Boost Mass and the Mechanics of Accelerated Black Holes

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Dedicated to Rafael Sorkin for his many contributions to our understanding of physics.

ABSTRACT

In this paper we study the concept of the boost mass of a spacetime and investigate how variations in the boost mass enter into the laws of black hole mechanics. We define the boost mass as the gravitational charge associated with an asymptotic boost symmetry, similar to how the ADM mass is associated with an asymptotic time translation symmetry. In distinction to the ADM mass, the boost mass is a relevant concept when the spacetime has stress energy at infinity, and so the spacetime is not asymptotically flat. We prove a version of the first law which relates the variation in the boost mass to the change in the area of the black hole horizon, plus the change in the area of an acceleration horizon, which is necessarily present with the boost Killing field, as we discuss. The C-metric and Ernst metric are two known analytical solutions to Einstein-Maxwell theory describing accelerating black holes which illustrate these concepts.

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1 Introduction

The close association between symmetries and conservation laws is a truth universally acknowledged amongst physicists. Invariance under time translation is associated with a conserved energy, or mass; spatial translations with conserved momentum; and rotational symmetry with conserved angular momentum. Perhaps the most famous symmetry of Minkowski spacetime is its invariance under Lorentz transformations or *boosts*. However, a *boost charge* is conspicuously absent from the preceding list. The purpose of this paper is to study this seemingly neglected type of charge in the context of general relativity. In general relativity, spacetime symmetries are generated by Killing vector fields. If a spacetime with stress-energy $T_{ab}$ has a Killing vector $V^a$, one way to obtain a conserved charge is via the conserved current $j^a = T^{ab}V_b$. The integral of the normal component of $j^a$ over a spacelike hypersurface is conserved. However, conserved quantities also exist for spacetimes that have symmetries only asymptotically. In asymptotically flat spacetimes, for example, the ADM mass, momentum and angular momentum are conserved and defined by surface integrals evaluated at spatial infinity [1]. Abbott and Deser [2] showed that similar conserved charges, also defined as boundary integrals at infinity, exist for any class of spacetimes that are asymptotic at spatial infinity to a fixed background metric having symmetries. For each Killing vector $V^a$ of this reference metric $g_{ab}^{(ref)}$, there is a conserved charge which we can denote generally as $Q [V^a, g_{ab}^{(ref)}]$. Regge and Teitelboim [3] showed how the original ADM charges arise as boundary terms in the Hamiltonian formulation of general relativity with asymptotically flat boundary conditions. Hawking and Horowitz [4] found similar results for the case of a general asymptotic background. They define gravitational charges by varying the Einstein action.

It is clear then that for asymptotically flat spacetimes, there exists a conserved charge, corresponding to the boost symmetry of Minkowski spacetime, which we will call $M_{boost}$. It is also relatively simple to understand why the boost mass $M_{boost}$ is neglected in most considerations of asymptotically flat spacetimes. The ADM mass, $M_{ADM}$, and the boost mass, $M_{boost}$, are, of course, zero for Minkowski spacetime. One can show that if $M_{ADM}$ is increased from zero, then $M_{boost}$ becomes infinite. This happens essentially because the boost Killing vectors of Minkowski spacetime diverge at spatial infinity, while the time translation Killing vector is constant. According to the positive energy theorem $M_{ADM}$ vanishes only in Minkowski spacetime [5]. Therefore, once the metric deviates from Minkowski spacetime, the boost mass necessarily becomes infinite. This is discussed in more detail in section 4.2. We then see that $M_{boost}$ is uninteresting for asymptotically flat spacetimes and that to examine
its properties, we need to find a more appropriate physical setting \(^2\).

If the background spacetime contains stress-energy in the asymptotic region, then the story is different in an interesting way. For example, suppose that the background spacetime is a static, straight cosmic string. This is not asymptotically flat at spatial infinity, because the string extends to infinity\(^3\). We will model the cosmic string metric by flat spacetime minus a wedge, and denote the resulting metric by \(\eta^{(\nu)}_{ab}\), where \(\nu\) parameterizes the missing angle. The metric \(\eta^{(\nu)}_{ab}\) is symmetric with respect to a time translation Killing vector \(T^a\), with respect to translations along the string and rotations around it, and also with respect to boosts along the string generated by the Killing vector \(\xi^a\). In this paper, we will define gravitational charges for spacetimes that are asymptotic to \(\eta^{(\nu)}_{ab}\). Of course, if the infinite string itself is compared with Minkowski spacetime, then it will have an infinite mass. However, the charges we compute represent the finite residual contributions after subtracting off the contribution of the infinite string. The development parallels the asymptotically flat case. If the metric is everywhere exactly \(\eta^{(\nu)}_{ab}\), then the charges \(Q[V^a, \eta^{(\nu)}_{ab}]\) vanish\(^4\). Also as in the asymptotically flat case, we will continue to call the charge \(Q[T^a]\) the ADM mass, and \(Q[\xi^a]\) the boost mass. However, it is now possible to find spacetimes that have \(M_{\text{ADM}} = 0\), but are not everywhere equal to \(\eta^{(\nu)}_{ab}\)!

For example, imagine clipping a segment of the string from the interior, and adding a ball of mass of some sort to each free end. If this is done so that, roughly speaking, we are just moving mass around, then \(M_{\text{ADM}}\) remains zero, but \(M_{\text{boost}}\) becomes nonzero. The C-metric provides a well known example of such a rearrangement, in which the balls of mass at the string ends are themselves black holes [8, 9]. As a second example, let the background reference metric be the Melvin spacetime, which has a non-vanishing magnetic field everywhere [10]. Again, this is not asymptotically flat, and because the background has stress-energy, we can again imagine rearranging the mass in the interior, such that the monopole moment of the mass distribution is unchanged in the far field. Since \(M_{\text{ADM}}\) measures this monopole moment, it would still vanish. On the other hand, the boost mass \(M_{\text{boost}}\), we will see, essentially measures the dipole moment of the mass distribution and would be both non-infinite and generically nonzero. The Ernst spacetime is an example of this sort of rearrangement,

\(^2\)References [6, 7] define the boost energy as the volume integral of the time component of the local current \(\int dv (n_a j^a)\). This boost energy is not the same as the boundary term that serves as our definition of \(M_{\text{boost}}\). This is discussed in section 2.2.

