the models using N point Veneziano amplitudes ²⁾, ...). In general, the spin effects have been neglected but they are presumably rather important. For instance, in the $K^- p \rightarrow \Lambda \pi^+ \pi^-$ case, the authors ³⁾ have considered only one amplitude instead of four.

The main purpose of this work is to study the kinematical singularities and the kinematical constraints between helicity amplitudes in the centre-of-mass for the case of three outgoing particles. (All the particles are massive.)

Indeed for $n > 3$, the relations between all the scalar products are no longer linear since in this case all the 5×5 Gram determinants of the four-vector p_i , $i = 1 \dots N$ $N > 5$, must vanish ⁴⁾. Then it is hopeless to find good bases even in the spinless case.

Svensson ⁵⁾, by using the Trueman ⁶⁾ method and more recently Johannesson ⁷⁾ by a quite similar method using the analytical properties of spinor amplitudes, have already worked on the multiparticle processes. The singularities have been classified and explicitly isolated by giving the expansion of the helicity amplitudes near each singularity surface. But they have not built regularized helicity amplitudes and derived explicitly the kinematical constraints between them.

Since the crossing matrix has been extensively studied by, for instance, Capella ⁸⁾ and Chih Kwan Chen and Kuo Hsian Wang ⁹⁾ we have not included the crossing in our work.

We have used all along this work Raynal's method ¹⁰⁾. Indeed, in the five-body case an expansion such as the Joos ¹¹⁾⁻¹³⁾ expansion of spinor amplitudes in the four-body case does not

generally exist, and the method of Cohen-Tannoudji, Morel and Navelet ¹⁴⁾ does not apply. The method to find the kinematical singularity of the helicity amplitudes consists in writing them as linear combinations of the spinor amplitudes $\mathcal{M}_{A_i}(p_i)$ namely

$$M_{\lambda_i} = C_{\lambda_i}^{A_i} \mathcal{M}_{A_i}.$$

The $\mathcal{M}_{A_i}(p_i)$ are analytic functions in the set of the four-momentum components restricted to the mass shell and obey the covariance property. Their only singularities originate in dynamics. For a given singularity of the $C_{\lambda_i}^{A_i}$'s, it is always possible, apart from the boundary of the physical region ^{*}), to choose a reference frame such that the $\mathcal{M}_{A_i}$ are analytic functions of the five independent invariants in the vicinity of this singularity. Then all the kinematical singularities are those of the $C_{\lambda_i}^{A_i}$'s.

Section II is devoted to the search of the kinematical singularities of the helicity amplitudes and to an explicit building of the regularized helicity amplitudes. (Technical details are given in Appendices A, B and C.)

In Section III, we show how to obtain the kinematical constraints between these regularized helicity amplitudes.

All our results are explicitly tested (in Appendix D) on the few cases in which the relations between invariant and helicity amplitudes are known.

^{*}) To get rid of this singularity which appears even in the spinless case, we have assumed that parity conservation is valid for these multiparticle processes.

II. KINEMATICAL SINGULARITIES OF TWO-BODY → THREE BODY HELICITY AMPLITUDES

1. Notations and Conventions

A. Spinor Amplitudes

Using the same notation as C-T.M.N. ¹⁴⁾, we recall briefly the definition of spinor states $|p, s, A\rangle$ (resp $|p, s, \dot{A}\rangle$) of a particle of mass m , spin s and four-momentum p .

By definition these states transform under a Lorentz transformation Λ by $U(\Lambda)|p, s, A\rangle = D^s(\Lambda)_A^{A'} |\Lambda p, s, A'\rangle$ resp $(U(\Lambda)|p, s, \dot{A}\rangle = D^s(\Lambda^*)_{\dot{A}}^{\dot{A}'} |\Lambda p, s, \dot{A}'\rangle$ where Λ^* is the 2×2 matrix complex conjugate of Λ .

From these states we define the spinor amplitudes $\mathcal{M}_{A_3 A_4 A_5; A_1 A_2}(p_3 p_4 p_5; p_1 p_2)$ for the processus $1+2 \rightarrow 3+4+5$ where particle i has spin s_i mass m_i and four-momentum p_i

$$\mathcal{M}_{A_3 A_4 A_5; A_1 A_2}(p_3 p_4 p_5; p_1 p_2) = \langle p_3 s_3 \dot{A}_3 p_4 s_4 \dot{A}_4 p_5 s_5 \dot{A}_5 / T / p_1 s_1 A_1 p_2 s_2 A_2 \rangle$$

where T is defined from the S matrix by

$$S = 1 + i (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4 - p_5) T$$

Now, the Lorentz invariance of the interaction is translated into the following covariance formula.

$$\mathcal{M}_{A_3 A_4 A_5; A_1 A_2}(p_3 p_4 p_5; p_1 p_2) = D^{s_1}(\Lambda)_{A_1}^{A'_1} D^{s_2}(\Lambda)_{A_2}^{A'_2} D^{s_3}(\Lambda)_{A_3}^{A'_3} \times \\ \times D^{s_4}(\Lambda)_{A_4}^{A'_4} D^{s_5}(\Lambda)_{A_5}^{A'_5} \mathcal{M}_{A'_5 A'_4 A'_3; A'_1 A'_2}(p'_5 p'_4 p'_3; p'_1 p'_2)$$

(II.1)

for any Lorentz transformation Λ .

The main advantage of these spinor amplitudes is that they possess analyticity properties, some of which can be derived from axiomatic field theory where amplitudes emerge as Fourier transforms of vacuum expectation values of time ordered products of field operators.

The spinor amplitudes are distributions defined in a set E of dimension $3n-4=11$ (p_i real ($i=1,2,3,4,5$) $p_i^0 > 0$, $p_i^2 = m_i^2$ and $p_1+p_2=p_3+p_4+p_5$ ¹²⁾) and is a boundary value of an holomorphic function defined in a domain $\bar{E}$ (p_i complex, $p_i^2 = m_i^2$ and $p_1+p_2=p_3+p_4+p_5$).

B. Helicity States and Amplitudes in the Centre-of-mass Frame

As usual ¹⁵⁾, we attach to each p_i three space-like vectors $n_j(p_i)$ $j=1,2,3$ for the helicity in the centre-of-mass, $n_3(p_i)$ is chosen in the two-plane P , p_i where $P=p_1+p_2=p_3+p_4+p_5$

$$n_3^\mu(p_i) = - \frac{\begin{vmatrix} m_i^2 & P \cdot p_i \\ p_i^\mu & P^\mu \end{vmatrix}}{m_i \sqrt{\Delta_2(P, p_i)}} \quad (\text{II.2})$$

where $\Delta_n(p_1, p_2, \dots, p_n)$ is the Gram determinant $\begin{vmatrix} p_1 p_2 \dots p_n \\ p_1 p_2 \dots p_n \end{vmatrix}$ up to a sign $(-1)^n$ (see Appendix A).

Since the sum of spin $s_1+s_2+s_3+s_4+s_5$ is an integer, we can always choose particle 3 such that $s_1+s_2+s_3$ is an integer. Then, for particles 1,2,3 we have defined

$$n_i^\mu(p_i) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_i) p_\rho(p_2) p_\sigma(p_3)}{\sqrt{\Delta_3(P, p_i, p_2, p_3)}} \quad i = 1, 2, 3 \quad (\text{II.3})$$

and for particles 4 and 5

$$n_i^\mu(p_i) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_i) p_\rho(p_4) p_\sigma(p_5)}{\sqrt{\Delta_3(P, p_i, p_4, p_5)}} \quad i = 4, 5 \quad (\text{II.4})$$

The interest of this choice will appear very soon ^{*)}.

The third space-like vector $n_1(p_i)$ is given by

$$n_1^\mu(p_i) = \varepsilon^{\mu\nu\rho\sigma} n_2(p_i)_\nu n_3(p_i)_\rho \frac{(p_i)_\sigma}{m_i}$$

Let now $(\overset{0}{t}, \overset{0}{n}_1, \overset{0}{n}_2, \overset{0}{n}_3)$ be the standard frame and $L(p_i)$ be a Lorentz transformation obeying

$$L(p_i) \begin{Bmatrix} \overset{0}{t} \\ \overset{0}{n}_1 \\ \overset{0}{n}_2 \\ \overset{0}{n}_3 \end{Bmatrix} = \begin{Bmatrix} p_i/m_i \\ n_1(p_i) \\ n_2(p_i) \\ n_3(p_i) \end{Bmatrix}$$

Then, the helicity state in the centre-of-mass reads

$$|p_i, L(p_i), s_i, \lambda_i\rangle = D^{s_i}(L(p_i))^{A_i}_{\lambda_i} |p_i, s_i, A_i\rangle$$

$$\text{and } \langle p_i, L(p_i), s_i, \lambda_i | = D^{s_i}(L(p_i)\varepsilon)^{A_i}_{\lambda_i} \langle p_i, s_i, A_i |$$

where $\varepsilon = i\sigma_2$ and where λ_i is any eigenvalue of the operator

$$- \frac{1}{m_i} W \cdot n_3(p_i)$$

(W is the polarization four-vector of the particle i.)

^{*)} With this convention particles 4 and 5 have the same 2-axis whereas in Svensson's definition the 2-axis of particle i ($i=4,5$) is

$$n_i^\mu(p_i) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_i)_\rho (p_i)_\sigma}{\sqrt{\Delta_3(p, p_4, p_5)}}$$

We can now express the helicity amplitudes in terms of spinor amplitudes, namely

$$M_{\lambda_3 \lambda_4 \lambda_5; \lambda, \lambda_2} = D^{s_3}(L(p_3)\varepsilon)^{A_3}_{\lambda_3} D^{s_4}(L(p_4)\varepsilon)^{A_4}_{\lambda_4} D^{s_5}(L(p_5)\varepsilon)^{A_5}_{\lambda_5} \times \\ \times M_{A_3 A_4 A_5, A_1 A_2}(p_3 p_4 p_5; p_1 p_2) D^{s_1}(L(p_1))^{A_1}_{\lambda_1} D^{s_2}(L(p_2))^{A_2}_{\lambda_2}$$

(II.5)

2. Outline of the Method

A. Choice of the Analytic Manifold in the Space of Invariants

Following Jacobson¹⁶⁾, we consider for a reaction involving n massive particles $(\sum_i \varepsilon_i m_i \neq 0 \quad \varepsilon_i = \pm 1)$ ¹⁷⁾, the $(n-1) \times (n-1)$ symmetric matrix Z having as elements all the scalar products of the four vectors $p_1, p_1, \dots, p_{n-1}$ (p_n is eliminated using energy-momentum conservation). The elements of Z are restricted by the relation $p_n^2 = m_n^2$, namely $f_0(Z) \equiv (p_1 + p_2 - p_3 - \dots - p_{n-1})^2 - m_n^2 = 0$.

Since the vector space has dimension 4, the rank of Z is at most 4. For complex values of the vectors, Z may be thought as a point in a $N = \frac{1}{2}(n-1)(n-2)$ dimensional complex space E_c^N . It was proved by Jacobson that the set

$$C_N = \{ Z \in E_c^N, f_0(Z) = 0, \text{rank } Z = 4 \} \quad (n \geq 6)$$

(II.6)

is an analytic manifold of dimension $3n-10$. In the following we shall assume the domain (II.6) as the analytic manifold

[For the case $n=4,5$ the set C_N is defined by

$$\{ Z \in E_c^N, f_0(Z) = 0, \text{rank } Z \leq 4 \}.$$

B. Raynal's Method ¹⁰⁾

We briefly recall the principle of Raynal's method.

Let us call π the projection

$$p_i \in E \xrightarrow{\pi} \begin{aligned} s &= (p_1 + p_2)^2 \\ t_1 &= 2p_1 \cdot p_3 \\ t_2 &= 2p_1 \cdot p_4 \\ t_3 &= 2p_3 \cdot p_5 \\ t_4 &= 2p_2 \cdot p_4 \end{aligned}$$

The helicity amplitude $M_{\lambda_3 \lambda_4 \lambda_5; \lambda_1 \lambda_2}$ is a distribution defined in the domain $\pi(E)$ of dimension 5.

Let us assume that $\mathcal{M}_{A_i}(p_j)$ is analytical in a neighbourhood of a point $\overset{\circ}{p}_j$; if we choose a function

$$\text{space of invariants} \xrightarrow{f_i} \text{space of momenta}$$

such that

$$\pi \circ f_i = \text{identity} \quad \text{and} \quad f_i \circ \pi(\overset{\circ}{p}_j) = \overset{\circ}{p}_i$$

(II.7)

then Eq. (II.5) can be written

$$M_{\{\lambda_i\}}(s, t_k) = \sum_{A_i} \pi_i \mathcal{D}^{s_i} [(L(i, f_j(s, t_k)) \varepsilon_i)]^{A_i}_{\lambda_i} \mathcal{M}_{A_i}(f_j(s, t_k))$$

$$\begin{aligned} \varepsilon_i &= 1 \quad \text{if } i=1, 2 \\ &= i\sigma_i \quad \text{if } i=3, 4, 5 \end{aligned}$$

(II.8)

If f_j is chosen analytical in a neighbourhood of $\pi(\overset{\circ}{p}_i)$, $\mathcal{M}_{A_i}(f_j(s, t_k))$ is also an analytical function of (s, t_k) in a neighbourhood of $\pi(\overset{\circ}{p}_i)$ and singularities of kinematical type are contained in the functions

$$\mathcal{D}^{s_i} [(L(i, f_j(s, t_k)) \varepsilon_i)]^{A_i}_{\lambda_i} \text{ namely } \begin{cases} \Delta_4(p, p_1, p_3, p_4) = 0 \\ \Delta_3(p, p_1, p_3) = 0 \\ \Delta_3(p, p_4, p_5) = 0 \\ \Delta_2(p, p_i) = 0 \quad i=1, 3, 4, 5 \end{cases}$$

*) With this choice of invariants, $2p_3 p_4 + s + m_3^2 + m_4^2 - (t_1 + t_2 + t_3 + t_4) = m_5^2$.

Furthermore, one can here reverse the relation (II.8) and get

$$M_{Ai}(g_j(s, t_R)) = \sum_{\lambda_i} \pi_i D^{j_i}(\varepsilon_i L^{-1}(i, g_j(s, t_R)))^{\lambda_i} M_{\lambda_i}(s, t_R) \quad (II.9)$$

To accomplish such a program, it is still necessary that a function g satisfying the conditions (II.7), analytical in a neighbourhood of $\pi(p_i^0)$ exists for every point p_i^0 . The condition for that is that the rank of the function π be maximum for every point p_i^0 : but the rank falls for the points where

$$\varepsilon^{\mu\nu\sigma} I_\mu(p_1)_\nu(p_3)_\sigma(p_4)_\sigma = 0$$

Indeed, this difficulty arises when we have more than four particles in the reaction and cannot be avoided even in the scalar case where $M(p_3 p_4 p_5; p_1 p_2)$ can be written as ¹⁸⁾

$$M(p_3 p_4 p_5; p_1 p_2) = A_S(p_3 p_4 p_5; p_1 p_2) + \varepsilon^{\mu\nu\sigma} I_\mu(p_1)_\nu(p_3)_\sigma(p_4)_\sigma + A_{PS}(p_3 p_4 p_5; p_1 p_2)$$

But if parity conservation holds, this trouble can be overcome as we shall see later. (In the scalar case, for instance, either A_S or A_P does vanish if parity is conserved.)

For the sake of completeness we recall a theorem given by Hall and Wightman ¹⁹⁾ about the application π .

Theorem: Let p_i and p_i' be any two sets of n four-vector such that $X_{ij} = p_i p_j = p_i' p_j'$ ($i, j = 1, \dots, n$). Then, if the rank of the $n \times n$ matrix X is three or four (or for $n \leq 2$ if X is non-singular), then there exists a complex Lorentz transformation, Λ , such that

$$\Lambda p_i = p_i' \quad i = 1, 2, \dots, n-1, n \quad (II.10)$$

If the rank of X is two or one (and $n > 2$ or $n > 1$ respectively) a Λ satisfying (II.10) will not exist, in general, but there always exists a Λ satisfying $\Lambda p_i = p_i' + \alpha_i \omega$ where α_i are complex numbers and ω is a vector of zero length orthogonal to p_i' and Λp_i $i = 1, \dots, n-1, n$.

Taking $n=5$, there is a one by one correspondance between a point in the space of invariant and an orbit in the space of four-vector (namely a set of four-vector having the same invariants and which can be deduced from each other by a Lorentz transformation) if the Gram determinant $\Delta_4(P, p_1, p_3, p_4)$ is different from zero. The same result holds if $\Delta_4(P, p_1, p_3, p_4) = 0$ but if at least one Gram determinant $\Delta_3(P, p_j, p_k)$ is different from zero^{*)}. In these two situations the domain of analyticity of the spinor amplitudes is π saturated^{12), 13)}.

To summarize, the kinematical singularities lie for points where

- i) there exists no $\mathcal{P}_i(s, t_k)$ which is analytic, namely at $\Delta_4(P, p_1, p_3, p_4) = 0$;
- ii) the four vectors which defined the helicity tetrad of each particle are singular, namely at $\Delta_3(P, p_1, p_3) = 0$, $\Delta_3(P, p_4, p_5) = 0$ and $\Delta_2(P, p_i) = 0$ $i = 1, 3, 4, 5$.