\(^3\)It is asymptotically locally flat if one moves to spatial infinity only in directions transverse to the string.

\(^4\)If the reference metric is clear from the context, we will simply write \(Q[V^a]\), rather than \(e.g. Q[V^a, \eta^{(\nu)}_{ab}]\) in the following.
in which two charged black holes are accelerated apart by the background magnetic field. We will discuss these two examples, the C-metric and the Ernst metric, below in sections 5 and 6 respectively.

The reader may wonder how we distinguish between a boost Killing vector and a time translation Killing vector in a general spacetime. We will call $\xi^a$ a boost Killing vector if it is timelike in a region, which includes a part of infinity, and which is bounded in the interior by an acceleration horizon. By an acceleration horizon we will mean a surface where $\xi^a$ becomes null, and which has noncompact spatial slices. These definitions are motivated by the behavior of the boost Killing vector $\xi^a = x(\frac{\partial}{\partial t})^a + t(\frac{\partial}{\partial x})^a$ in Minkowski spacetime and are discussed in more detail in section 4.1.

After studying $M_{\text{boost}}$ in the context of these examples, we will turn our attention to generalizations of the first law of black hole mechanics, or thermodynamics, that take into account variations, $\delta M_{\text{boost}}$, in the boost mass. The usual first law of black hole thermodynamics applies to black holes in asymptotically flat spacetimes, for which the generator of the black hole horizon is a Killing vector $T^a$ that becomes time translation at infinity. The Killing vector $T^a$ is timelike outside the black hole and becomes null on the horizon. The first law links variations of properties of the spacetime evaluated at infinity to variations evaluated at the black hole horizon. For example, in the simplest uncharged and non-rotating case, we have

$$\delta M_{\text{ADM}} = \frac{\kappa_{bh}}{8\pi} \delta A_{bh}$$

where $M_{\text{ADM}}$ is evaluated at infinity and the horizon area $A_{bh}$ is evaluated at the horizon. Now suppose instead that the generator of the black hole horizon is a Killing vector $\xi^a$, which is null on the horizon, timelike in a region outside the black hole and asymptotes to a boost at infinity. When we say that $\xi^a$ is a boost we will mean that there is also an acceleration horizon, on which the Killing vector $\xi^a$ is null. An analogue of the usual first law, the variations about this black hole spacetime will involve $\delta M_{\text{boost}}$ at infinity, $\delta A_{bh}$ at black hole horizon, and also $\delta A_{\text{acc}}$, the variation of the area of the acceleration horizon. The contribution from the acceleration horizon arises because it is an additional boundary on which the Killing field becomes null. In section 4.3 we derive such a first law for variations in the boost mass

$$\delta M_{\text{boost}} = \frac{1}{8\pi} \kappa_{bh} \delta A_{bh} + \frac{1}{8\pi} \kappa_{\text{acc}} \delta A_{\text{acc}}.$$ 

This equation is written for purely gravitational case, in which there are no stress-
energy sources and no variation in the electric charge. Such additional terms are included in section 4.3. A similar expression is discussed by Jacobson and Parentani [12]. There are differences between this earlier paper and this one and we will discuss this issue in section 4.3.

In section 4.2 we give the expression for $M_{\text{boost}}$ when the spacetime is asymptotically Rindler spacetime minus a wedge $\eta^{(\nu)}$, with a specified missing angle and an acceleration parameter. We identify the fall-off conditions on the metric such that $M_{\text{boost}}$ is finite. Later, a simple example of the theorem is provided by working out $\delta M_{\text{boost}}$ when the variation is with respect to a nearby C-metric. The asymptotically Melvin case is considered in section 6.

2 Perturbative Constraints on Charges

In this section we show that gravitational charges can be defined in a useful way for spacetimes which approach a reference metric, whenever the reference spacetime has a Killing field, not just in the asymptotically flat case. The definition of the charge that we present is useful, because we can then prove a relation analogous to the usual first law of black holes for variations in the charge. Readers familiar with the result of equation (17) at the end of this section, may skip this section without loss of continuity.

2.1 Basic Hamiltonian Formalism

In this subsection we set up the Hamiltonian formalism of General Relativity. More details of the Hamiltonian formalism are given in references [13, 14].

The calculations are little involved but the idea is simple. Let the spacetime $(M, g_{ab})$ be foliated by a family of spacelike slices $(\Sigma_t)$ with a timelike vector field $\frac{\partial}{\partial t}$, and a unit normal field $n^a = -N \nabla^a t$. Let $g_{ab}$ be a Lorentzian metric satisfying Einstein’s Equation $G_{ab} = 8\pi T_{ab}$, and $\nabla_a$ be the derivative operator compatible with $g_{ab}$, i.e., $\nabla_c g_{ab} = 0$.

The spacetime metric $g_{ab}$ induces a spatial metric $s_{ab}$ on the constant time spacelike hypersurfaces $\Sigma_t$,

$$g_{ab} = s_{ab} - n^a n_b, \quad n^a s_{ab} = 0,$$

and $n \cdot n = -1$.

Here we will consider Einstein-Maxwell theory. The formalism can be generalized for any energy momentum tensor $T^{ab}$, describing a matter field which has a well
defined Hamiltonian formalism. In Einstein-Maxwell theory, a point in the phase space is specified by the initial data \((s_{ab}, \pi^{ab}, \tilde{A}_a, E^a)\) on a spacelike surface \(\Sigma\), where \(\tilde{A}_a = s_c^a A_c\) is the projection onto \(\Sigma\) of the spacetime gauge potential \(A_a\). \(\pi^{ab}\) is the momentum conjugate to \(s_{ab}\), and is related to the extrinsic curvature \(K_{ab}\) of \(\Sigma\)

\[
\pi^{ab} = \sqrt{s}(K^{ab} - s^{ab} K),
\]

where \(s = \text{det}[s_{ab}]\). The momentum conjugate to \(\tilde{A}_a\) is proportional to the electric field \(E^a\).