Furthermore, since $n_3(p_i)$ changes sign when one changes the determination of $\Delta_2(P, p_i)$, one can guess that in order to free the helicity amplitudes from kinematical singularities it will be necessary to associate $M_{\{\lambda\}, \lambda_i}$ to $M_{\{\lambda\}, -\lambda_i}$.

*) In the case $n=4$, on the boundary of the physical region, $\Delta_3(p_1, p_2, p_3) = 0$ which means that the rank of the matrix is equal to 2, then, as noticed by Raynal there are three orbits corresponding to one point on this boundary and the application π is not saturated.

On the contrary since the singularities in Δ_3 affect only the second axis of the helicity tetrad, these singularities are expected to be factorized out.

To treat the case of one singularity of the ii) type, we have to find a set of $\mathcal{P}_i(s, t_k)$ which is analytic in the neighbourhood of this singularity. Since each component of p_i is the scalar product of p_i by one of the four vectors of a reference frame, it is enough to choose a reference frame, the four vectors of which are not singular. We shall extensively use the centre-of-mass frame since we study helicity amplitudes in the centre-of-mass. The peculiar case $m_1 = m_2$ will be treated separately.

3. Kinematical Singularities at $\Delta_4(P, p_1, p_3, p_4) = 0$

As quoted before, there does not exist any $\mathcal{P}_i(s, t_k)$ which is analytic at $\Delta_4(P, p_1, p_3, p_4) = 0$ which is the equation of the boundary of the physical region.

The product $\varepsilon^{\mu\nu\sigma\tau} p_\mu(p_1)_\nu (p_3)_\sigma (p_4)_\tau$ is a pseudo-scalar and its square is equal to $\Delta_4(P, p_1, p_3, p_4)$. Then, by parity, $M_{\{\lambda\}}(s, t_k, \sqrt{\Delta_4})$ will be related to $M_{\{-\lambda\}}(s, t_k, -\sqrt{\Delta_4})$, namely

$$M_{\{\lambda\}}(s, t_k, \sqrt{\Delta_4(P, p_1, p_3, p_4)}) = \eta_{\{\lambda\}} M_{\{-\lambda\}}(s, t_k, -\sqrt{\Delta_4(P, p_1, p_3, p_4)}) \quad (\text{II.11})$$

where

$$\eta_{\{\lambda\}} = \prod_{i=1}^5 \eta_i (-1)^{s_i + \lambda_i}$$

where η_i is the intrinsic parity of particle i .

If parity conservation does not hold it is impossible to make explicit, as in the four-body scattering, the singularity of the $m_{A_i}(\mathcal{P}_j(s, t_k))$ on the boundary of the physical region. Whereas, if parity conservation is valid, [see Eq. (II.11)], we define

two functions $F_{\{\lambda\}}^i$ ($i=1,2$) which are kinematically regular at $\Delta_4(P, p_1, p_3, p_4) = 0$, namely

$$\begin{aligned} F_{\{\lambda\}}^1 &= M_{\{\lambda\}} + \eta_{\{\lambda\}} M_{\{-\lambda\}} \\ F_{\{\lambda\}}^2 &= \frac{M_{\{\lambda\}} - \eta_{\{\lambda\}} M_{\{-\lambda\}}}{\sqrt{\Delta_4(P, p_1, p_3, p_4)}} \end{aligned} \quad (\text{II.12})$$

The $F_{\{\lambda\}}^i$ now obey the relations

$$F_{\{\lambda\}}^1(s, t_k) = \eta_{\{\lambda\}} F_{\{-\lambda\}}^1(s, t_k) \quad (\text{II.13})$$

and

$$F_{\{\lambda\}}^2(s, t_k) = -\eta_{\{\lambda\}} F_{\{-\lambda\}}^2(s, t_k)$$

4. Kinematical Singularities at $\Delta_3(P, p_1, p_3) = 0$ and $\Delta_3(P, p_4, p_5) = 0$

To treat these singularities we need two different centre-of mass frames R_1 and R_2 which are not singular at $\Delta_3(P, p_4, p_5) = \Delta_3(P, p_3, p_4)$ and $\Delta_3(P, p_1, p_3)$ respectively.

Frame R_1 $\hat{p}_1$ is chosen as 3-axis and $\vec{p}_1 \times \vec{p}_3$ as 2-axis. $t_0 = P/\sqrt{s}$,

$$n_2^\mu = \frac{\varepsilon^{\mu\nu\sigma} P_\nu (p_1)_\sigma (p_3)_\sigma}{\sqrt{\Delta_3(P, p_1, p_3)}}, \quad n_3^\mu = \frac{\begin{vmatrix} p^2 & p \cdot p_1 \\ p^\mu & p_1^\mu \end{vmatrix}}{\sqrt{s} \sqrt{\Delta_2(P, p_1)}}$$

and

$$n_1^\mu = - \frac{\begin{vmatrix} p^2 & p \cdot p_1 & p \cdot p_3 \\ p \cdot p_1 & p_1^2 & p_1 \cdot p_3 \\ p^\mu & p_1^\mu & p_3^\mu \end{vmatrix}}{\sqrt{\Delta_3(P, p_1, p_3)} \sqrt{\Delta_2(P, p_1)}}$$

(II.14)

In the physical region, $\sqrt{s}$, $\sqrt{\Delta_2(P, p_1)}$ and $\sqrt{\Delta_3(P, p_1, p_3)}$ are the positive determinations of the square-root.

Frame R_2 $\hat{\vec{p}}_3$ is chosen as 3-axis and $\hat{\vec{p}}_3 \times \vec{p}_4 = \vec{p}_4 \times \vec{p}_5$ as 2-axis.

$$\hat{\vec{e}}_0 = \frac{\vec{P}}{\sqrt{s}},$$

$$\hat{e}_2^\mu = \frac{\varepsilon^{\mu\nu\sigma} P_\nu (p_3)_\sigma (p_4)_\sigma}{\sqrt{\Delta_3(P, p_3, p_4)}}, \quad \hat{n}_3^\mu = \frac{\begin{vmatrix} p^2 & P \cdot p_3 \\ p^\mu & p_3^\mu \end{vmatrix}}{\sqrt{s} \sqrt{\Delta_2(P, p_3)}}$$

and

$$\hat{n}_1^\mu = - \frac{\begin{vmatrix} p^2 & P \cdot p_3 & P \cdot p_4 \\ P \cdot p_3 & p_3^2 & p_3 \cdot p_4 \\ p^\mu & p_3^\mu & p_4^\mu \end{vmatrix}}{\sqrt{\Delta_3(P, p_3, p_4)} \sqrt{\Delta_2(P, p_3)}}$$

(II.15)

A. Singularity at $\Delta_3(P, p_4, p_5) = 0 = \Delta_3(P, p_3, p_4)$

In the frame R_1 , $\mathcal{M}_{Ai}(\varphi_j(s, t_k))$ is analytic at $\Delta_3(P, p_4, p_5) = 0$. Furthermore, only the 2-axis of the helicity tetrad of particle 4 and 5 is singular at $\Delta_3(P, p_3, p_4) = \Delta_3(P, p_4, p_5) = 0$

$$(n_2^\mu(p_4) = n_2^\mu(p_5) = \frac{\varepsilon^{\mu\nu\sigma} P_\nu (p_4)_\sigma (p_5)_\sigma}{\sqrt{\Delta_3(P, p_4, p_5)}}).$$

Then, we can write

$$M_{\{\lambda_i\}} = D^{s_4}(L(p_4)\varepsilon)^{A_4}_{\lambda_4} D^{s_5}(L(p_5)\varepsilon)^{A_5}_{\lambda_5} \mathcal{G}_{A_4 A_5}(\varphi_j(s, t_k)) \quad (\text{II.16})$$

where $\mathcal{C}_{A_4 A_5}$ is analytic in the neighbourhood of $\Delta_3(P, p_4, p_5) = 0$

$$L(p_i) = R_i B\left(\frac{P}{\sqrt{s}} \rightarrow R_i^{-1} \frac{p_i}{m_i}\right) \quad i=4,5$$

B is the "boost" which takes the time-axis onto $(R_i^{-1} p_i)/m_i$. R_i is the rotation of Euler angle $(\varphi_i, \theta_i, \gamma_i)$ which takes the 2-axis $\widehat{p_1 \times p_3}$ onto $\widehat{p_4 \times p_5}$ and the 3-axis $\widehat{p_1}$ onto $\widehat{p_i}$.

θ_i and φ_i are the usual polar and azimuthal angle of particle i and γ_i is given (in Appendix B) by

$$\cos \gamma_i = \frac{\begin{vmatrix} P & p_4 & p_5 \\ P & p_1 & p_i \end{vmatrix}}{\sqrt{\Delta_3(P, p_4, p_5)} \sqrt{\Delta_3(P, p_1, p_i)}} \quad i=4,5$$

$$\sin \gamma_i = \frac{\varepsilon(P, p_1, p_3, p_4) \sqrt{\Delta_2(P, p_i)}}{\sqrt{\Delta_3(P, p_4, p_5)} \sqrt{\Delta_3(P, p_1, p_i)}}$$

(II.17)

where $\begin{vmatrix} p_a \dots p_n \\ q_a \dots q_n \end{vmatrix}$ is the Gram determinant of the n four-vectors p_i and q_j , and

$$\varepsilon(p, q, r, s) = \varepsilon^{\mu\nu\rho\sigma} p_\mu q_\nu r_\rho s_\sigma \quad (\varepsilon_{0123} = +1)$$

In the frame R_1 θ_i and φ_i are analytic as the reference frame four-vectors. Since only γ_i are singular we deduce from formula (II.16)

$$M_{\{\lambda_i\}} \approx e^{i(\lambda_4 \gamma_4 + \lambda_5 \gamma_5)} \mathcal{C}_{\{\lambda_i\}} \quad (II.18)$$

when $\mathcal{C}_{\lambda_i}$ is analytic at $\Delta_3(P, p_4, p_5) = 0$.

Actually the symmetric determinant $\Delta_3(P, p_4, p_5)$ can be factorized⁵⁾, namely $\Delta_3(P, p_4, p_5) = D_3^+(P, p_4, p_5) D_3^-(P, p_4, p_5)$ (see Appendix A), and in the Appendix B we prove that $\gamma_4 + \gamma_5$

(resp $\gamma_4 - \gamma_5$) is regular at $D_3^-(P, p_4, p_5) = 0$ [resp $D_3^+(P, p_4, p_5) = 0$].

From (II.18) we deduce that

$$M_{\{\lambda_i\}} \approx e^{i(\lambda_4 \pm \lambda_5) \frac{(\gamma_4 \pm \gamma_5)}{2}}$$

in the neighbourhood of $D_3^\pm(P, p_4, p_5) = 0$

(II.19)

But

$$e^{i \frac{(\gamma_4 \pm \gamma_5)}{2}} \approx \left(\sqrt{D_3^\pm(P, p_4, p_5)} \right)^2$$

where $\eta = \pm 1$, following the determination of $\mathcal{E}(P, p_1, p_3, p_4) = \sqrt{\Delta_4(P, p_1, p_3, p_4)}$ at $D_3^\pm(P, p_4, p_5) = 0$ (see Appendix A). Therefore,

$$M_{\{\lambda_i\}} \approx \left(\sqrt{D_3^\pm(P, p_4, p_5)} \right)^2 (\lambda_4 \pm \lambda_5)$$

(II.20)

To regularize at $\Delta_4(P, p_1, p_3, p_4) = 0$, we have to mix up two amplitudes $M_{\{\lambda_i\}}$ and $M_{\{-\lambda_i\}}$. Then,

$$F_{\{\lambda_i\}}^i \approx \frac{1}{\sqrt{D_3^\mp(P, p_4, p_5)}} |\lambda_4 \mp \lambda_5| \left\{ \alpha + \beta \left(\sqrt{D_3^\mp(P, p_4, p_5)} \right)^2 |\lambda_4 \mp \lambda_5| \right\}$$

(II.21)

where α and β are regular at $\Delta_3(P, p_4, p_5) = 0$.

If $\lambda_4 \pm \lambda_5$ were half integer, a $\sqrt{D_3^\pm(P, p_4, p_5)}$ singularity would not be avoided. Then, we see why we have chosen particles 4 and 5 such that $s_4 + s_5$ is an integer ^{*)}.

*) For the Svensson's helicity amplitudes, this singularity cannot be avoided for each final fermion i ($i = 4, 5, \dots, n$) once the regularization (II.12) at $\Delta_4(P, p_1, p_3, p_4) = 0$ has been done.

For convenience, we define θ_{45} as the angle of the pure rotation around $\vec{p}_4 \times \vec{p}_5$ which takes $\hat{p}_4$ into $\hat{p}_5$ in the centre-of-mass frame

$$\cos \theta_{45} = - \frac{\begin{vmatrix} P & p_4 \\ P & p_5 \end{vmatrix}}{\sqrt{\Delta_2(P, p_4)} \sqrt{\Delta_2(P, p_5)}}, \quad \sin \theta_{45} = \frac{\sqrt{s} \Delta_3(P, p_4, p_5)}{\sqrt{\Delta_2(P, p_4)} \sqrt{\Delta_2(P, p_5)}} \quad \text{for } 0 \leq \theta_{45} \leq \pi \quad (\text{II.22})$$

Then, by formula (A.5) of Appendix A,

$$\cos\left(\frac{\theta_{45}}{2}\right) \approx \sqrt{D_3^-(P, p_4, p_5)} \quad \sin\left(\frac{\theta_{45}}{2}\right) \approx \sqrt{D_3^+(P, p_4, p_5)}$$

and from formula (II.21) we deduce that

$$G_{\{\lambda_i\}}^i = \cos \frac{\theta_{45}}{2}^{|\lambda_4 - \lambda_5|} \sin \frac{\theta_{45}}{2}^{|\lambda_4 + \lambda_5|} F_{\{\lambda_i\}}^i \quad (\text{II.23})$$

is regular at $D_3^\pm(P, p_4, p_5) = 0$.

Note Formula (II.20) shows that constraints do exist at $\Delta_3(P, p_4, p_5) = 0$ since, following the determination of $\mathcal{E}(P, p_1, p_3, p_4) = \sqrt{\Delta_4(P, p_1, p_3, p_4)}$, helicity amplitudes may vanish as

$$(\sqrt{D_3^\pm(P, p_4, p_5)})^{|\lambda_4 \pm \lambda_5|}$$

B. Singularity at $\Delta_3(P, p_1, p_3) = 0$

In the frame R_2 , $\mathcal{M}_{A_1}(g_j(s, t_k))$ is analytic at $\Delta_3(P, p_1, p_3) = 0$. Furthermore, only the second axis of the helicity tetrad of particle 1, 2 and 3 is singular at $\Delta_3(P, p_1, p_3) = 0$

$$(n_2^\mu(p_1) = n_2^\mu(p_2) = n_2^\mu(p_3) = \frac{\varepsilon^{\mu\nu\sigma\tau} p_\nu(p_1) g_\sigma(p_3) \sigma}{\sqrt{\Delta_3(P, p_1, p_3)}})$$

Then, we can write

$$M_{\lambda_1 \lambda_2 \lambda_3} = D^{S_3}(L(p_3) \varepsilon)_{\lambda_3}^{A_3} D^{S_1}(L(p_1))_{\lambda_1}^{A_1} D^{S_2}(L(p_2))_{\lambda_2}^{A_2} C_{A_3 A_1 A_2}^{(g_3(s, t_E))} \quad (\text{II.24})$$

where $CA_3A_1A_2$ is regular at $\Delta_3(P, p_1, p_3) = 0$

$$L(p_i) = R B\left(\frac{P}{\sqrt{s}} \rightarrow R^{-1} \frac{p_i}{m_i}\right) \epsilon_i \quad \begin{matrix} \epsilon_i = 1 & \text{for } i=1, \\ \epsilon_i = -i\sigma_2 & \text{for } i=2 \end{matrix}$$

$$L(p_3) = R_3 B\left(\frac{P}{\sqrt{s}} \rightarrow R_3^{-1} \frac{p_3}{m_3}\right)$$

R takes the 2-axis $\vec{p}_3 \times \vec{p}_4$ onto the second axis of the helicity tetrad $\vec{p}_1 \times \vec{p}_3$ and the third axis $\hat{p}_j$ onto $\hat{p}_1$ and has the Euler angle $(\varphi_1 + \pi, -\theta_{13}, 0)$ where θ_{13} is the angle of the pure rotation around $\vec{p}_1 \times \vec{p}_3$ who takes $\hat{p}_1$ onto $\hat{p}_3$, and φ_1 is the azimuthal angle of particle 1.

R_3 is the rotation around $\hat{p}_3$ which takes $\vec{p}_3 \times \vec{p}_4$ onto $\vec{p}_1 \times \vec{p}_3$ and has the Euler angle $(\varphi_1 + \pi, 0, 0)$.

$$\cos \theta_{13} = - \frac{\left| \begin{matrix} P & p_3 \\ p & p_4 \end{matrix} \right|}{\sqrt{A_2(P, p_1)} \sqrt{A_2(P, p_3)}}$$

$$\sin \theta_{13} = \frac{\sqrt{s} A_3(P, p_1, p_3)}{\sqrt{A_2(P, p_1)} \sqrt{A_2(P, p_3)}} \quad \text{if } 0 \leq \theta_{13} \leq \pi$$

$$\cos \varphi_1 = \frac{\left| \begin{matrix} P & p_3 & p_4 \\ p & p_3 & p_1 \end{matrix} \right|}{\sqrt{A_3(P, p_3, p_4)} \sqrt{A_3(P, p_3, p_1)}}$$

$$\sin \varphi_1 = \frac{E(P, p_3, p_4, p_1) \sqrt{A_2(P, p_3)}}{\sqrt{A_3(P, p_3, p_4)} \sqrt{A_3(P, p_3, p_1)}} \quad \text{if } 0 \leq \varphi_1 \leq \pi$$

(II.25)

From (II.25), φ_1 and θ_{13} are singular at $\Delta_3(P, p_1, p_3) = 0$ and in the vicinity of $\Delta_3(P, p_1, p_3) = 0$, the helicity amplitude

$$M_{\{\lambda_i\}}^M = e^{i\lambda_3\varphi_1} \sum_{A_1 A_2} e^{-i(A_1 + A_2)\varphi_1} \sin \frac{\theta_{13}}{2} \frac{|A_1 - \lambda_1| + |A_2 + \lambda_2|}{2} \frac{|A_1 + \lambda_1| + |A_2 + \lambda_2|}{2} C_{A_1 A_2}'$$

(II.26)

where CA_1A_2 is regular at $\Delta_3(P, p_1, p_3) = 0$

B1. Singularity at $D_3^+(P, p_1, p_3) = 0$

From formula (II.25) we deduce

- i) $e^{i\varphi_1} \approx (\sqrt{\Delta_3(P, p_1, p_3)})^{\pm 1}$ depending on the determination of $\sqrt{\Delta_4(P, p_1, p_3, p_4)}$;
 ii) $\sin \frac{\theta_{13}}{2} \approx \sqrt{D_3^+(P, p_1, p_3)}$.

Then (II.26) reads

$$M_{\lambda_i} \approx e^{i\{\lambda_3 - (\lambda_1 - \lambda_2)\}\varphi_1} \sum_{A_1, A_2} \sin \frac{\theta_{13}}{2} \frac{|A_1 - \lambda_1|}{e^{-i(A_1 - \lambda_1)\varphi_1}} \sin \frac{\theta_{13}}{2} \frac{|A_2 + \lambda_2|}{e^{-i(A_2 + \lambda_2)\varphi_1}} C_{A_1, A_2} \quad (II.27)$$

with

$$e^{-i(A_1 - \lambda_1)\varphi_1} \sin \frac{\theta_{13}}{2} \frac{|A_1 - \lambda_1|}{\approx \sqrt{D_3^+(P, p_1, p_3)}} \{ |A_1 - \lambda_1| - \eta(A_1 - \lambda_1) \}$$

$$e^{-i(A_2 + \lambda_2)\varphi_1} \sin \frac{\theta_{13}}{2} \frac{|A_2 + \lambda_2|}{\approx \sqrt{D_3^+(P, p_1, p_3)}} \{ |A_2 + \lambda_2| - \eta(A_2 + \lambda_2) \}$$

where $\eta = \pm 1$

The exponent $|A_1 - \lambda_1| \pm (A_1 - \lambda_1)$ [resp $|A_2 + \lambda_2| \pm (A_2 + \lambda_2)$] is always even and positive. Furthermore, it reaches the value zero. Then, the sum in formula (II.27) is regular at $D_3^+(P, p_1, p_3) = 0$ and

$$M_{\lambda_i} \approx e^{i\{\lambda_3 - (\lambda_1 - \lambda_2)\}\varphi_1} \quad (II.28)$$

in the neighbourhood of $D_3^+(P, p_1, p_3) = 0$.

B2. Singularity at $D_3^-(P, p_1, p_3) = 0$

Now, $\sin \frac{\theta_{13}}{2} \approx \sqrt{D_3^-(P, p_1, p_3)}$

and with exactly the same method, we find that

$$M_{\lambda_i} \approx e^{i(\lambda_3 + (\lambda_1 - \lambda_2))\varphi_1}$$

in the neighbourhood of $\mathcal{D}_3^-(P, p_1, p_3) = 0$

(II.29)

As before we have to mix $M_{\{\lambda_i\}}$ and $M_{\{-\lambda_i\}}$ to regularize at $\Delta_4(P, p_1, p_3, p_4) = 0$. To summarize, we can define the amplitudes

$$H_{\{\lambda_i\}}^i = \sin \frac{\theta_{13}}{2}^{|\lambda_3 - (\lambda_1 - \lambda_2)|} \cos \frac{\theta_{13}}{2}^{|\lambda_3 + (\lambda_1 - \lambda_2)|} \sin \frac{\theta_{45}}{2}^{|\lambda_4 + \lambda_5|} \cos \frac{\theta_{45}}{2}^{|\lambda_4 - \lambda_5|} F_{\{\lambda_i\}}^i$$

(II.30)

which are regular at $\Delta_4(P, p_1, p_3, p_4) = 0$, $\Delta_3(P, p_4, p_5) = 0$ and $\Delta_3(P, p_1, p_3) = 0$.

Notes i) Here again, formulae (II.28) and (II.29) show that constraints do exist at $\Delta_3(P, p_1, p_3) = 0$ since, following the determination of $\mathcal{E}(P, p_1, p_3, p_4) = \sqrt{\Delta_4(P, p_1, p_3, p_4)}$, the helicity amplitudes may vanish as

$$\sqrt{\Delta_3^\pm(P, p_1, p_3)}^{|\lambda_3 \mp (\lambda_1 - \lambda_2)|}$$

ii) This regularisation is possible (no square-root singularity remaining) because we have chosen particle 3 such that $s_1 + s_2 + s_3$ is an integer which is not always the case in the Svensson's definition.

5. Singularity at $\Delta_2(P, p_i) = 0$

A. Singularity at $\Delta_2(P, p_3) = 0$

In the reference frame 1, the $\mathcal{M}_{A_i}(\mathcal{P}_i(s, t_j))$ are analytic at $\Delta_2(P, p_3) = 0$. Furthermore, only the 3-axis of the helicity tetrad of particle 3 is singular. Then, we can write

$$\mathcal{M}_{\lambda_3; \{\lambda_i\}} = (-1)^{s_3 + \lambda_3} d^{s_3/\theta_{13}}_{-\lambda_3} B_3^{-\lambda_3} C_{A_3}$$

where C_{A_3} is analytic at $\Delta_2(P, p_3) = 0$, $B_3 = (\omega_{3+m_3} - |\vec{p}_3| \sigma_z) / (\sqrt{2m_3(\omega_{3+m_3})})$, and $\{\lambda_i\}$ labels the set of the remaining helicity ($i = 1, 2, 4, 5$).

By definition

$$\mathcal{D}_2^\pm(p, p_3) = \sqrt{s} m_3 \pm p \cdot p_3 = \sqrt{s} \{ m_3 \pm \omega_3 \}$$

and

$$\Delta_2(p, p_3) = \mathcal{D}_2^+(p, p_3) \mathcal{D}_2^-(p, p_3)$$

one turns around $\mathcal{D}_2^\pm(p, p_3) = 0$, induces the following change

$$|\vec{p}_3| \rightarrow -|\vec{p}_3|, \quad B_s \begin{matrix} -\lambda_3 \\ -\lambda_3 \end{matrix} \rightarrow \begin{pmatrix} (-1)^{s_3} \\ 1 \end{pmatrix} B_s \begin{matrix} \lambda_3 \\ \lambda_3 \end{matrix}$$

$$\theta_{13} \rightarrow \theta_{13} + \varepsilon \pi$$

where $\varepsilon = \pm 1$ depending on the determination of $\sqrt{\Delta_3(p, p_1, p_3)}$

$$d^{s_3}(\theta_{13}) A_{-\lambda_3} \rightarrow (-1)^{s_3 + \varepsilon \lambda_3} d^{s_3}(\theta_{13}) A_{\lambda_3}$$

and, if we define

$$M'_{\lambda_3; \{\lambda_i\}} = \sin \frac{\theta_{13}}{2}^{| \lambda_3 - (\lambda_1 - \lambda_2) |} \cos \frac{\theta_{13}}{2}^{| \lambda_3 + (\lambda_1 - \lambda_2) |} M_{\lambda_3; \{\lambda_i\}} \quad (\text{II.31})$$

it is easy to check that

$$M'_{\lambda_3; \{\lambda_i\}} (-\sqrt{\mathcal{D}_2^-(p, p_3)}) = (-1)^{s_3 + \lambda_1 - \lambda_2} M'_{-\lambda_3; \{\lambda_i\}} (\sqrt{\mathcal{D}_2^-(p, p_3)})$$

$$M'_{\lambda_3; \{\lambda_i\}} (-\sqrt{\mathcal{D}_2^+(p, p_3)}) = (-1)^{s_3 - (\lambda_1 - \lambda_2)} M'_{-\lambda_3; \{\lambda_i\}} (\sqrt{\mathcal{D}_2^+(p, p_3)}) \quad (\text{II.32})$$

Note Since we have regularized out at $\sqrt{\Delta_3(p, p_1, p_3)} = 0$, the phase does not depend on ε .

Furthermore, near $\Delta_2(p, p_3) = 0$, $d^s(\theta)$ being a homogeneous polynomial of degree $2s$ in $\sin \theta/2$ and $\cos \theta/2$, one finds that

$$M'_{\lambda_3; \{\lambda_i\}} \approx (\sqrt{\Delta_2(p, p_3)})^{-m_3} \quad \text{where } m_3 = s_3 + \text{Max} \{ |\lambda_3|, |\lambda_1 - \lambda_2| \} \text{ is an integer} \quad (\text{II.33})$$

To summarize, the following combinations of $H^i_{\lambda_i, \{\lambda_i\}}$ defined by formula (II.30) are free of singularity at $\sqrt{\Delta_2(P, p_3)} = 0$

$$\begin{aligned} & [H^i_{\lambda_3, \{\lambda_i\}} + \eta_3 H^i_{-\lambda_3, \{\lambda_i\}}] \sqrt{\mathcal{D}_2^-(P, p_3)}^{m_3^-} \sqrt{\mathcal{D}_2^+(P, p_3)}^{b_3} \\ & [H^i_{\lambda_3, \{\lambda_i\}} - \eta_3 H^i_{-\lambda_3, \{\lambda_i\}}] \sqrt{\mathcal{D}_2^-(P, p_3)}^{m_3^+ - 1} \sqrt{\mathcal{D}_2^+(P, p_3)}^{d_3} \end{aligned} \quad (\text{II.34})$$

where

$$\begin{aligned} \eta_3 &= (-1)^{s_3 + (\lambda_1 - \lambda_2)} \\ m_3 &= s_3 + \text{Max} \{ |\lambda_3|, |\lambda_1 - \lambda_2| \} \\ N^\pm &= \begin{cases} N & \text{if } N \text{ is even} \\ N \pm 1 & \text{if } N \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} \text{if particle 3 is a boson} \quad & b_3 = m_3^-, \quad d_3 = m_3^+ - 1 \\ \text{if particle 3 is a fermion} \quad & b_3 = m_3^+ - 1, \quad d_3 = m_3^- \end{aligned}$$

B. Singularity at $\Delta_2(P, p_4) = 0$

We apply the same kind of method in the reference frame R_2 .

$$M_{\lambda_4, \{\lambda_i\}} = (-1)^{s_4 - \lambda_4} d^{s_4}(\theta_4) A_{-\lambda_4}^{A_4} B_{-\lambda_4}^{-\lambda_4} C_{A_4}$$

where C_4 is analytic at $\Delta_2(P, p_4) = 0$, θ_4 is the angle of the pure rotation around $\vec{p}_3 \times \vec{p}_4$ which takes $\hat{p}_3$ onto $\hat{p}_4$.

$$\cos \theta_4 = - \frac{|\vec{p}_3 \times \vec{p}_4|}{\sqrt{\Delta_2(P, p_3)} \sqrt{\Delta_2(P, p_4)}} \quad \sin \theta_4 = \frac{\sqrt{s \Delta_3(P, p_3, p_4)}}{\sqrt{\Delta_2(P, p_3)} \sqrt{\Delta_2(P, p_4)}}$$

If we define θ_5 as the angle of the pure rotation around $\vec{p}_3 \times \vec{p}_4$ which takes $\hat{p}_3$ onto $\hat{p}_5$, $\theta_5 - \theta_4 = \theta_{45} \pmod{2\pi}$ which means that $\theta_{45} + \theta_4$ is regular at $\Delta_2(P, p_4) = 0$.

If one turns around $D_2^\pm(P, p_4) = 0$, one induces the following change

$$|\vec{p}_4| \rightarrow -|\vec{p}_4|, \quad B_4^{-\lambda_4} \rightarrow \begin{pmatrix} (-1)^{\epsilon_{s_4}} & \\ & 1 \end{pmatrix} B_4^{\lambda_4}$$

$\theta_4 \rightarrow \theta_4 + \epsilon\pi$, $\theta_{45} \rightarrow \theta_{45} - \epsilon\pi$, $\epsilon = \pm 1$ following the determination of $\sqrt{\Delta_3(P, p_3, p_4)}$.

Then, for the amplitude

$$M''_{\lambda_4, \{\lambda_i\}} = \sin \frac{\theta_{45}}{2}^{|\lambda_4 + \lambda_5|} \cos \frac{\theta_{45}}{2}^{|\lambda_4 - \lambda_5|} M_{\lambda_4, \{\lambda_i\}}$$

we have

$$\begin{aligned} M''_{\lambda_4, \{\lambda_i\}}(-\sqrt{D_2^-(P, p_4)}) &= (-1)^{s_4 + \lambda_5} M''_{-\lambda_4, \{\lambda_i\}}(\sqrt{D_2^-(P, p_4)}) \\ M''_{\lambda_4, \{\lambda_i\}}(-\sqrt{D_2^+(P, p_4)}) &= (-1)^{s_4 - \lambda_5} M''_{-\lambda_4, \{\lambda_i\}}(\sqrt{D_2^+(P, p_4)}) \end{aligned} \quad (\text{II.35})$$

and

$$M''_{\lambda_4, \{\lambda_i\}} \simeq \sqrt{\Delta_2(P, p_4)}^{-m_4} \quad (\text{II.36})$$

where

$$m_4 = s_4 + \text{Max} \{ |\lambda_4|, |\lambda_5| \}$$

is an integer.

To summarize, the following combinations of $H^i_{\{\lambda_i\}}$ defined by formula (II.30) are free of singularity at $\sqrt{\Delta_2(P, p_4)} = 0$

$$\begin{aligned} & [H^i_{\lambda_4, \{\lambda_i\}} + \eta_4 H^i_{-\lambda_4, \{\lambda_i\}}] \sqrt{D_2^-(P, p_4)}^{m_4^-} \sqrt{D_2^+(P, p_4)}^{b_4} \\ & [H^i_{\lambda_4, \{\lambda_i\}} - \eta_4 H^i_{-\lambda_4, \{\lambda_i\}}] \sqrt{D_2^-(P, p_4)}^{m_4^+ - 1} \sqrt{D_2^+(P, p_4)}^{d_4} \\ & \eta_4 = (-1)^{s_4 + \lambda_5}, \quad m_4 = s_4 + \text{Max} \{ |\lambda_4|, |\lambda_5| \} \end{aligned} \quad (\text{II.37})$$

if particle 4 is a boson $b_4 = m_4^-$ $d_4 = m_4^+ - 1$
 if particle 4 is a fermion $b_4 = m_4^+ - 1$ $d_4 = m_4^-$

C. Singularity at $\Delta_2(P, p_5) = 0$

The same proof is valid except that now $\Theta_{45} - \Theta_5$ is regular at $\Delta_2(P, p_5) = 0$

$$\begin{aligned} M''_{\lambda_5; \{\lambda_i\}}(-\sqrt{\mathcal{D}_2^-(P, p_5)}) &= (-1)^{s_5 - \lambda_4} M''_{-\lambda_5; \{\lambda_i\}}(\sqrt{\mathcal{D}_2^-(P, p_5)}) \\ M''_{\lambda_5; \{\lambda_i\}}(-\sqrt{\mathcal{D}_2^+(P, p_5)}) &= (-1)^{s_5 + \lambda_4} M''_{-\lambda_5; \{\lambda_i\}}(\sqrt{\mathcal{D}_2^+(P, p_5)}) \\ M''_{\lambda_5; \{\lambda_i\}} &\simeq \sqrt{\Delta_2(P, p_5)}^{-m_5} \end{aligned} \quad (\text{II.38})$$

where

$$m_5 = s_5 + \text{Max} \{ |\lambda_4|, |\lambda_5| \}$$

(II.39)

is an integer and the following combinations of $H^i_{\{\lambda_i\}}$ defined by formula (II.30) are free of kinematical singularity at

$$\Delta_2(P, p_5) = 0$$

$$\begin{aligned} [H^i_{\lambda_5; \{\lambda_i\}} + \eta_5 H^i_{-\lambda_5; \{\lambda_i\}}] \sqrt{\mathcal{D}_2^-(P, p_5)}^{m_5^-} \sqrt{\mathcal{D}_2^+(P, p_5)}^{b_5} \\ [H^i_{\lambda_5; \{\lambda_i\}} - \eta_5 H^i_{-\lambda_5; \{\lambda_i\}}] \sqrt{\mathcal{D}_2^-(P, p_5)}^{m_5^+ - 1} \sqrt{\mathcal{D}_2^+(P, p_5)}^{d_5} \end{aligned}$$

(II.40)

where

$$\eta_5 = (-1)^{s_5 - \lambda_4}$$

$$m_5 = s_5 + \text{Max} \{ |\lambda_4|, |\lambda_5| \}$$

if particle 5 is a boson $b_5 = m_5^-$ $d_5 = m_5^+ - 1$
 if particle 5 is a fermion $b_5 = m_5^+ - 1$ $d_5 = m_5^-$

D. Singularity at $\Delta_2(P, p_1) = 0$

In the reference frame R_2 , which is regular at $\Delta_2(P, p_1) = 0$, the helicity amplitude reads (see formula II.24)

$$M_{\{\lambda_i\}; \lambda_1 \lambda_2} = \sum_{A_1 A_2} d^{s_1}(-\theta_{13})^{A_1}_{\lambda_1} d^{s_2}(-\theta_{13})^{A_2}_{-\lambda_2} (-1)^{s_2 - \lambda_2} B_1^{\lambda_1}_{\lambda_1} \bar{B}_2^{\lambda_2}_{-\lambda_2} C_{A_1 A_2}$$

where $C_{A_1 A_2}$ is analytic at $\Delta_2(P, p_1) = 0$.