Initial data must satisfy the Einstein constraints, which are non-dynamical equations. In the Hamiltonian variables, the constraints on \(\Sigma\) are

\[
0 = C = \frac{1}{4\pi} \sqrt{s} D_a E^a,
\]

\[
0 = C_0 = \frac{1}{16\pi} \sqrt{s}[-R] + 2 E_a E^a + \tilde{F}_{ab} \tilde{F}^{ab} + \frac{1}{2s}(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2),
\]

\[
0 = C_a = -\frac{1}{8\pi} \sqrt{s}[D_b(\pi^b_a / \sqrt{s}) - 2 \tilde{F}_{ab} E^b],
\]

where \(D_a\) is the derivative operator on \(\Sigma\) compatible with \(s_{ab}\), \(R\) denotes the scalar curvature of \(s_{ab}\) and \(\tilde{F}_{ab} = 2D_{[a} \tilde{A}_{b]}\). The Hamiltonian \(\mathcal{H}_{tot}\) for Einstein-Maxwell theory is a sum of the constraints,

\[
\mathcal{H}_{tot} = NC_0 + N^a C_a + A_t C \equiv (N, N^a, A_t) \cdot H_{tot}
\]

Here \(N\), \(N^a\) and \(A_t\) are Lagrange multipliers, which may be prescribed arbitrarily. The variations of the Hamiltonian with respect to the Lagrange multipliers give the constraint equations, and the usual Hamilton’s equations give the evolution of the dynamical variables \((s_{ab}, \pi^{ab}, \tilde{A}_a, E^a)\). The vector \(\bar{w}^a = N n^a + N^a\) represents the flow of time in the spacetime and the Hamiltonian generates this time flow. The projection of \(\bar{w}\) onto \(\Sigma\) yields the shift vector \(N^a\) and the projection normal to \(\Sigma\) yields the lapse function \(N\). \(\mathcal{H}_{tot}\) is identically zero on solutions.

### 2.2 Definition of Gravitational Charges

Let \(g_{ab}^{(\text{ref})}\) be a fixed spacetime which we call the reference, and which has a Killing field \(V^a\). Let \(g_{ab}\) be a metric which asymptotes to the reference spacetime \(g_{ab}^{(\text{ref})}\). Let \({}^{(4)}\gamma_{ab} = g_{ab} - g_{ab}^{(\text{ref})}\) be the difference between the spacetime metric and the reference metric. Note that \({}^{(4)}\gamma_{ab}\) is the perturbation to the full spacetime metric,
distinguished from $\gamma_{ab}$, the perturbation to the spatial metric. The definition of $Q_V$, the gravitational charge associated with the asymptotic Killing field $V^a$, will depend on $\gamma_{ab}$ only in the asymptotic region, on $g_{ab}^{\text{ref}}$, and on the boundary $\partial \Sigma_{\text{asy}}$ of the volume $\Sigma$. The idea is that we define $Q_V$ as a boundary integral of the form

$$Q_V(g^{\text{ref}}, g) = \frac{1}{16\pi} \int_{\partial \Sigma_{\text{asy}}} B^c [s^{(\text{ref})}, \pi^{(\text{ref})}, \gamma, \delta \pi, V] da_c, \tag{9}$$

and we will choose the integrand $B^c$ such that there is a theorem of the form of the first law for the variations of $Q_V$. The expression for the boundary integrand $B^c$ is given in equations (12) and (13). We will often shorten the notation to $Q_V$.

So far we have not said anything about the rate at which $\gamma_{ab}$ must go to zero. Indeed, for a particular metric $g_{ab}$ and hence a particular $\gamma_{ab}$, the charge might be finite for one Killing vector, but infinite or zero for a different Killing vector. One of the interesting issues in sorting out the meaning of the boost mass, is to understand when it is infinite, or finite, or zero. We do this in section 4.2. Depending on the background and on the spacetime of interest, either the boost mass or the ADM mass will give useful information, but not both.

If the reference metric is Minkowski, then asymptotically the spacetime has all the Poincare symmetries, and we can define Killing charges corresponding to all generators of the Poincare group. The $Q_V$ are the different conserved charges, as Killing vector $V^a$ ranges over all the generators. The integral in equation (9) reduce to the usual ADM mass and angular momentum when the Killing vector is taken to be time translation or a rotation respectively. Regge and Teitleboim [3] have shown that the boundary terms satisfy the correct algebra of Minkowski spacetime.

For two perturbatively close metrics $g_{ab}$ and $g_{ab}^{(0)}$, we will next use this definition to prove a theorem about the variations of the Killing charge $\delta Q$ which is defined by the same expression as in equation (9), with $g_{ab} = g_{ab}^{(0)} + \lambda \gamma_{ab} + \mathcal{O}(\lambda^2)$, $\gamma_{ab}$ now being the perturbation to the metric $g_{ab}^{(0)}$. Both $g_{ab}$ and $g_{ab}^{(0)}$ asymptote to the metric $g_{ab}^{\text{ref}}$. In this case we consider $g_{ab}^{(0)}$ as our background.

We will see that $M_{\text{boost}}$, the Killing charge corresponding to the asymptotic boost symmetry of the background spacetime, will play an important role in this paper. As previously mentioned, references [6, 7] define a boost energy $E_{\text{boost}}$, in spacetimes which have a boost Killing vector everywhere. $E_{\text{boost}}$ is the integral over the volume $\Sigma$ of $n_a \xi^b T^a_b$, where $\xi^a$ is the boost Killing vector. When the background is de Sitter, anti-de Sitter or Minkowski spacetime, the construction of Abbott and Deser [2] shows that the boundary term defining the boost charge $Q_V$ in equation (9) is equal
to $E_{\text{boost}}$, plus a volume integral of nonlinear terms from the Einstein tensor. This construction would have to be repeated with $g_{ab}^{(0)}$ taken to be a cosmic string or magnetic field background, to see the same type of relation holds here.

2.3 Gauss’s Law for Perturbations

In this subsection we study solutions to the linearized Einstein equations. Let $g_{ab}$ and $g_{ab}^{(0)}$ both approach the reference $g_{ab}^{(ref)}$, and consider the case when the two metrics are perturbatively close everywhere. Suppose that $g_{ab}^{(0)}$ has a Killing field $V^a$. $V^a$ is a Killing field throughout the spacetime, not just asymptotically. Then perturbations about the zeroth-order spacetime satisfy a Gauss’s Law type constraint. First we summarize the derivation of this result. Then in section 3 we outline how this result yields the first law of black hole mechanics, when $g_{ab}^{(0)}$ is asymptotically flat and $V^a$ is a time translation Killing vector of a static black hole spacetime. Finally in section 4.3 we apply the constraint to derive the first law of black hole mechanics when $g_{ab}^{(0)}$ is asymptotic to flat spacetime minus a wedge $\eta^{(0)}$, and $g_{ab}^{(0)}$ has a boost Killing vector $\xi^a$.