D1. Singularity at $s = (m_1 + m_2)^2$

Turning around the threshold induces the following changes

$$|\vec{p}_1| \rightarrow -|\vec{p}_1|, \quad B_i^{\lambda_i}_{\lambda_i} \rightarrow B_i^{-\lambda_i}_{-\lambda_i} \quad i=1,2$$

$\theta_{13} \rightarrow \theta_{13} + \varepsilon \pi$ $\varepsilon = \pm 1$ following the determination of $\sqrt{\Delta_3(P, p_1, p_3)}$.

It follows, using the definition (II.31)

$$M'_{\{\lambda_i\}; \lambda_1 \lambda_2}(-\varphi_{12}) = (-1)^{s_1 - s_2 - \lambda_3} M'_{\{\lambda_i\}; -\lambda_1 - \lambda_2}(\varphi_{12})$$

(II.41)

where $\varphi_{12} = \sqrt{s - (m_1 + m_2)^2}$

and $M'_{\{\lambda_i\}; \lambda_1 \lambda_2} \simeq \varphi_{12}^{-m_{12}}$

where $m_{12} = s_1 + s_2 + \text{Max} \{|\lambda_3|, |\lambda_1 - \lambda_2|\}$ is an integer.

D2. Singularity at $s = (m_1 - m_2)^2$ ($m_1 \neq m_2$)

In this case $\omega_i + m_i$ does vanish at the pseudothreshold if $m_j > m_i$ ($j \neq i = 1$ or 2). Then,

$$m_1 > m_2 \quad M'_{\{\lambda_i\}; \lambda_1 \lambda_2}(-\psi_{12}) = \eta_{12} (-1)^{2s_2} M'_{\{\lambda_i\}; -\lambda_1 - \lambda_2}(\psi_{12})$$

$$m_2 > m_1 \quad M'_{\{\lambda_i\}; \lambda_1 \lambda_2}(-\psi_{12}) = \eta_{12} (-1)^{2s_1} M'_{\{\lambda_i\}; -\lambda_1 - \lambda_2}(\psi_{12})$$

$$\text{where} \quad \psi_{12} = \sqrt{s - (m_1 - m_2)^2} \quad (\text{II.42})$$

$$\text{and} \quad \eta_{12} = (-1)^{s_1 - s_2 - \lambda_3}$$

$$\text{and} \quad M'_{\{\lambda_i\}; \lambda_1 \lambda_2} \approx \psi_{12}^{-m_{12}}$$

To summarize, the following combinations of $H^i_{\{\lambda_i\}}$ defined by formula (II.30) are free of kinematical singularity at $\Delta_2(P, p_1) = 0$

$$[H^i_{\{\lambda_i\}; \lambda_1 \lambda_2} + \eta_{12} H^i_{\{\lambda_i\}; -\lambda_1 - \lambda_2}] \varphi_{12}^{m_{12}} \psi_{12}^{b_1}$$

$$[H^i_{\{\lambda_i\}; \lambda_1 \lambda_2} - \eta_{12} H^i_{\{\lambda_i\}; -\lambda_1 - \lambda_2}] \varphi_{12}^{m_{12}^+ - 1} \psi_{12}^{d_1}$$

$$\eta_{12} = (-1)^{s_1 - s_2 - \lambda_3} \quad m_{12} = s_1 + s_2 + \text{Max}\{|\lambda_3|, |\lambda_1 - \lambda_2|\}$$

$$\text{Case } B_1 \bar{B}_2 \text{ or } B_1 F_2 \quad m_{B_1} < m_{F_2} \quad b_1 = m_{12}^- \quad d_1 = m_{12}^+ - 1$$

$$\text{Case } F_1 \bar{F}_2 \text{ or } F_1 B_2 \quad m_{F_1} < m_{B_2} \quad b_1 = m_{12}^+ - 1 \quad d_1 = m_{12}^-$$

(II.43)

6. Singularity at $s=0$

Following Raynal ¹⁰⁾, we choose a third frame R_3 which is the laboratory frame of, for instance, particle 3

$$t = p_3/m_3, \quad n_2^\mu = \frac{\varepsilon^{\mu\nu\sigma} (p_3)_\nu (p_4)_\sigma (p_5)_\sigma}{\sqrt{\Delta_3(p_3, p_4, p_5)}} \quad n_3^\mu = \frac{\begin{vmatrix} m_3^2 & p_3 \cdot p_4 \\ p_3^\mu & p_4^\mu \end{vmatrix}}{m_3 \sqrt{\Delta_2(p_3, p_4)}}$$

Then, the functions $\mathcal{F}_i(s, t_k)$ are obviously free of kinematical singularity. Furthermore, if $m_1 \neq m_2$, the vectors of the helicity tetrad of each particle are not singular at $s=0$. Then, the helicity amplitudes are free of kinematical singularity at $s=0$. Of course, this result is very long to get if one chooses as Svensson ⁵⁾ or Johannesson ⁷⁾ a centre-of-mass frame where the time axis, namely $P/\sqrt{s}$ of the reference frame is singular.

However, to regularize the helicity amplitudes at $\Delta_3(P, p_1, p_3)=0$ and at $\Delta_3(P, p_4, p_5)=0$, we have introduced singularity at $s=0$ since $\sin \theta_{13}$ and $\sin \theta_{45}$ behave like $\sqrt{s}$. From

$$H_{\{\lambda_i\}}^i \approx \sin \frac{\theta_{13}}{2} \frac{|\lambda_3 - \lambda_1 - \lambda_2|}{\cos \frac{\theta_{13}}{2}} \frac{|\lambda_3 + \lambda_1 - \lambda_2|}{\sin \frac{\theta_{45}}{2}} \frac{|\lambda_4 + \lambda_5|}{\cos \frac{\theta_{45}}{2}} \frac{|\lambda_4 - \lambda_5|}{\sin \frac{\theta_{45}}{2}} F_{\{\lambda_i\}}^i,$$

we deduce

$$H_{\{\lambda_i\}}^i \approx \sqrt{s} \left[|\lambda_3 - \varepsilon_1 (\lambda_1 - \lambda_2)| + |\lambda_4 + \varepsilon_2 \lambda_5| \right]$$

(II.44)

where ε_1 (resp ε_2) is equal to +1 if $\sin \theta_{13}/2$ (resp $\sin \theta_{45}/2$) behaves like $\sqrt{s}$ and is equal to -1 if $\cos \theta_{13}/2$ (resp $\cos \theta_{45}/2$) behaves like $\sqrt{s}$.

Now, to regularize the $H_{\{\lambda_i\}}^i$ amplitudes at $\Delta_2(P, p_j)=0$, we have to associate two amplitudes of opposite helicity λ_j . Then, the total amplitude, once we have regularized at each $\Delta_2(P, p_j)=0$ will behave like

$$\sqrt{s} | |\lambda_3| - |\lambda_1 - \lambda_2| | + | |\lambda_4| - |\lambda_5| | \left\{ \alpha(s, t_R) + \sqrt{s}^{2 \min(|\lambda_4|, |\lambda_5|)} \beta(s, t_R) + \right. \\ \left. + \sqrt{s}^{2 \min(|\lambda_3|, |\lambda_1 - \lambda_2|)} \gamma(s, t_R) + \sqrt{s}^{2 \min(|\lambda_4|, |\lambda_5|) + 2 \min(|\lambda_3|, |\lambda_1 - \lambda_2|)} \delta(s, t_R) \right\} \quad (II.45)$$

where $\alpha, \beta, \gamma, \delta$ are regular at $s=0$.

The important thing to notice is that if particle 4 and/or 3 are fermions, then a remaining $\sqrt{s}$ singularity cannot be avoided. This situation is very similar to that which occurs in the two-body boson-fermion scattering.

7. Summary and Conclusions

It is a bit long to write explicitly the 16 regularized helicity amplitudes. We sketch here only the method to get these amplitudes

- i) To regularize at $\Delta_4(P, p_1, p_3, p_4)$ we need parity invariance and we have defined (see Section II.3) two linear combinations of helicity amplitudes

$$F_{\{\lambda_i\}}^1 = M_{\{\lambda_i\}} + \eta_{\{\lambda_i\}} M_{\{-\lambda_i\}} \\ F_{\{\lambda_i\}}^2 = \frac{M_{\{\lambda_i\}} - \eta_{\{\lambda_i\}} M_{\{-\lambda_i\}}}{\sqrt{\Delta_4(P, p_3, p_1, p_4)}}$$

which are free of singularity at $\Delta_4(P, p_1, p_3, p_4) = 0$.

- ii) Since the singularity at $\Delta_3(P, p_1, p_3)$ and $\Delta_3(P, p_4, p_5)$ factorized out, we have defined the two following regularized functions

$$H_{\{\lambda_i\}}^\alpha = \sin \frac{\theta_{13}}{2} \frac{|\lambda_3 - (\lambda_1 - \lambda_2)|}{\cos \frac{\theta_{13}}{2}} \frac{|\lambda_3 + (\lambda_1 - \lambda_2)|}{\sin \frac{\theta_{45}}{2}} \frac{|\lambda_4 + \lambda_5|}{\cos \frac{\theta_{45}}{2}} \frac{|\lambda_4 - \lambda_5|}{2} F_{\{\lambda_i\}}^\alpha \\ (\alpha=1,2)$$

where θ_{13} (resp θ_{45}) is the angle of the pure rotation around $\hat{p}_1 \times \hat{p}_3$ (resp $\hat{p}_4 \times \hat{p}_5$) which takes $\hat{p}_1$ (resp $\hat{p}_4$) onto $\hat{p}_3$ (resp $\hat{p}_5$).

- iii) Then, we must combine these $H^{\alpha}_{\{\lambda_i\}}$ in order to get amplitudes which are also regular at $\Delta_2(P, p_j) = 0$ $j=1,3,4,5$. This can be done by using successively formulae (II.34), (II.37) (II.40) and (II.43) and get the final amplitudes $\mathcal{F}^{\alpha}_{\{\lambda_i\}}$.

For instance, the following combination

$$\begin{aligned} G^{\alpha}_{\{\lambda_i\}; \lambda_4 \lambda_5} &= H^{\alpha}_{\{\lambda_i\}; \lambda_4 \lambda_5} + (-1)^{s_4 - \lambda_5} H^{\alpha}_{\{\lambda_i\}; -\lambda_4 \lambda_5} + (-1)^{s_5 + \lambda_4} [H^{\alpha}_{\{\lambda_i\}; \lambda_4 - \lambda_5} + (-1)^{s_4 + \lambda_5} H^{\alpha}_{\{\lambda_i\}; -\lambda_4 - \lambda_5}] \\ &= H^{\alpha}_{\{\lambda_i\}; \lambda_4 \lambda_5} + (-1)^{s_5 + \lambda_4} H^{\alpha}_{\{\lambda_i\}; \lambda_4 - \lambda_5} + (-1)^{s_4 - \lambda_5} [H^{\alpha}_{\{\lambda_i\}; -\lambda_4 \lambda_5} + (-1)^{s_5 - \lambda_4} H^{\alpha}_{\{\lambda_i\}; -\lambda_4 - \lambda_5}] \end{aligned}$$

is uniform both at $\Delta_2(P, p_4) = 0$ and $\Delta_2(P, p_5) = 0$.

- iv) As previously quoted, this procedure has introduced a $\sqrt{s}$ singularity which does not exist in the helicity amplitudes.

Formula (II.45) shows that we must divide $\mathcal{F}^{\alpha}_{\{\lambda_i\}}$ by the factor $\sqrt{s}^{||\lambda_3| - |\lambda_1 - \lambda_2|| + ||\lambda_4| - |\lambda_5||}$. Nevertheless a $\sqrt{s}$ singularity will remain if particle 3 (and/or) 4 are fermions.

Note The parity conservation relations between the $H^{\alpha}_{\{\lambda_i\}}$, namely

$$\begin{aligned} H^1_{\{\lambda_i\}} &= \eta_{\{\lambda_i\}} H^1_{\{-\lambda_i\}} \\ H^2_{\{\lambda_i\}} &= -\eta_{\{\lambda_i\}} H^2_{\{-\lambda_i\}} \end{aligned}$$

implies that we need only a linear combination of eight $H^{\alpha}_{\{\lambda_i\}}$ (or 16 $M^{\alpha}_{\{\lambda_i\}}$) to regularize at $\Delta_2(P, p_j)$ $j=1,3,4,5$. For instance, if we have regularized at $\Delta_2(P, p_1)$, $\Delta_2(P, p_4)$ and $\Delta_2(P, p_5)$, then the singularity at $\Delta_2(P, p_3) = 0$ must factorize out.

8. Case $m_1 = m_2 = m$ (we assume as usual $s_1 = s_2 = s_{in}$)

The general study applies to all singularities but the pseudothreshold. Indeed, since $m_1 = m_2$, $\Psi_{12} = 0$ coincides with $s = 0$.

In the laboratory frame R_3 of particle 3, the functions $\rho_i(s, t_k)$ are obviously free of kinematical singularity and so are the spinor amplitudes. Nevertheless, the helicity amplitudes are singular since the 3-axis of the helicity tetrad of particle 1 and 2 are in this peculiar mass case singular at $s = 0$
 $[\Delta_2(P, p_1) = \Delta_2(P, p_2) = s(s - 4m^2)]$.

In the vicinity of $s = 0$ we write

$$M_{\lambda_1 \lambda_2}; \lambda_1, \lambda_2 = D^{s_1}(L(p_1))^{A_1}_{\lambda_1} D^{s_2}(L(p_2))^{A_2}_{\lambda_2} C_{A_1 A_2} \quad (\text{II.46})$$

where $C_{A_1 A_2}$ is analytic at $s = 0$.

If one turns around $s = 0$, the 3-axis of particle 1 and 2 $n_3(p_i)$ is changed in its opposite whereas p_i/m_i and $n_2(p_i)$ are invariant.

As a consequence

$$L(p_i(-\sqrt{s})) = \varepsilon_i L(p_i(\sqrt{s})) Y \quad (\text{II.47})$$

where Y is a rotation of π around the 2-axis n_2^0 and $\varepsilon_i = \pm 1$ ($i = 1, 2$). In order to relate ε_1 and ε_2 we use the following remark

$$L(p_1) Y L^{-1}(p_2) = L(P \rightarrow p_1) L^{-1}(P \rightarrow p_2) = \frac{m^2 + \sigma p_1 \tilde{\sigma} p_2}{\sqrt{2m^2(m^2 + p_1 \cdot p_2)}} \quad (\text{II.48})$$

where $L(P \rightarrow p_i)$ is the "boost" which takes $P = p_1 + p_2$ onto p_i . Since $m^2 + p_1 \cdot p_2 = s/2$, we deduce at once, using formulae (II.47) and (II.48)

$$\begin{aligned} L(p_1(-\sqrt{s})) \gamma L^{-1}(p_2(-\sqrt{s})) &= - L(p_1(\sqrt{s})) \gamma L^{-1}(p_2(\sqrt{s})) \\ &= \epsilon_1 \epsilon_2 L(p_1(\sqrt{s})) \gamma \gamma \gamma^{-1} L^{-1}(p_2(\sqrt{s})) \end{aligned}$$

(II.49)

Then $\epsilon_1 \epsilon_2 = -1$ and

$$\begin{aligned} M_{\{i\}; \lambda_1 \lambda_2}(-\sqrt{s}) &= (-1)^{\sin + \epsilon_1 \lambda_1} (-1)^{\sin - \epsilon_1 \lambda_2} D^{\sin}_{(L(p_1(\sqrt{s})))}{}^{A_1}_{-\lambda_1} D^{\sin}_{(L(p_2(\sqrt{s})))}{}^{A_2}_{\lambda_2} \times \\ &\quad \times C_{A_1 A_2} \\ &= (-1)^{\lambda_1 + \lambda_2} M_{\{i\}; -\lambda_1 - \lambda_2} \quad *) \end{aligned}$$

(II.50)

As far as the pole is concerned, it is shown in Appendix C that the matrix elements

$$D^{\sin}_{(L(p_i))}{}^{A_i}_{\lambda_i}$$

behave like $(1/\sqrt{s})^{\sin}$ in the vicinity of $s=0$.