Let $(s_{ab}(\lambda), \pi^{ab}(\lambda), \tilde{A}_a(\lambda), E^a(\lambda))$ be a one parameter family of solutions to the Einstein-Maxwell theory with perturbative expansion $s_{ab} = s_{ab}^{(0)} + \lambda h_{ab} + \mathcal{O}(\lambda^2) + \ldots$, and $\tilde{A}_a = \tilde{A}_a^{(0)} + \lambda \tilde{A}_a^{(1)} + \mathcal{O}(\lambda^2) + \ldots$, and similarly for the corresponding conjugate momenta. $s_{ab}^{(0)}$ is the spatial metric induced by $g_{ab}^{(0)}$ on the constant time hypersurface $\Sigma$. So, the set $(p^{(0)}, q^{(0)})$ is a solution to the Einstein-Maxwell equation with a Killing vector; $(p^{(1)}, q^{(1)})$ solve the equations linearized about the zeroth order solutions, and so on. In particular, $(p^{(1)}, q^{(1)})$ solve the linearized constraints $\delta H_{tot} = 0$.

Let $F, \beta^a$ be an arbitrary function and vector on $\Sigma$, and consider the linear combination of the constraints $(F, \beta^a, A_t) \cdot H_{tot}(s_{ab}, \pi^{ab}, \tilde{A}_a, E^a) = 0$. Then the perturbative fields $(p^{(1)}, q^{(1)})$ are solutions to the following linearized constraints,

$$ (F, \beta^a, A_t) \cdot \delta H_{tot} \cdot (p^{(1)}, q^{(1)}) = 0. \quad (10) $$

We can rewrite this equation in terms of the adjoint operator and a total derivative,

$$ (p^{(1)}, q^{(1)}) \cdot \delta H_{tot}^* \cdot (F, \beta^a, A_t) + D_a B^a = 0, \quad (11) $$

where $B^a$ is a function of the $(p^{(1)}, q^{(1)})$, the Lagrange multipliers, and of course the

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5More precisely, the Einstein tensor is formally expanded in $(^4g)_{ab}$, and let $G_{ab}^{(NL)} = G_{ab} - G_{ab}^{(L)}$, where $G_{ab}^{(L)}$ is the term linear in $\gamma$. Then $Q_{\text{boost}} = E_{\text{boost}} + \int_{\Sigma} G_{ab}^{(NL)} h^a \xi^b$. 

background spacetime. The boundary term vector $B^a$ is the sum of a gravitational piece and a contribution from the matter fields, $B^a[\mathcal{E}^{(\text{ref})}, \pi^{(\text{ref})}, h, \delta \pi, V] = B^a_G + B^a_M$, where

$$B^a_G = F(D^a h - D_b h^{ab}) - h D^a F + h^{ab} D_b F + \frac{\beta^b}{\sqrt{s}}(\pi^{cd} h_{cd} s^a - 2 \pi^{ac} h_{bc} - 2 \delta \pi^a),$$

(12)

$$B^a_M = -\frac{1}{\sqrt{s}} A_t \delta p^a + 4 F_{\tilde{T}ab} \delta \tilde{A}_b + \frac{2}{\sqrt{s}} \beta^{[a} p^{b]} \delta \tilde{A}_b.$$ (13)

Here $h = h_{ab} s^{ab}$ and $p^a = -4N\sqrt{s} F^a$ is the electromagnetic momentum conjugate to $A_a$.

Now, Hamilton’s equations for the background spacetime are

$$\left(\dot{s}^{(0)}, -\dot{\pi}^{(0)}, \dot{A}^{(0)}, -\dot{E}^{(0)}\right) = \delta H^*_{\text{tot}} \cdot (F, \beta^a, A_t),$$ (14)

where $\dot{f}$ is the lie derivative of $f$ along a vector field $V^a = F n^a + \beta^a$.

Thus if $V^a$ is a Killing field of the background spacetime, the Lie derivatives vanish and therefore $F$ and $\beta^a$ are solutions to the differential equation $\delta H^*_{\text{tot}} \cdot (F, \beta^a, A_t) = 0$. Equation (14) then implies that all perturbations about the background spacetime $g_{ab}^{(0)}$ must satisfy the source free Gauss’s Law type constraint

$$D_a B^a = 0.$$ (15)

So far we have been discussing pure Einstein-Maxwell theory. But, it is simple to include additional perturbative sources $\delta T^{ab}$. Then the identity modifies to $D_a B^a = \delta S$, where $\delta S = -16\pi V^a n^b \delta T_{ab}$. Using Stoke’s law over the volume $\Sigma$, we get the following integral form of the constraint,

$$-\int_{\Sigma} \sqrt{s} \delta S = \int_{\partial \Sigma} d\Sigma_c B^c,$$ (16)

where $\partial \Sigma$ are all the boundaries of the volume $\Sigma$. Equations (15) and (16) are the main results of this section and key ingredients to the construction of the first law of black hole mechanics in different asymptotic backgrounds with different Killing fields.

Analogous to equation (9), we define the perturbative charge $\delta Q_V$. Equation (16) can then be rewritten as

$$\delta Q_V(g, g^{(0)}) \equiv \frac{1}{16\pi} \int_{\delta \Sigma_\infty} d\Sigma_c B^c = -\frac{1}{16\pi} \sum_i \int_{\delta \Sigma_i} d\Sigma_c B^c - \frac{1}{16\pi} \int_{\Sigma} \sqrt{s} \delta S.$$ (17)
where the sum is over all boundaries other than the one at infinity. We emphasize that both $Q_V$ and $\delta Q_V$ depend on $g^{(\text{ref})}_{ab}$, since both $g_{ab}$ and $g^{(0)}_{ab}$ asymptotes to $g^{(\text{ref})}_{ab}$.

### 3 Usual First Law of Black Hole Mechanics

Readers familiar with the derivation of the first law of black hole mechanics can skip this section without loss of continuity.

As a pedagogical application of the concepts discussed in the last section, consider the case of an asymptotically flat, stationary axisymmetric black hole spacetime satisfying Einstein’s equation. We assume that the black hole event horizon is a bifurcate Killing horizon with the bifurcation surface $\delta \Sigma_b$. Let $t^\mu$ and $\phi^\mu$ denote the Killing fields on this spacetime which asymptotically approach time translation and rotation at spatial infinity respectively. So the volume $\Sigma$ is bounded by the internal compact boundary $\delta \Sigma_b$ and spatial infinity. In this case there exists a linear combination $V^a = t^a + \Omega \phi^a$, which is the generator of the horizon, and defines the angular velocity $\Omega$ of the horizon. $V^a$ vanishes on the bifurcation surface.

Using the asymptotically flat boundary conditions for the perturbations to this spacetime, the boundary terms at spatial infinity simplify. Substituting $t^\mu$ and $\phi^\mu$ into equation (17) and using $F \to 1$ at spatial infinity one finds the standard expressions for change in mass and angular momentum respectively,

\[
16\pi \delta M_{\text{ADM}} = \int_{\partial \Sigma_\infty} da_c (-D^c h + D_b h^{cb}), \tag{18}
\]

\[
16\pi \Omega \delta J = -\Omega \int_{\partial \Sigma_\infty} da_c \frac{2\phi^b \delta \pi^c_b}{\sqrt{s}}. \tag{19}
\]

On the bifurcation surface of the horizon $V^a$ vanishes, and the gravitational boundary term becomes

\[
\int_{\partial \Sigma_{bh}} da_c (-h D^c F + h^{cb} D_b F) = 2\kappa_{bh} \delta A_{bh}, \tag{20}
\]

where $\kappa_{bh}$ and $A_{bh}$ are the surface gravity \(^1\) and the area of the black hole horizon respectively. $A_t = V^b A_b$ vanishes on the bifurcation surface, and $A_t \to 1$ at spatial infinity. Assembling the boundary terms into equation (17) we have that for any

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\(^1\)Surface gravity $\kappa$ is defined by the equation $V^b \nabla_b V_a = \kappa V_a$, where $V^a$ is the generator of the horizon.
asymptotically flat solutions to the linearized equations

$$\delta M_{ADM} = \Omega \delta J + \frac{1}{8\pi} \kappa_{bh} \delta A_{bh} - A_t \delta Q + \int_\Sigma V^a n_b \delta T_b^a.$$  \hspace{1cm} (21)

The above equation is the standard form of the first law of black hole mechanics [15, 16]. It will be of interest to compare the source term for $\delta M_{ADM}$ to the source term for $\delta M_{boost}$.

4 Boost Mass and First Law for Black Hole

In the previous section we outlined the derivation of the first Law of black hole mechanics in an asymptotically flat spacetime. This involved using the Gauss's Law for perturbations in the integral form. The boundary terms are different if the spacetime is not asymptotically flat, and/or has additional internal boundaries. In turn, this mean that the boundary terms may have different physical interpretations than in the asymptotically flat case. We will now address these issues when the Killing field is boost $\xi^a$.

4.1 Boost Killing Vectors and Acceleration Horizons

Let us turn to the definition of a boost Killing vector and an acceleration horizon. First consider Minkowski spacetime, $ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2$. $T^a = \frac{\partial}{\partial t'}$ is a time translation Killing vector, and is timelike everywhere. The boost Killing vector $\xi^a = z' \frac{\partial}{\partial t'} + t' \frac{\partial}{\partial z'}$ is timelike in the wedges $z'^2 > t'^2$ with $z' > 0$, and $z'^2 > t'^2$ with $z' < 0$. $\xi^a$ is null on the surfaces $z' = \pm t'$. Pick one wedge, say $z' > 0$. Then the region in which $\xi^a$ is timelike is bounded by infinity and two null surfaces which intersect at $z' = 0$ and extend to null infinity. We will use the term “acceleration horizon”, $\mathcal{H}_{acc}$ of $\xi^a$ to refer to this null boundary of one connected region in which $\xi^a$ is timelike. A key feature of $\mathcal{H}_{acc}$ is that its spatial sections are noncompact. For example, at $t' = 0$, $\mathcal{H}_{acc}$ is the plane $z' = 0$ which extends to spacelike infinity. In Rindler spacetime, which is just Minkowski spacetime written in the coordinates of an observer who undergoes constant acceleration, this is usually called the Rindler horizon. We use the term “acceleration horizon” instead, since it is commonly used when analyzing the C- and Ernst metrics.

Motivated by the example of flat spacetime, and by the behavior of the analogous Killing fields in the C-metric and the Ernst metric, we will call a Killing field a “boost” if it is time-like in some region of the spacetime which is bounded by a part of infinity,
and by a null surface which is spatially noncompact. We will call this null surface the “acceleration horizon” \( H_{\text{acc}} \). On a constant time surface, a black hole horizon \( H_{\text{bh}} \) is compact with \( S^2 \) topology, whereas \( H_{\text{acc}} \) is noncompact with planar topology. Also, following the common usage, we will often refer to the spatial sections of a horizon as just the “horizon”. (For example, in the first law one talks about the change in the area of the horizon of the black hole, meaning the area of a spatial cross section.) Hence we will refer to the black hole horizon as compact, and the acceleration horizon as noncompact.

To summarize, a boost vector comes with an acceleration horizon. The acceleration horizon is noncompact, and this raises finiteness questions in the first law involving changes in the area of \( H_{\text{acc}} \), which we will address in section 4.2.

4.2 \( M_{\text{boost}} \) for Asymptotically Rindler Spacetime with a Missing Angle

In this section we define the Boost mass for spacetimes that are asymptotically Rindler spacetime with a missing angle. The definition of \( M_{\text{Boost}} \) only requires that spacetime have a boost Killing vector asymptotically.

Consider flat space minus a wedge spacetime with angular deficit parameter \( \nu \)

\[
\begin{align*}
 ds^2 &= -dt'^2 + dz'^2 + d\rho^2 + \nu^2 \rho^2 d\phi^2.
\end{align*}
\]

For brevity, as in the introduction we will simply write this as \( n_{ab}^{(\nu)} \). Let \( z' = z \cosh(\kappa_{\text{acc}} t) \) and \( t' = z \sinh(\kappa_{\text{acc}} t) \). Then the previous equation becomes Rindler spacetime with a missing angle, given by the two parameter metric

\[
\begin{align*}
 ds^2 &= -\kappa_{\text{acc}}^2 z^2 dt^2 + dz^2 + d\rho^2 + \nu^2 \rho^2 d\phi^2.
\end{align*}
\]

The range of \( \phi \) is \( 0 < \phi \leq 2\pi \). We will be concerned with the region \( z > 0 \). Here \( \kappa_{\text{acc}} \) is the surface gravity of the acceleration horizon and \( \nu \) is the angular deficit parameter around the \( \phi \) axis. This metric describes the spacetime outside an infinite straight static string and we will refer to it as a cosmic string.