If, for the sake of simplicity particles 4 and 5 are spinless, the following functions are free of kinematical singularity at $s=0$

$$\begin{aligned} &[H^i_{\lambda_3; \lambda_1 \lambda_2} + (-1)^{\lambda_1 + \lambda_2} H^i_{\lambda_3; -\lambda_1 - \lambda_2}] \sqrt{s}^{(2\sin)-} \\ &[H^i_{\lambda_3; \lambda_1 \lambda_2} - (-1)^{\lambda_1 + \lambda_2} H^i_{\lambda_3; -\lambda_1 - \lambda_2}] \sqrt{s}^{(2\sin)+-1} \end{aligned}$$

(II.51)

*) Note that this result is exactly the same as in the two-body scattering [see Appendix A of Ref. 14].

Unfortunately, if particles 4 and 5 are not spinless, an extra $\sqrt{s}$ singularity does appear in the

$$\sin \frac{\theta_{45}}{2}^{|\lambda_4 + \lambda_5|} \cos \frac{\theta_{45}}{2}^{|\lambda_4 - \lambda_5|}$$

factors and we have

$$H_{\{\lambda_i\}; \lambda_1, \lambda_2}^i(-\sqrt{s}) = (-1)^{\lambda_1 + \lambda_2} (-1)^{\lambda_4 + \epsilon_2 \lambda_5} H_{\{\lambda_i\}; -\lambda_1 - \lambda_2}^i(\sqrt{s}) \quad (\text{II.52})$$

where $\epsilon_2 = +1$ if $\sin \theta_{45}/2$ behaves like $\sqrt{s}$ and -1 if $\cos \theta_{45}/2$ behaves like $\sqrt{s}$. Then, as usual, the helicity amplitudes can be regularized only if particles 4 and 5 are bosons since, in this case, the phase is independent of ϵ_2 . In the fermion-fermion case, a $\sqrt{s}$ singularity cannot be avoided [see also formula (II.45)].

III. KINEMATICAL CONSTRAINTS AND TRANSVERSITY AMPLITUDES

1. Principle of the Method

In order to study helicity amplitudes at $\Delta_2(P, p_j) = 0$ ($j = 1, 3, 4, 5$) we have chosen a reference frame such that the set of $\mathcal{P}_j(s, t_k)$ is analytic in the vicinity of this singularity. So, any spinor amplitude is analytic at this point. But, such an amplitude can be expressed as a function of the helicity amplitudes [see Eq. (II.8) and (II.9)], namely

$$\mathcal{M}_{A_i}(\mathcal{P}_j(s, t_k)) = \sum_{\lambda_i} \prod_i D^{s_i}(\varepsilon_i L^{-1}(i, \mathcal{P}_j(s, t_k)))^{\lambda_i} M_{\lambda_i}(s, t_k) \quad (\text{III.1})$$

With the results of the previous section we can build the helicity amplitudes regularized at this point and bearing the regularized amplitudes in (III.1), we get an expansion which is no longer analytic at $\Delta_2(P, p_i) = 0$ as $\mathcal{M}_{A_i}(\mathcal{P}_j(s, t_k))$ since extra poles appear. Then, as in the two-body case we get a set of constraints between regularized helicity amplitudes since these poles must be cancelled out. Raynal¹⁰⁾ has proved that this method gives all the kinematical constraints but it is very difficult to use Eq. (III.1) to cancel the poles so, as in the two-body scattering, we use a generalization of the transversity amplitudes the behaviour of which at $\Delta_2(P, p_i)$ gives all the kinematical constraints.

2. Transversity Amplitudes

We briefly recall here the properties of transversity amplitudes²⁰⁾. If the helicity frame of the particle i is defined by the tetrad $p_i, n_1^h(p_i), n_2^h(p_i), n_3^h(p_i)$ then, the "transversity" frame can be defined by $p_i/m_i, n_1^t(p_i), n_2^t(p_i) = -n_3^h(p_i), n_3^t(p_i) = n_2^h(p_i)$ (we interchange the 3- and the 2-axis).

So, transversity states are related to helicity states by

$$|p_i, L_t(p_i), s_i, \tau_i\rangle = D^{s_i}(R)^{\lambda_i} \tau_i |p_i, L_h(p_i), s_i, \lambda_i\rangle \quad (\text{III.2})$$

where R is a rotation through $-\pi/2$ around the first axis with the following properties

$$\begin{aligned} R \sigma_2 R^{-1} &= -\sigma_3 & R^{-1} \sigma_2 R &= \sigma_3 \\ R^{-1} \sigma_3 R &= -\sigma_2 & R \sigma_3 R^{-1} &= \sigma_2 \end{aligned}$$

(III.3)

Transversity amplitudes are related to helicity amplitudes through

$$T_{\{\tau_f\}, \{\tau_i\}} = \prod_{i,f} D^{s_i}(R)^{\lambda_i}_{\tau_i} D^{s_f}(R^{-1})^{\lambda_f}_{\tau_f} M_{\{\lambda_f\}, \lambda_i}$$

(III.4)

where i (resp f) labels the initial (resp final) particles.

Using the relation

$$D^{s_f}(R^{-1})^{A_f}_{A'_f} = e^{i\pi s_f} D^{s_f}(R)^{A_f}_{-A'_f},$$

the covariance property (II.1) and Eq. (II.5), transversity amplitudes are expressed as functions of the spinor amplitudes, namely

$$T_{\{\tau_f\}, \{\tau_i\}} = \prod_f \epsilon^{s_f - \tau_f} D^{s_f}(R L(f) R^{-1})^{A_f}_{\tau_f} \prod_i \overline{D}^{s_i}(R^{-1} L(i) R)^{A_i}_{\tau_i} \times M_{-A_f, A_i}(R^{-1} p_f, R^{-1} p_i)$$

(III.5)

when $L(i)$ and $L(f)$ are the Lorentz transformations of the helicity states. This relation is very useful to study directly the analytical properties of the transversity amplitudes.

A. Singularity at $\Delta_4(P, p_1, p_3, p_4) = 0$

From the parity conservation condition for helicity amplitudes [Eq. (II.11)] and from

$$(-1)^{s+\lambda} D^s(R)^{\lambda}_{\tau} = e^{i\pi\tau} D^s(R)^{-\lambda}_{-\tau},$$

we deduce

$$T_{\{\tau_i\}}(-\sqrt{\Delta_4}) = \eta^{(-1)^{\tau_1+\tau_2-\tau_3-\tau_4-\tau_5}} T_{\{\tau_i\}}(\sqrt{\Delta_4}) \quad (\text{III.6})$$

where

$$\eta = \prod_{i=1}^5 \eta_i$$

that is if

$$\eta^{(-1)^{\tau_1+\tau_2-\tau_3-\tau_4-\tau_5}} = +1$$

$T_{\{\tau_i\}}$ is regular at $\Delta_4(P, p_1, p_3, p_4) = 0$ and if

$$\eta^{(-1)^{\tau_1+\tau_2-\tau_3-\tau_4-\tau_5}} = -1$$

$T_{\{\tau_i\}} / \sqrt{\Delta_4(P, p_1, p_3, p_4)}$ is regular.

B. Singularity at $\Delta_3(P, p_4, p_5) = 0$

As already seen previously (see Section II.4), γ_4 and γ_5 are the only functions which are singular at $\Delta_3(P, p_4, p_5) = 0$.

From the expansion (III.5) and formula (III.3), the transversity amplitudes near $\Delta_3(P, p_4, p_5) = 0$ read

$$T_{\tau_4 \tau_5; \{\tau_i\}} = (-1)^{s_4 - \tau_4 + s_5 - \tau_5} d^{s_4}(\gamma_4)^{A_4}_{\tau_4} d^{s_5}(\gamma_5)^{A_5}_{\tau_5} C_{A_4 A_5} \quad (\text{III.7})$$

where $C_{A_4 A_5}$ is analytic.

Near $D_3^\pm(P, p_4, p_5) = 0$, $\gamma_4 \mp \gamma_5$ is regular. Then, one turn around the $D_3^\pm$ induces the following change $\gamma_4 \rightarrow \gamma_5 + \varepsilon\pi$, $\gamma_5 \rightarrow \gamma_5 \pm \varepsilon\pi$ where $\varepsilon = \pm 1$ and

$$T_{\tau_4 \tau_5; \{\tau_i\}}(-\sqrt{D_3^\pm(P, p_4, p_5)}) = (-1)^{s_4 + s_5 + \tau_4 \pm \tau_5} T_{-\tau_4 - \tau_5; \{\tau_i\}}(\sqrt{D_3^\pm(P, p_4, p_5)}) \quad (\text{III.8})$$

From formula (III.7) we deduce the behaviour of the transversity amplitude near $\Delta_3(P, p_4, p_5) = 0$

$$T_{\tau_4 \tau_5, \{\tau_i\}} \approx (\sqrt{\Delta_3(P, p_4, p_5)})^{-(s_4 + s_5)} \quad (\text{III.9})$$

C. Singularity at $\Delta_3(P, p_1, p_3) = 0$

We use expansion (III.5) in the frame R_1 where φ_1 and θ_{13} are singular. One turn around $D_3^{\pm}(P, p_1, p_3)$ [resp $D_3^-(P, p_1, p_3)$] induces the following change

$$\begin{aligned} \varphi_1 &\rightarrow \varphi_1 + \varepsilon \pi \\ \theta_{13} &\rightarrow -\theta_{13} \quad (\text{resp } -\theta_{13} + 2\pi) \end{aligned}$$

Then,

$$T_{\tau_3 \tau_1 \tau_2 \{\tau_i\}} (-\sqrt{D_3^{\pm}(P, p_1, p_3)}) = (-1)^{s_1 + s_2 + s_3 + \tau_3 \pm (\tau_1 - \tau_2)} T_{-\tau_3 - \tau_1 - \tau_2 \{\tau_i\}} (\sqrt{D_3^{\pm}(P, p_1, p_3)}) \quad (\text{III.10})$$

and

$$T_{\tau_3 \tau_1 \tau_2 \{\tau_i\}} \approx (\sqrt{\Delta_3(P, p_1, p_3)})^{-(s_1 + s_2 + s_3)} \quad (\text{III.11})$$

near $\Delta_3(P, p_1, p_3) = 0$.

D. Singularity at $\Delta_2(P, p_i) = 0 \quad i = 1, 3, 4, 5$

As an example we treat the case $\Delta_2(P, p_1) = 0$.

In the reference frame R_2 , expansion (III.5) reads

$$T_{\tau_1 \tau_2 \{\tau_i\}} = e^{i(A_1 + A_2)\theta_{13}} d^{s_1(B_1^y)}_{\tau_1} d^{s_2(B_2^y)}_{\tau_2} e^{-i\pi\tau_2} C_{A_1 A_2}$$

where $C_{A_1 A_2}$ is analytic at $\Delta_2(P, p_1) = 0$ and

$$B_i^y = \frac{n_i + \omega_i + 2i|\vec{p}_1| \tau_2}{\sqrt{2m_i(\omega_i + m_i)}} \quad \begin{aligned} \ell_i &= +1 \Leftrightarrow i=1 \\ \ell_i &= -1 \Leftrightarrow i=2 \end{aligned}$$

At the threshold $\varphi_{12}=0$ $B_i^y = 1$ and

$$T_{\tau_1 \tau_2 \{\tau_i\}} \simeq e^{i(\tau_1 + \tau_2) \theta_{13}} \quad \text{near } \varphi_{12}=0 \quad (\text{III.12})$$

At the pseudothreshold $\psi_{12}=0$ $B_i^y = 1$ if

$$\varepsilon_{ij} = \frac{m_i - m_j}{|m_i - m_j|} = +1$$

$$= \pm i\tau_2 \quad \text{if } \varepsilon_{ij} = -1$$

and

$$T_{\tau_1 \tau_2 \{\tau_i\}} \simeq e^{i\varepsilon_{12}(\tau_1 - \tau_2) \theta_{13}} \quad \text{near } \psi_{12}=0 \quad (\text{III.13})$$

Note As expected, it is exactly the same behaviour as in the two-body scattering.

For the other $\Delta_2(P, p_i)$ we proceed exactly in the same way and get

$$T_{\tau_3 \{\tau_i\}} \simeq e^{\mp i\tau_3 \theta_{13}} \quad \text{near } D_2^\pm(P, p_3) = 0 \quad (\text{III.14})$$

$$T_{\tau_4 \{\tau_i\}} \simeq e^{\pm i\tau_4 \theta_{45}} \quad \text{near } D_2^\pm(P, p_4) = 0 \quad (\text{III.15})$$

$$T_{\tau_5 \{\tau_i\}} \simeq e^{\mp i\tau_5 \theta_{45}} \quad \text{near } D_2^\pm(P, p_5) = 0 \quad (\text{III.16})$$

3. Kinematical Constraints

Equations (III.12) and (III.16) show that in the vicinity of $\Delta_2(P, p_i) = 0$, transversity amplitudes have pole or zero since $e^{i\theta}$ behaves like $\sqrt{\Delta_2(P, p_i)}^{\pm 1}$ depending on the determination of $\sqrt{\Delta_3(P, p_i, p_k)}$.

For instance, as in the two-body case [see Eq. (IV.11) of Ref. 14)],

$$\begin{aligned}
 \text{near } \varphi_{12}=0 \quad & \sum_{12}^{s_1+s_2} T_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2} \approx \varphi_{12}^{s_1+s_2 \pm (\tau_1+\tau_2)} \\
 & \text{if } e^{i\theta_{13}} \approx \varphi_{12}^{\pm 1} \quad (\text{III.17}) \\
 \text{near } \psi_{12}=0 \quad & \sum_{12}^{s_1+s_2} T_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2} \approx \psi_{12}^{s_1+s_2 \pm \varepsilon_{12}(\tau_1-\tau_2)} \\
 & \text{if } e^{i\theta_{13}} \approx \psi_{12}^{\pm 1}
 \end{aligned}$$

which means that linear combinations of helicity amplitudes and of their derivatives do vanish at $\varphi_{12}=0$ or $\psi_{12}=0$. ($S_{12} = \varphi_{12} \cdot \psi_{12}$).

But a new difficulty arises in the five-body problem when we want to write constraints between regularized helicity amplitudes. Indeed, to treat completely the constraints at $\Delta_2(P, p_1)$ or $\Delta_2(P, p_3)=0$ [resp $\Delta_2(P, p_4)$ or $\Delta_2(P, p_5)$] we have first to regularize transversity amplitudes at $\Delta_4(P, p_1, p_3, p_4)=0$, $\Delta_3(P, p_4, p_5)=0$, $\Delta_2(P, p_4)=0$ and $\Delta_2(P, p_5)=0$ [resp $\Delta_4(P, p_1, p_3, p_4)=0$, $\Delta_3(P, p_1, p_3)=0$, $\Delta_2(P, p_1)=0$ and $\Delta_2(P, p_3)=0$]. This can be done in the following way using the results of the previous section.

A. Case $\Delta_2(P, p_1)=0$

The regularization of transversity amplitudes at $\Delta_4(P, p_1, p_3, p_4)=0$ is straightforward if parity conservation holds [see formula (III.6)]. To regularize at $\Delta_3(P, p_4, p_5)=0$, we have to associate transversity amplitudes $T_{\{\tau_{ij}\}; \tau_4 \tau_5}$ and $T_{\{\tau_{ij}\}; -\tau_4 -\tau_5}$ which behave in the vicinity of $D_2^\pm(P, p_4)=0$ [resp $D_2^\pm(P, p_5)=0$] as

$$\left(\sqrt{D_2^\pm(P, p_4)} \right)^{\pm \varepsilon \tau_4} \left(\text{resp } \left(\sqrt{D_2^\pm(P, p_5)} \right)^{\pm \varepsilon \tau_5} \right), \quad \varepsilon = \pm 1$$

Then, if particle 4 is a fermion the linear combinations

$$\mathcal{L}_{\{\tau_{ij}\}; \tau_4 \tau_5}^\pm = T_{\{\tau_{ij}\}; \tau_4 \tau_5} \pm T_{\{\tau_{ij}\}; -\tau_4 -\tau_5}$$

will exhibit a $\sqrt{D_2^\pm(P, p_4)}$ [resp $\sqrt{D_2^\pm(P, p_5)}$] singularity, namely

$$\mathcal{E}_{\{\tau_4\}, \tau_4 \tau_5}^\pm \simeq \frac{1}{\sqrt{D_2^\pm(P, p_4)} |\tau_4|} \left\{ \alpha + \beta \sqrt{D_2^\pm(P, p_4)}^{2/|\tau_4|} \right\}$$

in the vicinity of $D_2^\pm(P, p_j) = 0$ ($j = 4$ or 5) where α and β are analytic at $D_2^\pm(P, p_j) = 0$.