\( \vec{T} = \frac{\partial}{\partial t'} = \left( \frac{1}{z \kappa_{\text{acc}}} \cosh(t \frac{\partial}{\partial z'}) - \sinh(t \frac{\partial}{\partial z'}) \right) \) is a Killing vector which translates in the \( t' \) time and is timelike everywhere. \( \vec{\xi} = z' \frac{\partial}{\partial \rho} + t' \frac{\partial}{\partial z'} = \frac{1}{\kappa_{\text{acc}}} \frac{\partial}{\partial t} \) is another Killing vector, which translates in the Rindler time \( t \). It is only timelike in the region \( S, z'^2 > t'^2 \)

\(^6\)However, for metrics which are not everywhere equal to the reference such as C or Ernst metric, the geometry specifies a value of \( \kappa_{\text{acc}} \)
and becomes null on the surfaces $z' = \pm t'$. $\xi^a$ is usually called a boost Killing vector.

We now consider a spacetime with metric $g_{ab}$ which approaches $\eta_{ab}^{(\nu)}$ as $\rho \to \infty$ in $z' \geq 0$ region, i.e, $g_{ab} = \eta_{ab}^{(\nu)} + \gamma_{ab}$ and $\gamma_{ab}$ goes to zero asymptotically. Hence, $g_{ab}$ is not asymptotically flat, since the background spacetime contains the stress energy of an infinite cosmic string.

We define the boost charge of $g_{ab}$ by the general expression in equation (9), with $\vec{V}$ taken to be the asymptotic boost Killing vector of $g_{ab}$. The definition of $Q_{\text{boost}}(\vec{\xi})$ only depends upon the spacetime fields in the asymptotic region; $\vec{\xi}$ need not be a Killing field everywhere. This may seem to be at odds with the fact the acceleration horizon of a boost Killing vector extends into the interior of the spacetime. The point is that the background spacetime $\eta_{ab}^{(\nu)}$ does have a boost Killing field $\vec{\xi}^{(0)}$ everywhere, with an acceleration horizon. $\vec{\xi}$ is asymptotic to this background $\vec{\xi}^{(0)}$.

With $z = R\cos \theta$ the metric in equation (23) becomes

$$ds^2 = -\kappa_{acc}^2 R^2 \cos^2 \theta dt^2 + dR^2 + R^2 (d\theta^2 + \nu^2 \sin^2 \theta d\phi^2); \quad 0 < \theta \leq \frac{\pi}{2}. \quad (24)$$

Let $\Sigma$ be a $t =$ constant spacelike slice which extends to spatial infinity. The unit normal vector to this surface is given by $\vec{n} = (\kappa_{acc} R \cos \theta)^{-1} \frac{\partial}{\partial t}$. Now as $\vec{\xi} = F_{\text{boost}} \vec{n}$, the lapse function for the boost Killing field is $F_{\text{boost}} = R \cos \theta$ and the shift vector is zero. The area element on $\partial \Sigma$ is $da^a = \nu R^2 \sin \theta d\theta d\phi \left( \frac{\partial}{\partial t} \right)^a$. In these coordinates the $S^2$ sphere has an area proportional to $R^2$. Following the general definition of equation (9), the expression for the boost Killing charge with the background of flat space minus a wedge spacetime of equation (23), is given by

$$16\pi Q_{\text{boost}} = -\int_{\partial \Sigma, \infty} da_c [F(D^c \gamma - D_b \gamma^{cb}) - \gamma D^c F + \gamma_{b}^c D^b F], \quad (25)$$

Writing this out in terms of partial derivatives we get

$$16\pi Q_{\text{boost}} = -\nu \lim_{R \to \infty} \int_0^{2\pi} d\phi \int_0^\frac{\pi}{2} d\theta [-\gamma^b_R R \sin \theta + R \cos \theta \partial_R \gamma - \frac{1}{R} \cot \theta \partial_c (R^2 \sin \theta \gamma^c_R)] R^2 \sin \theta, \quad (26)$$

where $\gamma = \text{Tr}[\gamma_{ab}]$. Instead of integrating over the whole $S^2$ sphere at spatial infinity, we integrate only over half of the sphere, i.e $0 < \theta \leq \frac{\pi}{2}$, which bounds the region where the Killing field is timelike.

$Q_{\text{boost}}$ does not have the units of mass, rather it is dimensionless. This is because we have written the boost Killing vector $\vec{\xi}$ so that it is dimensionless, like a rotation, while the dimension of time translation $[\vec{T}] = \text{time}^{-1}$. However since $Q_{\text{boost}}$ is associated with time translation in Rindler time, it is nice to call it mass. To be consistent
with units we define $M_{\text{boost}} = \kappa_{\text{acc}} Q_{\text{boost}}$. So $M_{\text{boost}}$ is the charge associated with the rescaled vector $\vec{\xi} = \kappa_{\text{acc}} \vec{\xi}$. Hereafter, we will refer to $M_{\text{boost}}$ (rather than $Q_{\text{boost}}$) which is more suited to comparison with $M_{\text{ADM}}$ and stating the first law in a familiar form, except for when analyzing an example calculation of $\delta M_{\text{boost}}$ in section 5.3.

We now discuss the normalization of the boost vector $\vec{\xi}$, which involves the role of the parameter $\kappa_{\text{acc}}$ in equation (23). Define the acceleration $a^c = \xi^b \nabla_b \xi^c$, and let $\vec{\xi} = N \frac{\partial}{\partial T}$. Then $|\frac{a^c a^c}{\kappa_{\text{acc}}^2 \xi^b \xi^b}| = \kappa_{\text{acc}}^2 N^2$. We fix the normalization of $\vec{\xi}$, and hence the surface gravity of the acceleration horizon, by specifying the (dimensionful) parameter $\kappa_{\text{acc}}$, and requiring that $|\frac{a^c a^c}{\kappa_{\text{acc}}^2 \xi^b \xi^b}| = \kappa_{\text{acc}}^2$. For a particular metric describing an accelerating mass, the physics may pick out a preferred value of $\kappa_{\text{acc}}$. For example, a Rindler particle located at $z = \frac{1}{\kappa_{\text{acc}}}$ has four velocity $\vec{\xi}$, and constant acceleration $\vec{a} \cdot \vec{a} = \kappa_{\text{acc}}^2$. Kinnersley and Walker [8] have shown that the trajectories of the acceleration black holes in the C-metric are like those for constant acceleration Rindler particles, in the small mass limit.

A particular spacetime may have more than one symmetry. Depending on how a spacetime approaches the background, different charges may be zero, finite or infinite. In particular for a given metric $g_{ab}$, not both of $M_{\text{ADM}}$ and $M_{\text{boost}}$ will be finite. This follows from the analysis of the boundary terms.

The ADM mass, results from the substitution of $T$ in equation (9). On a constant $t$ slice the lapse function for the time translation Killing field, $F_{TT} = \cosh t$, is one power less in radial coordinates in comparison to the boost Killing field. Let $\gamma^j_i$ be the components of $\gamma_{ab}$ in an orthonormal frame. Therefore, whereas for the $M_{\text{ADM}}$ to be finite we need $\gamma^j_i$ to fall-off as $R^{-1}$, for the Boost charge to be finite we need $\gamma^j_i$ to fall off as $R^{-2}$. This is evident from the expressions of equation (18) and equation (26). Hence for a given $\gamma^j_i$, $M_{\text{ADM}}$ and $M_{\text{boost}}$ will not both be finite and nonzero. If the perturbations fall off as $R^{-1}$, $M_{\text{ADM}}$ is finite but $M_{\text{boost}}$ is infinite; if they fall off as $R^{-2}$ the $M_{\text{ADM}}$ is zero but $M_{\text{boost}}$ is finite. The same analysis of the fall off conditions for the finiteness of $M_{\text{ADM}}$ and $M_{\text{boost}}$ will also apply for the perturbative charges $\delta M_{\text{ADM}}$ and $\delta M_{\text{boost}}$. We will see that the C and Ernst spacetimes naturally yield examples with $M_{\text{boost}} \neq 0$, but $M_{\text{ADM}} = 0$. More generally, Bicak and Schmidt [9] have shown that besides the C metric, there are boost and rotation symmetric spacetimes corresponding to general sources moving on boost-symmetric orbits.

### 4.3 First Law for $\delta M_{\text{boost}}$

We now derive the first law of black hole mechanics for $\delta M_{\text{boost}}$. Consider a background spacetime $g_{ab}^{(0)}$ which is asymptotic to the Rindler metric with a missing angle, given
We consider $\nu$, which determines the mass per unit length of the string and $\kappa_{acc}$, the surface gravity of the acceleration horizon, to be fixed at infinity. We assume that the metric $g_{ab}^{(0)}$ has a boost Killing vector $\vec{\xi} = \frac{\partial}{\partial t}$ throughout the spacetime, with an associated acceleration horizon $\mathcal{H}_{acc}$, on which the Killing field $\xi^a$ goes null. In addition suppose that $g_{ab}^{(0)}$ has a black hole, and that the black hole horizon is also generated by $\xi^a$. Since the boost Killing vector generates both the horizons, the resulting first law will relate the variations of the areas of the different horizons to the variation of $M_{boost}$, the Killing charge corresponding to $\xi^a$, instead of the usual ADM mass.

Now, we will study the perturbations about this background spacetime $g_{ab}^{(0)}$ i.e., $g_{ab} = g_{ab}^{(0)} + {}^4h_{ab}$ where the perturbations $^4h_{ab}$ satisfy the linearized Einstein equations, and hence the linearized constraints. Consider a spacelike slice $\Sigma$ which intersects the black hole horizon on the bifurcation sphere and the acceleration horizon on the bifurcation surface. Substituting the Killing field $\xi^a$ in the constraint equation (16), the gravitational boundary terms at spatial infinity gives us $\delta M_{boost}$. The expression for $\delta M_{boost}$ is the same as the expression for $M_{boost}$ given in equation (26), with $\gamma_{ab}$ replaced by $h_{ab}$. That is

$$16\pi \delta M_{boost} = -\nu \lim_{R \to \infty} \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \left[ -h_{R}^0 R \sin \theta + R \cos \theta \partial_R h - \frac{1}{R} \cot \theta \partial_c (R^2 \sin \theta h^c_c) \right] R^2 \sin \theta.$$  

(27)

There are differences in the proof of the first law for $\delta M_{ADM}$ and $\delta M_{boost}$. These differences are important when the differential Gauss' Law of equation (15) is converted to the integral form of equation (16). In the proof of the first Law for a black hole in an asymptotically flat spacetime with a time translation Killing vector, there is a boundary at spatial infinity, and a boundary at the black hole horizon. With a boost Killing vector, in addition to the compact black hole horizon we also have a noncompact acceleration horizon, which extends to spatial infinity. Further, the spacetime is not flat at infinity. Since there is an infinite amount of string at infinity, even $\delta M_{boost}$ would be infinite if we were comparing to Minkowski spacetime. However, the definition of $\delta M_{boost}$ in equation (27) compares the perturbed spacetime to the background of a cosmic string. The result may still be infinite, but it may also be finite.

There are two additional boundaries in this case, one at each horizon. $\xi^a$ vanishes on the bifurcation surface of either horizon. The derivative of the lapse function $F$ along the normal to the bifurcation surface is proportional to the surface gravity $\kappa$ of the respective horizon. Since $\kappa$ is constant over the bifurcation surface, this can
be taken out of the integral and the gravitational boundary term on each horizon reduces to

$$\int_{\mathcal{H}} da_c (-h D_c^a F + h^{cb} D_b F) = \kappa \int_{\mathcal{H}} (h_{x_1}^1 + h_{x_2}^2) da = 2\kappa \delta A.$$  \hspace{1cm} (28)$$

Here $x^1$ and $x^2$ are coordinates on the horizon and $da$ is the area element of the spatial metric on the horizon.