To get rid of this trouble, we propose the following recipe. Let us define

$$\widehat{T}_{\{\tau_i\}, \tau_4 \tau_5} = e^{-i \tau_4 \theta_{45}} \overline{T}_{\{\tau_i\}, \tau_4 \tau_5} \quad (\text{III.18})$$

Then, using (III.8), we deduce

$$\widehat{T}_{\{\tau_i\}, \tau_4 \tau_5} (-\sqrt{\Delta_3(P, p_4, p_5)}) = (-1)^{s_4 + \tau_4 + s_5 + \tau_5} \overline{T}_{\{\tau_i\}, -\tau_4 - \tau_5} (\sqrt{\Delta_3(P, p_4, p_5)}) \quad (\text{III.19})$$

Hence, the two linear combinations

$$\frac{\widehat{T}_{\{\tau_i\}, \tau_4 \tau_5} + (-1)^{s_4 + s_5 + \tau_4 + \tau_5} \overline{T}_{\{\tau_i\}, -\tau_4 - \tau_5}}{N} = \widehat{T}_{\{\tau_i\}, \tau_4 \tau_5}^1$$

and,

$$\frac{\widehat{T}_{\{\tau_i\}, \tau_4 \tau_5} - (-1)^{s_4 + s_5 + \tau_4 + \tau_5} \overline{T}_{\{\tau_i\}, -\tau_4 - \tau_5}}{N \sqrt{\Delta_3(P, p_4, p_5)}} = \widehat{T}_{\{\tau_i\}, \tau_4 \tau_5}^2 \quad (\text{III.20})$$

where $N = +1$ [resp $\sqrt{\Delta_4(P, p_1, p_3, p_4)}$] if

$$\prod_i \varepsilon_i (-1)^{\tau_1 + \tau_2 - \tau_3 - \tau_4 - \tau_5}$$

is equal to 1 (resp -1), are uniform at $\Delta_3(P, p_4, p_5)$ and $\Delta_4(P, p_1, p_3, p_4) = 0$ ^{*}.

*) Note that from this definition

$$\widehat{T}_{\{\tau_i\}, \tau_5 \tau_4}^\alpha = \varepsilon_\alpha (-1)^{s_4 + s_5 + \tau_4 + \tau_5} \widehat{T}_{\{\tau_i\}, -\tau_5 - \tau_4}^\alpha$$

$\varepsilon_1 = +1, \quad \varepsilon_2 = -1$

The main interest of these amplitudes is that

$$\hat{T}_{\{i\}}^{\pm}, \tau_4 \tau_5 \simeq (\sqrt{D_2^{\pm}(P, p_4)})^{\eta(\tau_4 \mp \tau_5)} \quad \text{near } D_2^{\pm}(P, p_4) = 0$$

and

$$\hat{T}_{\{i\}}^{\pm}, \tau_4 \tau_5 \simeq (\sqrt{D_2^{\pm}(P, p_5)})^{\eta'(\tau_5 \pm \tau_4)} \quad \text{near } D_2^{\pm}(P, p_5) = 0$$

$$(\eta, \eta' = \pm 1)$$

which shows that now the exponent of $\sqrt{D_2^{\pm}(P, p_i)}$ is always an integer. As a consequence the two functions

$$\hat{\mathcal{E}}_{\{i\}}^{\alpha} = \sqrt{D_2^{-}(P, p_4)}^{2/\tau_4} \sqrt{D_2^{+}(P, p_5)}^{1/\tau_4 + \tau_5} \sqrt{D_2^{-}(P, p_5)}^{1/\tau_5 - \tau_4} \hat{T}_{\tau_5 \tau_4 \tau_3; \tau_1 \tau_2}^{\alpha} \quad (III.21)$$

are uniform at $\Delta_4(P, p_1, p_3, p_4) = 0$, $\Delta_3(P, p_4, p_5) = 0$, $\Delta_2(P, p_4) = 0$ and $\Delta_2(P, p_5) = 0$ and have obviously the same behaviour as $\tau_3 \tau_4 \tau_5; \tau_1 \tau_2$ near $D_2^{\pm}(P, p_3) = 0$ and $\Delta_2(P, p_1) = 0$.

Similarly, we define

$$\hat{\hat{T}}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3 = e^{i(\tau_1 + \tau_2) \theta_{13}} \hat{T}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3 \quad (III.22)$$

which can be regularized at $\Delta_4 = 0$, $\Delta_3(P, p_1, p_3) = 0$, $\Delta_2(P, p_1) = 0$ and $\Delta_2(P, p_3) = 0$. From Eq. (III.10) we deduce that the two linear combinations

$$\frac{\hat{\hat{T}}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3 + (-1)^{s_1 + s_2 + s_3 + \tau_1 - \tau_2 + \tau_3} \hat{\hat{T}}_{\{i\}}^{\pm}, -\tau_1 - \tau_2 - \tau_3}{N} = \hat{\hat{T}}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3$$

$$\frac{\hat{\hat{T}}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3 - (-1)^{s_1 + s_2 + s_3 + \tau_1 - \tau_2 + \tau_3} \hat{\hat{T}}_{\{i\}}^{\pm}, -\tau_1 - \tau_2 - \tau_3}{N \sqrt{\Delta_3(P, p_1, p_3)}} = \hat{\hat{T}}_{\{i\}}^{\pm}, \tau_1 \tau_2 \tau_3 \quad (III.23)$$

are uniform at $\Delta_3(P, p_4, p_5) = 0$ and $\Delta_4(P, p_1, p_3, p_4) = 0$ and that the two functions

$$\hat{\mathcal{E}}_{\{i\}}^{\alpha} = \psi_{12}^{1/(4-\varepsilon_{12})} \tau_1 + (4+\varepsilon_{12})\tau_2 / \sqrt{D_4^{+}(P, p_3)}^{1/\tau_1 + \tau_2 - \tau_3} \sqrt{D_2^{-}(P, p_3)}^{1/\tau_1 + \tau_2 + \tau_3} \hat{\hat{T}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^{\alpha} \quad (III.24)$$

are uniform at $\Delta_2(P, p_3) = 0$ and $\Delta_2(P, p_1) = 0$ and have the same behaviour as $T_{\{\tau_i\}}$ near $\Delta_2(P, p_4) = 0$ and $\Delta_2(P, p_5) = 0$.

Once the two sets of transversity amplitudes $\hat{\mathcal{C}}_{\{\tau_i\}}^\alpha$ and $\hat{\mathcal{C}}_{\{\tau_i\}}^{\alpha'}$ have been defined, then the constraints can be derived between regularized helicity amplitudes. For instance, in the case of the constraints at $\Delta_2(P, p_1) = 0$, we only have to replace in formula (III.17) $T_{\{\tau_i\}}$ by $\hat{\mathcal{C}}_{\{\tau_i\}}^\alpha$.

We summarize these results in the following formulae

$$\begin{aligned} \text{Near } s = (m_1 + m_2)^2 \quad (\varphi_{12} = 0) \quad \varphi_{12}^{s_1+s_2} \hat{\mathcal{C}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^\alpha &\simeq \varphi_{12}^{s_1+s_2 \pm (\tau_1 + \tau_2)} \\ \text{if } t_2 \theta_{13} &= \mp i \text{ at } \varphi_{12} = 0 \\ \text{Near } s = (m_1 - m_2)^2 \quad (\psi_{12} = 0) \quad \psi_{12}^{s_1+s_2} \hat{\mathcal{C}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^\alpha &\simeq \psi_{12}^{s_1+s_2 \pm \varepsilon_{12} (\tau_1 - \tau_2)} \\ \text{if } t_2 \theta_{13} &= \mp i \text{ at } \psi_{12} = 0 \end{aligned} \quad (\text{III.25})$$

$$\begin{aligned} \text{Near } \sqrt{s} = \mp \frac{p \cdot p_3}{m_3} \quad (D_2^\pm(p, p_3) = 0) \quad \sqrt{D_2^\pm(p, p_3)}^{s_3} \hat{\mathcal{C}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^\alpha &\simeq \sqrt{D_2^\pm(p, p_3)}^{s_3 \pm \varepsilon \tau_3} \\ \text{if } t_2 \theta_{13} &= \varepsilon i \text{ at } D_2^\pm(p, p_3) = 0 \\ (\varepsilon = \pm 1) & \end{aligned} \quad (\text{III.26})$$

$$\begin{aligned} \text{Near } \sqrt{s} = \mp \frac{p \cdot p_4}{m_4} \quad (D_2^\pm(p, p_4) = 0) \quad \sqrt{D_2^\pm(p, p_4)}^{s_4} \hat{\mathcal{C}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^\alpha &\simeq \sqrt{D_2^\pm(p, p_4)}^{s_4 \mp \varepsilon \tau_4} \\ \text{if } t_2 \theta_{45} &= \varepsilon i \text{ at } D_2^\pm(p, p_4) = 0 \\ (\varepsilon = \pm 1) & \end{aligned} \quad (\text{III.27})$$

$$\begin{aligned} \text{Near } \sqrt{s} = \mp \frac{p \cdot p_5}{m_5} \quad (D_2^\pm(p, p_5) = 0) \quad \sqrt{D_2^\pm(p, p_5)}^{s_5} \hat{\mathcal{C}}_{\tau_3 \tau_4 \tau_5; \tau_1 \tau_2}^\alpha &\simeq \sqrt{D_2^\pm(p, p_5)}^{s_5 \pm \varepsilon \tau_5} \\ \text{if } t_2 \theta_{45} &= \varepsilon i \text{ at } D_2^\pm(p, p_5) = 0 \\ (\varepsilon = \pm 1) & \end{aligned} \quad (\text{III.28})$$

IV. SUMMARY AND CONCLUSIONS

We have first obtained all the kinematical singularities of the helicity amplitudes by using the very elegant method of Raynal ¹⁰⁾. Then, by assuming parity conservation, we have built amplitudes which are analytic in the image of the analyticity domain of the spinor amplitudes in the space of the four-momenta, on the mass shell which is saturated outside the boundary of the physical region. However, due to our regularization we have introduced an extra $\sqrt{s}$ singularity which cannot be removed in the $BF \rightarrow FBB$, $BF \rightarrow FFF$, $FF \rightarrow BFF$ and $BB \rightarrow BFF$ case.

Finally, by using the behaviour of the transversity amplitudes near $\Delta_2(P, p_i) = 0$, we have derived explicitly the constraints by the means, as in the two-body cases, of relations between the regularized helicity amplitudes and some number of their derivatives with respect to s which hold at $\sqrt{s} = \pm(P \cdot p_i)/(m_i)$.

It may be noticed here that adding one particle to the two-body case leads to a tremendous increase of complications. For instance, parity conservation is needed and we must combine in the general case no less than 16 helicity amplitudes to get rid of the kinematical singularity. Nevertheless, we hoped that this work would be useful for the phenomenology of the multiparticle scattering and more precisely to the B_5 Veneziano model ²⁾, where the spin has to be treated correctly.

ACKNOWLEDGEMENTS

It is a pleasure to thank H. Ruegg for several fruitful discussions and H. Epstein for reading the manuscript.

APPENDIX A

Following Svensson ⁵⁾, we specify in this Appendix our notations for determinants. Consider two sets of n four-vectors $\{q_i\}$ and $\{p_i\}$ $i = 1 \dots n$. The determinant of the $n \times n$ matrix $(q_i \cdot p_k)$ is called the Gram determinant and

$$\text{Det}(q_i \cdot p_k) = \begin{vmatrix} q_1 \dots q_n \\ p_1 \dots p_n \end{vmatrix} = \begin{vmatrix} p_1 \dots p_n \\ q_1 \dots q_n \end{vmatrix} = \begin{vmatrix} q_2 q_1 \dots q_n \\ p_2 p_1 \dots p_n \end{vmatrix} = - \begin{vmatrix} q_1 q_2 \dots q_n \\ p_2 p_1 \dots p_n \end{vmatrix} \quad (\text{A.1})$$

In a four-dimensional space all Gram determinants with $n \geq 5$ vanish. We note the symmetric Gram determinant

$$\Delta_n(p_1 \dots p_n) = (-1)^{n+1} \begin{vmatrix} p_1 \dots p_n \\ p_1 \dots p_n \end{vmatrix} \quad \Delta_0 = -1 \quad (\text{A.2})$$

We use extensively in our work the identity

$$\begin{vmatrix} q_1 \dots q_{n-2} \\ p_1 \dots p_{n-2} \end{vmatrix} \begin{vmatrix} q_1 \dots q_{n-2} q_{n-1} q_n \\ p_1 \dots p_{n-2} p_{n-1} p_n \end{vmatrix} = \begin{vmatrix} q_1 \dots q_{n-2} q_{n-1} \\ p_1 \dots p_{n-2} p_{n-1} \end{vmatrix} \begin{vmatrix} q_1 \dots q_{n-2} q_n \\ p_1 \dots p_{n-2} p_n \end{vmatrix} - \begin{vmatrix} q_1 \dots q_{n-2} q_{n-1} \\ p_1 \dots p_{n-2} p_n \end{vmatrix} \begin{vmatrix} q_1 \dots q_{n-2} q_n \\ p_1 \dots p_{n-2} p_{n-1} \end{vmatrix} \quad (\text{A.3})$$

Then, we deduce the factorization property of the symmetric determinant (A.2), namely

$$\Delta_{n-2}(p_1 \dots p_{n-2}) \Delta_n(p_1 \dots p_{n-2} p_{n-1} p_n) = \mathcal{D}_n^+(p_1 \dots p_n) \mathcal{D}_n^-(p_1 \dots p_n) \quad n \geq 2 \quad (\text{A.4})$$

where

$$\mathcal{D}_n^\pm(p_1 \dots p_n) = \sqrt{\Delta_{n-1}(p_1 \dots p_{n-2} p_{n-1})} \sqrt{\Delta_{n-1}(p_1 \dots p_{n-2} p_n)} \pm \begin{vmatrix} p_1 \dots p_{n-2} p_{n-1} \\ p_1 \dots p_{n-2} p_n \end{vmatrix} \quad (\text{A.5})$$

With these notations

$$\mathcal{D}_2^\pm(p_k, p_i) = m_i m_k \pm p_i p_k \quad (\text{A.6})$$

$$\mathcal{D}_3^\pm(p, p_i, p_k) = \sqrt{\Delta_2(p, p_i)} \sqrt{\Delta_2(p, p_k)} \pm \begin{vmatrix} p & p_i \\ p & p_k \end{vmatrix} \quad (\text{A.7})$$

$$\left[\varepsilon^{\mu\nu\sigma\tau} p_\mu(p_1) p_\nu(p_3) p_\sigma(p_4) p_\tau \right]^2 = \Delta_4(p, p_1, p_3, p_4) \quad (\text{A.8})$$

Note $\mathcal{D}_4^\pm(p, p_i, p_1, p_3) = \sqrt{\Delta_3(p, p_i, p_1)} \sqrt{\Delta_3(p, p_i, p_3)} \pm \begin{vmatrix} p & p_i & p_3 \\ p & p_i & p_1 \end{vmatrix} \quad i=4,5$

which shows that at $\Delta_3(p, p_4, p_3) = 0$,

$$\Delta_2(p, p_4) \Delta_4(p, p_1, p_2, p_4) = - \left[\begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_4 \end{vmatrix} \right]^2$$

Then $\text{tg } \gamma_4 = \pm i$ following the determination of $\sqrt{\Delta_4(p, p_1, p_3, p_4)}$.

APPENDIX B

In this Appendix we calculate the third Euler angle γ_i ($i=4,5$) of the rotation R which takes the 2-axis $\vec{p}_1 \times \vec{p}_3$ onto $\vec{p}_4 \times \vec{p}_5$ and the 3-axis $\hat{p}_1$ onto $\hat{p}_i$, and shows that $\gamma_4 \pm \gamma_5$ are regular at $D_3^\pm(P, p_4, p_5) = 0$. By definition, γ_i is the angle of the pure rotation around $\hat{p}_i$ which takes $\vec{p}_1 \times \vec{p}_i$ onto $\vec{p}_4 \times \vec{p}_5$.

Let (n_0, n_1, n_2, n_3) be the helicity tetrad of particle i and (n_0, n'_1, n'_2, n'_3) be the helicity tetrad with another 2 axis n'_2 (then n'_1) where

$$n'_2{}^\mu = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu (p_1)_\rho (p_i)_\sigma}{\sqrt{\Delta_3(P, p_1, p_i)}} \quad (B.1)$$

then

$$\cos \gamma_i = - n_2 \cdot n'_2 = - n_1 \cdot n'_1$$

$$\sin \gamma_i = - n_1 \cdot n'_2 = n'_1 \cdot n_2$$

$$\cos \gamma_i = - \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu (p_4)_\rho (p_5)_\sigma \varepsilon_{\mu\nu'\rho'\sigma'} p^{\nu'} (p_1)_{\rho'} (p_i)_{\sigma'}}{\sqrt{\Delta_3(P, p_4, p_5)} \sqrt{\Delta_3(P, p_1, p_i)}}$$

$$= \frac{\begin{vmatrix} P & p_4 & p_5 \\ P & p_1 & p_i \end{vmatrix}}{\sqrt{\Delta_3(P, p_4, p_5)} \sqrt{\Delta_3(P, p_1, p_i)}}$$

$\sin \gamma_i = n_2 n'_1$ where (B.2)

$$n'_1{}^\mu = \frac{\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\nu\alpha\rho\gamma} p^\alpha p_1^\beta p_i^\gamma (-m_i p_3)_\sigma (p_i)_\sigma}{\sqrt{\Delta_3(P, p_1, p_i)} \sqrt{\Delta_2(P, p_i)}} \frac{1}{m_i}$$

$$n_1'^\mu = - \frac{\begin{vmatrix} p^\mu & p^2 & p p_i \\ p_1^\mu & p_1 \cdot p & p_1 \cdot p_i \\ p_i^\mu & p_i \cdot p & m_c^2 \end{vmatrix}}{\sqrt{\Delta_3(p, p_1, p_i)} \sqrt{\Delta_2(p, p_i)}}$$

Then

$$\begin{aligned} \sin \gamma_i &= \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu(p_4) p_\rho(p_5) \sigma(p_1)_\mu \begin{vmatrix} p^2 & p p_i \\ p \cdot p_i & m_c^2 \end{vmatrix}}{\sqrt{\Delta_3(p, p_1, p_i)} \sqrt{\Delta_3(p, p_4, p_5)} \sqrt{\Delta_2(p, p_i)}} \\ &= \frac{\varepsilon(p, p_1, p_3, p_4) \sqrt{\Delta_2(p, p_i)}}{\sqrt{\Delta_3(p, p_1, p_i)} \sqrt{\Delta_3(p, p_4, p_5)}} \quad i = 4, 5 \end{aligned}$$

$$\cos(\gamma_4 \pm \gamma_5) = \frac{\begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_4 \end{vmatrix} \begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_5 \end{vmatrix} \mp \varepsilon^2(p, p_1, p_3, p_4) \sqrt{\Delta_2(p, p_4)} \sqrt{\Delta_2(p, p_5)}}{\Delta_3(p, p_4, p_5) \sqrt{\Delta_3(p, p_1, p_4)} \sqrt{\Delta_3(p, p_1, p_5)}} \quad (B.3)$$

By definition of $D_3^{\varepsilon'}(p, p_4, p_5)$ ($\varepsilon' = \pm 1$), [formula (A.7) of Appendix A]

$$\sqrt{\Delta_2(p, p_4)} \sqrt{\Delta_2(p, p_5)} = D_3^{\varepsilon'}(p, p_4, p_5) - \varepsilon' \begin{vmatrix} p & p_4 \\ p & p_5 \end{vmatrix}$$

from formula (A.3) of Appendix A

$$\begin{aligned} \begin{vmatrix} p & p_4 \\ p & p_5 \end{vmatrix} \begin{vmatrix} p & p_4 & p_5 & p_1 \\ p & p_5 & p_1 & p_4 \end{vmatrix} &= \begin{vmatrix} p & p_4 & p_5 \\ p & p_5 & p_1 \end{vmatrix} \begin{vmatrix} p & p_4 & p_1 \\ p & p_5 & p_4 \end{vmatrix} - \begin{vmatrix} p & p_4 & p_1 \\ p & p_5 & p_1 \end{vmatrix} \begin{vmatrix} p & p_4 & p_5 \\ p & p_5 & p_4 \end{vmatrix} \\ &= \begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_5 \end{vmatrix} \begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_4 \end{vmatrix} + \Delta_3(p, p_4, p_5) \begin{vmatrix} p & p_4 & p_1 \\ p & p_5 & p_1 \end{vmatrix} \end{aligned}$$

Then

$$\cos(\gamma_4 \pm \gamma_5) = \frac{-\varepsilon \varepsilon' \varepsilon^2(p, p_1, p_3, p_4) D_3^{\varepsilon'}(p, p_4, p_5) + \begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_5 \end{vmatrix} \begin{vmatrix} p & p_4 & p_5 \\ p & p_1 & p_4 \end{vmatrix} (1 \pm \varepsilon \varepsilon') - \varepsilon \varepsilon' \Delta_3 \begin{vmatrix} p & p_4 & p_1 \\ p & p_5 & p_1 \end{vmatrix}}{\Delta_3(p, p_4, p_5) \sqrt{\Delta_3(p, p_1, p_4)} \sqrt{\Delta_3(p, p_1, p_5)}} \quad (\varepsilon = \pm 1)$$

which shows that if $\varepsilon \varepsilon' = -1$, $D_3^{\varepsilon'}(P, p_4, p_5)$ does not appear in $\cos(\gamma_4 + \varepsilon \gamma_5)$. Namely $\cos(\gamma_4 \pm \gamma_5)$ is regular at $D_3^{\mp}(P, p_4, p_5) = 0$.

Note The γ_i are singular at $\Delta_3(P, p_1, p_i) = 0$, nevertheless the rotation $R(\varphi_i, \theta_i, \gamma_i)$ is regular at this point as it should be. By exactly the same kind of proof, it can be shown that $\varphi_i \pm \gamma_i$ is regular at $D_3^{\pm}(P, p_1, p_i)$ and, then, the matrix element of the 2×2 matrix R , namely

$$e^{\pm i(\gamma_i + \varphi_i)} \cos \frac{\theta_i}{2} \quad \text{and} \quad e^{\pm i(\gamma_i - \varphi_i)} \sin \frac{\theta_i}{2}$$

are regular $[\sin(\theta_i/2) \text{ (resp } \cos(\theta_i/2)) \text{ behaves like } \sqrt{D_3^{\pm}(P, p_1, p_i)} \text{ (resp } \sqrt{D_3^{\mp}(P, p_1, p_i)})}]$.

APPENDIX C

Behaviour near $s=0$ of $L(p_i)$ $i=1,2$ in the case $m_1=m_2$. Joos ¹¹⁾ has proved that the Lorentz transformation which takes the orthonormal tetrad (n_0, n_1, n_2, n_3) onto (n'_0, n'_1, n'_2, n'_3) has the following form

$$L = \pm \frac{\sigma n'_0 \tilde{\sigma} n_0 - \sigma n'_1 \tilde{\sigma} n_1 - \sigma n'_2 \tilde{\sigma} n_2 - \sigma n'_3 \tilde{\sigma} n_3}{N^{1/2}} \quad (C.1)$$

where σn (resp $\tilde{\sigma} n$) is equal to $\sigma^0 n^0 - \vec{\sigma} \vec{n}$ (resp $\sigma^0 n^0 + \vec{\sigma} \vec{n}$) and N is calculated in such a way that $\det L = +1$. Namely,

$$N^2 = 4 + 2 \sum_{\mu \neq \nu} [(\tilde{n}'_\mu \cdot n_\mu)(n'_\nu \cdot n_\nu) - (\tilde{n}'_\mu n_\nu)(n'_\nu \cdot n_\mu)] g^{\mu\mu} g^{\nu\nu} - 2i \sum_{\mu \neq \nu} \epsilon^{\alpha\beta\gamma\delta} (n_\mu)_\alpha (\tilde{n}'_\mu)_\beta (n_\nu)_\gamma (n'_\nu)_\delta g^{\mu\mu} g^{\nu\nu} \quad (C.2)$$

In our case, n'_3 (resp n'_1) the 3-axis (resp the 1-axis) of the helicity tetrad of particle i ($i=1,2$) behaves like $1/\sqrt{s}$ near $s=0$. Hence, apparently, N^2 behaves like $1/s$ since it contains a quadratic expression in n'_1 and n'_3 . But, actually, the residue of $1/s$ vanishes and N^2 behaves like $1/\sqrt{s}$.

Proof

a)

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta} (n_0)_\alpha (\tilde{n}'_3)_\beta (n_1)_\gamma (\tilde{n}'_1)_\delta &= \epsilon^{\alpha\beta\gamma\delta} (n_3)_\alpha (\tilde{n}'_3)_\beta (n_1)_\gamma \epsilon^{\delta\alpha'\beta'\gamma'} (\tilde{n}'_2)_{\alpha'} (\tilde{n}'_3)_{\beta'} (\tilde{n}'_0)_{\gamma'} \\ &= \begin{vmatrix} n_1 \cdot n'_2 & n_3 \cdot n'_3 & n_3 \cdot n'_0 \\ 0 & -1 & 0 \\ n_1 n'_2 & n_1 n'_3 & n_1 n'_0 \end{vmatrix} \\ &= - [(n_3 n'_2)(n_1 n'_0) - (n_3 n'_0)(n_1 n'_2)] \end{aligned}$$

which shows that this quantity behaves like a constant at $s=0$.

b)

$$\begin{aligned}
 (n_3 n'_3)(n_1 n'_1) &= (n_3 n'_3) \varepsilon^{\mu\nu\rho\sigma} (n_0)_\nu (n_2)_\rho (n_3)_\sigma \varepsilon_{\mu\nu\rho\sigma'} (n'_0)^\nu (n'_2)^\rho (n'_3)^{\sigma'} \\
 &= -(n_3 n'_3) \begin{vmatrix} n_0 & n_2 & n_3 \\ n'_0 & n'_2 & n'_3 \end{vmatrix} \\
 (n_3 n'_1)(n_1 n'_3) &= n_3^{\mu'} \varepsilon_{\mu'\nu'\rho'\sigma'} (n'_0)^\nu (n'_2)^\rho (n'_3)^{\sigma'} (n'_3)_\mu \varepsilon^{\mu\nu\rho\sigma} (n_0)_\nu (n_2)_\rho (n_3)_\sigma \\
 &= - \begin{vmatrix} n_3 n'_3 & 0 & 0 & -1 \\ 0 & n'_0 n_0 & n'_0 n_2 & n'_0 n_3 \\ 0 & n'_2 n_0 & n'_2 n_2 & n'_2 n_3 \\ -1 & n'_3 n_0 & n'_3 n_2 & n'_3 n_3 \end{vmatrix}
 \end{aligned}$$

which proves that

$$(n_3 n'_3)(n_1 n'_1) - (n_3 n'_1)(n_1 n'_3) = (n_0 n'_0)(n_2 n'_2) - (n'_0 n_2)(n'_2 n_0) \quad (C.3)$$

Formula (C.3) shows that this term behaves like a constant at $s=0$. Then N^2 behaves like $1/\sqrt{s}$ and L as $s^{-1/4}$.

APPENDIX D

In this Appendix, we test all the results obtained in this paper on examples for which one knows directly the relations between helicity amplitudes and invariant amplitudes which are free of kinematical singularity.

Actually, such invariant amplitudes exist in very few cases, namely, in the case 2 spin $\frac{1}{2}$ and 3 spin 0 particles and 1 spin 1 and 4 spin 0 particles ^{12), 21)}.

1. Case $\frac{1}{2}^+ + 0^+ \rightarrow \frac{1}{2}^+ + 0^+ + 0^+$

The following invariants form a good basis ²¹⁾

$$1, \not{p}_4, \not{p}_5, [\not{p}_4, \not{p}_5]_-$$

and

$$M_{\lambda_3 \lambda_1} = \bar{u}^{\lambda_3}(p_3) [A + B \not{p}_4 + C \not{p}_5 + D [\not{p}_4, \not{p}_5]_-] u_{\lambda_1}(p_1)$$

where

$$u_{\lambda_i}(p_i) = \frac{1}{\sqrt{2}} \begin{pmatrix} D^{1/2}(L(p_i)) \cdot \lambda_i \\ D^{1/2}(L^+(p_i)) \cdot \lambda_i \end{pmatrix}$$

and

$$\bar{u}^{\lambda_i}(p_i) = \frac{1}{\sqrt{2}} \left(D^{1/2}(L^-(p_i)) \cdot \lambda_i, D^{1/2}(L^+(p_i)) \cdot \lambda_i \right)$$

Since we only want to check our formulae, we assume for simplicity that C and D are equal to zero.

Then, we get, after a straightforward but tedious calculation in the R_1 frame

$$N_{13} M_{\lambda\lambda} = A \cos \frac{\theta_{13}}{2} [(m_3 + \omega_3)(m_1 + \omega_1) - |\vec{p}_1| |\vec{p}_3|] + B |\vec{p}_4| \left[\sin \frac{\theta_{13}}{2} \sin \theta_4 (|\vec{p}_3| (m_1 + \omega_1) e^{2i\lambda\varphi_4} + \right.$$

$$+ |\vec{p}_1| (m_3 + \omega_3) e^{-2i\lambda\varphi_4} + \frac{\omega\theta_{13}}{2} \omega\theta_4 (|\vec{p}_3|(m_1 + \omega_1) + |\vec{p}_1|(m_3 + \omega_3))]$$

$$N_{13} M_{\lambda-\lambda} = 2\lambda \left\{ A \sin \frac{\theta_{13}}{2} [(m_3 + \omega_3)(m_1 + \omega_1) + |\vec{p}_1||\vec{p}_3|] + B |\vec{p}_4| \left[\frac{\omega\theta_{13}}{2} \sin \theta_4 (|\vec{p}_3|(m_1 + \omega_1) e^{-2i\lambda\varphi_4} + \right. \right. \\ \left. \left. + |\vec{p}_1|(m_3 + \omega_3) e^{2i\lambda\varphi_4}) - \sin \frac{\theta_{13}}{2} \omega\theta_4 (|\vec{p}_3|(m_1 + \omega_1) - |\vec{p}_1|(m_3 + \omega_3)) \right] \right\}$$

where p_i ($\omega_i, \vec{p}_i$) has (φ_i, θ_i) for azimuthal and polar angles

$$N_{13} = \sqrt{2m_3(m_3 + \omega_3) 2m_1(m_1 + \omega_1)} = \sqrt{\frac{m_3 m_1}{s}} \sqrt{p_2^+(p_1, p_4)} \sqrt{p_2^+(p_1, p_3)}$$

i) Regularization at $\Delta_4(p, p_1, p_3, p_4) = 0$

Formula (II.11) reads

$$M_{\lambda_3 \lambda_1}(-\sqrt{\Delta_4}) = (-1)^{\lambda_3 - \lambda_1} M_{-\lambda_3 - \lambda_1}(\sqrt{\Delta_4})$$

Then,

$$F_{++}^1 = M_{++} + M_{--}, \quad F_{++}^2 = \frac{M_{++} - M_{--}}{\sqrt{\Delta_4}}, \quad \text{and} \quad F_{+-}^2 = \frac{M_{+-} + M_{-+}}{\sqrt{\Delta_4}}$$

$$F_{+-}^1 = M_{+-} - M_{-+}$$

must be regular at $\Delta_4(p, p_1, p_3, p_4)$. From the expression of $M_{\lambda_3; \lambda_1}$, we check that the only term which is singular at Δ_4 , namely

$$\sin \varphi_4 = \frac{\sqrt{\Delta_2(p, p_1)} \mathcal{E}(p, p_1, p_3, p_4)}{\sqrt{\Delta_3(p, p_1, p_3)} \sqrt{\Delta_3(p, p_1, p_4)}}$$

factorizes in $F_{+\pm}^2$ and does not appear in $F_{+\pm}^1$.

ii) Regularization at $\Delta_3(p, p_1, p_3) = 0$

We define now

$$G_{\lambda_3 \lambda_1}^i = \frac{\omega\theta_{13}}{2}^{|\lambda_3 + \lambda_1|} \sin \frac{\theta_{13}}{2}^{|\lambda_3 - \lambda_1|} F_{\lambda_3 \lambda_1}^i$$

Then G_{++}^1 (resp G_{+-}^1) is a function of $\cos^2(\theta_{13}/2)$ [resp $\sin^2(\theta_{13}/2)$] and of $\sin\theta_{13}\cos\varphi_4$ and $G_{\pm\pm}^2$ is a function of $\sin\theta_{13}\sin\varphi_4$ with

$$\cos\varphi_4 = \frac{\begin{vmatrix} p & p_1 & p_3 \\ p & p_1 & p_4 \end{vmatrix}}{\sqrt{\Delta_3(p, p_1, p_3)} \sqrt{\Delta_3(p, p_1, p_4)}}$$

Then $\sin\theta_{13}\sin\varphi_4$ and $\sin\theta_{13}\cos\varphi_4$ are regular at $\Delta_3(p, p_1, p_3)$ [$\sin\theta_{13}$ behaves like $\sqrt{\Delta_3(p, p_1, p_3)}$] and $G_{\pm\pm}^1 \lambda_3 \lambda_1$ are regular at Δ_4 and Δ_3 .

iii) Regularization at $\Delta_2(p, p_3)$

From formula (II.32)

$$\begin{aligned} G_{++}^1(-\sqrt{D_2^-(p, p_3)}) &= \mp G_{-+}^1(\sqrt{D_2^-(p, p_3)}) = \pm G_{+-}^1(\sqrt{D_2^-(p, p_3)}) \\ G_{++}^2(-\sqrt{D_2^-(p, p_3)}) &= \mp G_{-+}^2(\sqrt{D_2^-(p, p_3)}) = \mp G_{+-}^2(\sqrt{D_2^-(p, p_3)}) \end{aligned}$$

with

$$\begin{aligned} N_{13} G_{+, \pm}^1 &= A(1 \pm \cos\theta_{13})[(m_3 + \omega_3)(m_1 + \omega_1) \mp |\vec{p}_1| |\vec{p}_3|] + B |\vec{p}_4| [(|\vec{p}_3| (m_1 + \omega_1) \mp |\vec{p}_1| (m_3 + \omega_3)) \times \\ &\times \sin\theta_{13} \cos\varphi_4 \sin\theta_4 + (|\vec{p}_1| (m_3 + \omega_3) \pm |\vec{p}_3| (m_1 + \omega_1)) (1 \pm \cos\theta_{13}) \cos\theta_4] \end{aligned}$$

$$N_{13} G_{+, \pm}^2 = -i B |\vec{p}_4| (|\vec{p}_1| (m_3 + \omega_3) \pm |\vec{p}_3| (m_1 + \omega_1)) \frac{\sin\theta_{13} \sin\varphi_4 \sin\theta_4}{\sqrt{\Delta_4(p, p_1, p_3, p_4)}}$$

One turn around $\sqrt{D_2^-(p, p_3)} = 0$ induces the following change

$$|\vec{p}_3| \rightarrow -|\vec{p}_3|, \quad \sin\theta_{13} \rightarrow -\sin\theta_{13}, \quad \cos\theta_{13} \rightarrow -\cos\theta_{13}$$

Then $G_{\frac{1}{2}\frac{1}{2}}^1 \rightarrow G_{\frac{1}{2}-\frac{1}{2}}^1$ and $G_{\frac{1}{2}\frac{1}{2}}^2 \rightarrow -G_{\frac{1}{2}-\frac{1}{2}}^2$. For $\sqrt{D_2^+(p, p_3)}$ an extra sign appears due to the change $N_{13} \rightarrow -N_{13}$. Then $G_{\frac{1}{2}\frac{1}{2}}^1 \rightarrow -G_{\frac{1}{2}-\frac{1}{2}}^1$ and $G_{\frac{1}{2}\frac{1}{2}}^2 \rightarrow +G_{\frac{1}{2}-\frac{1}{2}}^2$.

iv) Regularization at $\Delta_2(P, p_1)$

Exactly the same procedure shows that formulae (II.41) and (II.42) are valid.

Note It is easy to check that the terms $|\vec{p}_4| \sin \theta_4 \cos \varphi_4$, $|\vec{p}_4| \cos \theta_4$ and $|\vec{p}_4| \sin \theta_4 \sin \varphi_4$ have no singularity at $\Delta_2(P, p_4) = 0$ and $\Delta_3(P, p_1, p_4) = 0$ since they are nothing but the scalar products $-p_4 \cdot \vec{n}_i$ ($i = 1, 2, 3$) where $\vec{n}_i$ are the vectors of the reference frame R_1 which are regular at $\Delta_2(P, p_4) = 0$ and $\Delta_3(P, p_1, p_4) = 0$.

2. Case $0^+ + 0^+ \rightarrow 0^+ \frac{1}{2} + \frac{1}{2}^+$

We have treated this case in order to check the regularization when particles 4 and 5 have spin. For the sake of simplicity we assume now that B, C and D are equal to zero.

The helicity amplitudes read

$$M_{\lambda_4 \lambda_5} = \bar{u}^{\lambda_4}(p_4) A v_{\lambda_5}(p_5) \quad \text{where} \quad v_{\lambda_i}(p_i) = \frac{1}{\sqrt{2}} \begin{pmatrix} D^{1/2}(L(p_i)E) \lambda_i \\ D^{1/2}(L^+(p_i)E^{-1}) \lambda_i \end{pmatrix}$$

In the centre-of-mass frame where $\hat{p}_4$ is along the 3-axis and $\hat{p}_5$ in the (3-1) plane,

$$M_{++} = -M_{--} = \frac{A}{N_{45}} \left[(\omega_4 + m_4) |\vec{p}_5| + (\omega_5 + m_5) |\vec{p}_4| \right] \sin \frac{\theta_{45}}{2}$$

$$M_{+-} = M_{-+} = \frac{A}{N_{45}} \left[-(\omega_4 + m_4) |\vec{p}_5| + (\omega_5 + m_5) |\vec{p}_4| \right] \cos \frac{\theta_{45}}{2}$$

i) Regularization $\Delta_4(P, p_1, p_3, p_4) = 0$

Formula (II.11) reads

$$M_{\lambda_4 \lambda_5}(-\sqrt{A_4}) = -(-1)^{2_4 - 2_5} M_{-\lambda_4 -\lambda_5}(\sqrt{A_4})$$

which means that $M_{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}}-M_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}}$ (resp $M_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}}+M_{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}}$) is regular at $\sqrt{\Delta}_4$. In our case ($B=C=D=0$), $M_{\frac{1}{2}\frac{1}{2}}=-M_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}}$ (resp $M_{\frac{1}{2}\frac{1}{2}}=M_{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}}$).

ii) Regularization $\Delta_3(P, p_4, p_5)=0$

We define

$$\hat{M}_{\lambda_4 \lambda_5} = \sin \frac{\theta_{45}}{2}^{\lambda_4 + \lambda_5} \omega \frac{\theta_{45}}{2}^{\lambda_4 - \lambda_5} M_{\lambda_4 \lambda_5}$$

Then,

$$\begin{aligned} M_{+\pm} &= \frac{A}{N_{45}} \frac{(1 \mp \omega \frac{\theta_{45}}{2})}{2} \left[\pm (\omega_4 + m_4) |\vec{p}_5| + (\omega_5 + m_5) |\vec{p}_4| \right] \\ &= \mp M_{-\mp} \end{aligned}$$

is obviously regular at $\cos \theta_{45} = 0$.

iii) Regularization at $\Delta_2(P, p_4)=0$

From (II.35)

$$\hat{M}_{\lambda_4 \lambda_5} \pm (-1)^{\lambda_4 + \lambda_5} \hat{M}_{-\lambda_4 \lambda_5}$$

is regular at $D_2^\mp(P, p_4)=0$. We check that

$$\hat{M}_{++} - \hat{M}_{-+} = \frac{A}{N_{45}} \left[(\omega_4 + m_4) |\vec{p}_5| - (\omega_5 + m_5) |\vec{p}_4| \omega \frac{\theta_{45}}{2} \right]$$

is regular at $D_2^-(P, p_4)=0$ $\left[\sqrt{\Delta}_2(P, p_4) \cos \theta_{45} \text{ is regular} \right]$ and that

$$\hat{M}_{++} + \hat{M}_{-+} = \frac{A}{N_{45}} \left[-(\omega_4 + m_4) |\vec{p}_5| \omega \frac{\theta_{45}}{2} + (\omega_5 + m_5) |\vec{p}_4| \right]$$

is regular at $D_2^+(P, p_4)=0$

$$(N_{45} \simeq \sqrt{\omega_4 + m_4} \simeq D_2^+(P, p_4))$$

iii) Regularization at $\Delta_2(P, p_5) = 0$

From (II.38) $\hat{M}_{\lambda_4 \lambda_5} \pm \epsilon_5 \hat{M}_{\lambda_4 - \lambda_5} \quad \epsilon_5 = (-1)^{s - \lambda_4}$

is regular at $D_2^\mp(P, p_5) = 0$ which means that $\hat{M}_{\frac{11}{22}} \pm \hat{M}_{-\frac{11}{22}}$ are regular at $D_2^\mp(P, p_5) = 0$ as one can check in their explicit expression.

3. Regularizations and constraints for the case $00 \rightarrow 100$

In this case the following covariants p_1, p_2, p_4 form a good basis ²¹⁾ and

$$M_{\lambda_3} = \epsilon_{(p_3)}^\mu M_{\lambda_3 \mu}$$

where $\epsilon_{(p_3)}^\mu$ is the polarization of the vector particle

$$(\epsilon^\mu(p_3)_{\lambda_3} \cdot (p_3)_\mu = 0)$$

and

$$M_\mu = \sum_{\bar{R}=1,2,4} A^{\bar{R}} p_{\bar{R}}$$

where the $A^{\bar{R}}$'s have no kinematical singularities. ($\prod_i n_i = -1$)

A. Kinematical Singularities

$$M_{\pm 1} = \mp \frac{(n_1(p_3) \mp i n_2(p_3))^\mu}{\sqrt{2}} M_{\mu}$$

$$M_0 = \frac{n_3^\mu(p_3)}{\sqrt{2}} M_\mu$$

where the $n_i(p_3)$ ($i=1,2,3$) are the space-like axis of the helicity tetrad of particle 3. Namely,

$$n_3^\mu(p_3) = - \frac{\begin{vmatrix} m_3^2 & p \cdot p_3 \\ p_3^\mu & p^\mu \end{vmatrix}}{m_3 \sqrt{\Delta_2(p, p_3)}}, \quad n_2^\mu(p_3) = \frac{\epsilon^{\mu\nu\rho\sigma} p_\nu(p_1) p_\rho(p_3) p_\sigma}{\sqrt{\Delta_3(p, p_1, p_3)}}$$

and

$$n_1^\mu(p_3) = \frac{\begin{vmatrix} p^\mu & p_3^\mu & p_1^\mu \\ p^2 & p \cdot p_3 & p \cdot p_1 \\ p \cdot p_3 & p_3^2 & p_3 \cdot p_1 \end{vmatrix}}{\sqrt{\Delta_3(p, p_1, p_3)} \sqrt{\Delta_2(p, p_3)}}$$

With these definitions

$$M_0 = - \sum_R A^R \frac{\begin{vmatrix} p_3 & p_R \\ p_3 & p \end{vmatrix}}{m_3 \sqrt{\Delta_2(p, p_3)}}$$

$$M_{\pm 1} = \frac{1}{\sqrt{2} \sqrt{\Delta_3(p, p_1, p_3)}} \sum_R A^R \left[i \varepsilon(k, p, p_1, p_3) \mp \frac{\begin{vmatrix} p & p_3 & p_1 \\ p & p_3 & p_R \end{vmatrix}}{\sqrt{\Delta_2(p, p_3)}} \right]$$

we can now test the regularization.

i) $\Delta_{-4}(p, p_1, p_3, p_4) = 0$

$$M_{\lambda_3}(-\sqrt{\Delta_4}) = -(-1)^{s_3 + \lambda_3} M_{-\lambda_3}(\sqrt{\Delta_4})$$

Then,

$$M_0(-\sqrt{\Delta_4}) = M_0(\sqrt{\Delta_4})$$

$$M_{\pm 1}(-\sqrt{\Delta_4}) = -M_{\mp 1}(\sqrt{\Delta_4})$$

which means that

$$M_0, \quad G_{+1}^1 = M_{+1} - M_{-1} = -\sqrt{2} \sum_R A^R \frac{\begin{vmatrix} p & p_3 & p_1 \\ p & p_3 & p_R \end{vmatrix}}{\sqrt{\Delta_2(p, p_3)} \sqrt{\Delta_3(p, p_1, p_3)}}$$

and

$$G_{+1}^2 = \frac{M_{+1} + M_{-1}}{\sqrt{\Delta_4(p, p_1, p_3, p_4)}} = -\frac{i \sqrt{2} A^4}{\sqrt{\Delta_3(p, p_1, p_3)}}$$

are regular at $\Delta_4 = 0$.

ii) $\Delta_{-3}(p, p_1, p_3) = 0$

From the regularization, the functions

$$H_{\lambda_3}^i = \frac{1}{2} \sin \theta_{13} |\lambda_3| G_{\lambda_i}^i$$

are regular at $\Delta_3(P, p_1, p_3) = 0$. Recalling that

$$\sin \theta_{13} = \frac{\sqrt{s} \sqrt{\Delta_3(P, p_1, p_3)}}{\sqrt{\Delta_2(P, p_1)} \sqrt{\Delta_2(P, p_3)}}$$

then

$$H_0^1 = M_0 \quad H_{+1}^1 = -\sqrt{s} \sum_k A_k \frac{\begin{vmatrix} P & p_3 & p_1 \\ P & p_3 & p_k \end{vmatrix}}{\sqrt{2} \sqrt{\Delta_2(P, p_1)} \Delta_2(P, p_3)}$$

and

$$H_{+1}^2 = \frac{-i \sqrt{s} A^4}{\sqrt{2} \sqrt{\Delta_2(P, p_3)} \sqrt{\Delta_2(P, p_1)}}$$

are regular at both $\Delta_4(P, p_1, p_3, p_4) = 0$ and $\Delta_3(P, p_1, p_3) = 0$.

iii) $\Delta_2(P, p_3) = 0$

Using formula (II.32)

$$H_{\lambda_3}^1(-\sqrt{\Delta_2(P, p_3)}) = (-1)^{s_3} H_{-\lambda_3}^1(+\sqrt{\Delta_2(P, p_3)}) = (-1)^{s_3+\lambda_3} H_{\lambda_3}^1(\sqrt{\Delta_2(P, p_3)})$$

and

$$H_{\lambda_3}^2(-\sqrt{\Delta_2(P, p_3)}) = (-1)^{s_3} H_{-\lambda_3}^2(+\sqrt{\Delta_2(P, p_3)}) = -(-1)^{s_3+\lambda_3} H_{\lambda_3}^2(\sqrt{\Delta_2(P, p_3)})$$

which means that H_{+1}^1 is uniform at $\Delta_2(P, p_3) = 0$ whereas H_{+1}^2 and H_0^1 does exhibit a $\sqrt{\Delta_2(P, p_3)}$ singularity.

Note The $\Delta_2(P, p_1)$ singularity does not exist in the M_{λ_3} . But, due to our procedure of regularization, this singularity appears in $H_{\lambda_3}^1$. From formulae (II.41) and (II.42)

$$H_{\lambda_3}^i(-\sqrt{\Delta_2(P, p_1)}) = (-1)^{\lambda_3} H_{\lambda_3}^i(\sqrt{\Delta_2(P, p_1)})$$

which shows that H_0^1 is regular at $\Delta_2(P, p_1) = 0$, whereas $H_{\pm 1}^1$ exhibits a $\sqrt{\Delta_2(P, p_1)}$ singularity. At $s=0$ the H_{λ_i} are regular from the general study and $\sqrt{s} |\lambda_3|$ factorizes out.

B. Kinematical Constraints at $\Delta_2(P, p_3) = 0$

From formula (III.26)

$$\sqrt{\Delta_2(P, p_3)}^{s_3} T_{\tau_3} \simeq \sqrt{D_2^-(P, p_3)}^{s_3 + \tau_3}$$

near $D_2^-(P, p_3) = 0$ if $\text{tg } \theta_{13} = +i$

$$\sqrt{\Delta_2(P, p_3)}^{s_3} T_{\tau_3} \simeq \sqrt{D_2^+(P, p_3)}^{s_3 + \tau_3}$$

near $D_2^+(P, p_3) = 0$ if $\text{tg } \theta_{13} = -i$

$$T_{\tau_3} = D^1(R^{-1})^{\lambda_3}_{\tau_3} M_{\lambda_3}$$

where

$$D^1(R^{-1}) = \begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ -i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & -i\sqrt{2} & 1 \end{pmatrix}$$

In the vicinity of $D_2^-(P, p_3) = 0$ ($\sqrt{s} m_3 = P \cdot p_3$)

$$\sqrt{\Delta_2(P, p_3)} T_1 \simeq D_2^-(P, p_3)$$

if $\text{tg } \theta_{13} = +i$ or

$$\frac{\sqrt{s} \sqrt{\Delta_3(P, p_1, p_3)}}{\begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix}} = -i$$

We must check that

$$\sqrt{\Delta_2(P, p_3)} T_1 = \sqrt{\Delta_2(P, p_3)} [M_{+1} - M_{-1} - i\sqrt{2} M_0]$$

vanishes at $D_2^-(P, p_3) = 0$ when $\sqrt{s} = (P \cdot p_3)/m_3$ and

$$\sqrt{\Delta_3(P, p_1, p_3)} = -i \frac{\begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix}}{\sqrt{s}}$$

From the explicit expression of M_{λ_i}

$$\sqrt{\Delta_2(p, p_3)} T_1 = \frac{-i\sqrt{2}}{m_3 \begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix}} \sum_k A^k \left[(p, p_3) \begin{vmatrix} p & p_3 & p_1 \\ p & p_3 & p_k \end{vmatrix} - \begin{vmatrix} p_3 & p_k \\ p_3 & p \end{vmatrix} \begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix} \right]$$

From formula (A.3)

$$\begin{vmatrix} p \\ p_3 \end{vmatrix} \begin{vmatrix} p & p_3 & p_1 \\ p_3 & p & p_k \end{vmatrix} = \begin{vmatrix} p & p_3 \\ p_3 & p \end{vmatrix} \begin{vmatrix} p & p_1 \\ p_3 & p_k \end{vmatrix} - \begin{vmatrix} p & p_3 \\ p_3 & p_k \end{vmatrix} \begin{vmatrix} p & p_1 \\ p_3 & p \end{vmatrix}$$

which shows that

$$\sqrt{\Delta_2(p, p_3)} T_1 = \frac{i\sqrt{2} \Delta_2(p, p_3)}{m_3 \left(\begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix} \right)} \sum_k A^k \begin{vmatrix} p & p_1 \\ p_3 & p_k \end{vmatrix}$$

vanish at $D_2^-(p, p_3) = 0$. Exactly the same kind of proof holds for the constraint at $D_2^+(p, p_3) = 0$, namely

$$\sqrt{\Delta_2(p, p_3)} T_1 \simeq D_2^+(p, p_3)$$

if
$$\frac{\sqrt{s} \sqrt{\Delta_3(p, p_1, p_3)}}{\begin{vmatrix} p & p_1 \\ p & p_3 \end{vmatrix}} = +i$$

and

$$\sqrt{s} = - \frac{p \cdot p_3}{m_3}$$

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