It is worthwhile to first consider the case when no black holes are present, and there is no charge. This allows us to focus on the new features due to the boost Killing vector. Using equations (27) and (28) in equation (16), it gives the first law for acceleration horizons,

$$\delta M_{\text{boost}} = \frac{1}{8\pi} \kappa \delta A_{\text{acc}} + \int_{\Sigma} \xi^n_{\alpha} n_b \delta T_{\alpha}^b.$$  \hspace{1cm} (29)$$

The first issue is to determine when the various terms in equation (29) are finite. The conditions for finiteness of $\delta M_{\text{boost}}$ are completely analogous to those for $M_{\text{boost}}$, namely that $\delta M_{\text{boost}}$ is finite when $h_{ij}^{2} \to \frac{1}{R^2}$ at spatial infinity, where the coordinate $R$ is defined in equation (24). Next, since the acceleration horizon itself is noncompact, finiteness of $\delta A_{\text{acc}}$ is an issue. Suppose there are no perturbative sources, i.e., $\delta T_{\alpha}^b = 0$. Equation (29) is an identity on the solutions to the linearized equations (about $g_{ab}^{(0)}$). Therefore, if $\delta M_{\text{boost}}$ is finite, $\delta A_{\text{acc}}$ must also be finite. From this point of view, a divergent $\delta A_{\text{acc}}$ is simply associated with a divergent $\delta M_{\text{boost}}$, which diverges more readily than $\delta M_{\text{ADM}}$ because the boost Killing vector diverges at infinity.

Examples of finite changes $\delta A_{\text{acc}}$ have been calculated in particular source free cases, using the C-metric and Ernst spacetime, in references [17] and [17], respectively. In the next section, we will compute $\delta M_{\text{boost}}$ in an example involving C-metric.

Now consider perturbative sources $\delta T_{\alpha}^b$. If the sources do not have compact support then fall-off conditions are necessary so that the volume integral of $\xi^n_{\alpha} n_b \delta T_{\alpha}^b$ is finite. For asymptotically flat spacetimes with a time translation Killing vector, the source integral goes like the monopole moment of $\delta \rho$. By contrast, with the boost Killing field, the source integral goes like $z \delta \rho$, at large $z$, which is a dipole moment of the source. So finiteness of $\delta M_{\text{boost}}$ requires that the dipole moment of the source is finite, whereas finiteness of $\delta M_{\text{ADM}}$ only requires a finite monopole moment. This is just the same condition that we have already seen for the respective rates of fall off of the metric perturbation $h_{ab}$ in the far field. The two are connected since $h_{ab}$ is the solution to a Poisson-type equation with source $\int_{\Sigma} \xi^n_{\alpha} n_b \delta T_{\alpha}^b$.

We also gain some understanding of the physical meaning of $\delta M_{\text{ADM}}$ and $\delta M_{\text{boost}}$
by focusing on the relation between these mass variations and the matter sources. In the weakly gravitating, but still relativistic limit, we have from equations (21) and (29)

$$\delta M_{ADM} \sim \int dv \delta T_{ii}$$

whereas,

$$\delta M_{boost} \sim \frac{1}{8\pi} \kappa_{Acc} \delta A_{acc} + \int dv (z \delta T_{ii} - t \delta T_{iz}).$$

Here the hats denote an ortho-normal frame. These relations help justify the names of the charges. The first is the Newtonian relation that follows from Poisson’s equation, if one judiciously defines the total mass of the system by a boundary integral of the gradient of the Newtonian potential. The source of the “Noether time-translation charge” is the mass density! The second relation says that the source of the “Noether boost-invariance charge” is the boost-moment of the stress-energy. (On a $t = 0$ surface this becomes a dipole moment of the energy density.) Because the boost mass is generated by a boost Killing vector, the $\delta A_{acc}$ term is still present. It would be interesting to know if there are any solutions to the Einstein’s equation in which $\delta A_{acc}$ is zero, and $\delta M_{boost}$ is due just to the source terms.

Having already considered the issues of convergence, adding the black hole to the first law for boost mass is simple. Again, using equation (16) with the additional internal black hole horizon boundary, we have the first law,

$$\delta M_{boost} = \frac{1}{8\pi} \kappa_{BH} \delta A_{BH} + \frac{1}{8\pi} \kappa_{Acc} \delta A_{acc} - \langle A_t \delta Q \rangle + \int \Sigma \frac{(\xi^a n_b \delta T_{ab})}{16\pi}.$$

where

$$\langle A_t \delta Q \rangle = \int_{\partial \Sigma_{\infty}} da_c \frac{1}{\sqrt{s}} A_c \delta \rho^c,$$

and $Q$ is the electric charge, $\delta Q = \int_{\partial \Sigma_{\infty}} da_c \delta \rho^c$. We have choosen the gauge potential such a way that it vanishes on the horizons. When $A_t$ is constant on the boundary at infinity then $\langle A_t \delta Q \rangle = A_t \delta Q$. Equation (32) is the desired first Law. It holds for any solution to linearized Einstein’s equation, when the background spacetime has a boost Killing vector with bifurcate black hole and acceleration horizon.

Based on the local notion of horizon entropy density, Jacobson and Parentani argue that the laws of black hole thermodynamics apply quite generally to any causal horizon. That work discusses the first Law including the change in the area of acceleration horizon (there called Rindler Horizon), in asymptotically flat spacetime where the background metric is Minkowski. We have seen that in this case the the boost mass of each spacetime is infinite. So the Minkowski background is not an
intersting case. For example, if one compares the spacetime of an isolated mass to Minkowski spacetime, the boost mass of the spacetime is infinite (and Minkowski spacetime has \( M_{\text{boost}} = 0 \)). Since \( \delta M_{\text{boost}} \) is infinite, \( \delta A_{\text{acc}} \) is also infinite. This is independent of the type of motion of that mass.

Note that it is possible that the difference \( \delta M_{\text{boost}} = M_{\text{boost}}^{(2)} - M_{\text{boost}}^{(1)} \) between two boost masses is finite for two asymptotically flat metrics \( g_{ab}^{(2)} \) and \( g_{ab}^{(1)} \). This just requires \( g_{ab}^{(1)} \) and \( g_{ab}^{(2)} \) have the same ADM masses. However, in this case we do not have the first law of equation (32) for \( \delta M_{\text{boost}} \). This is because neither \( g_{ab}^{(1)} \) or \( g_{ab}^{(2)} \) will have a boost Killing vector. If the spacetime is asymptotically flat but contains stress energy, it will not have a boost symmetry.

5 An Example with the C-metric

5.1 The C-metric

An example of interest, indeed the motivation for this work, is to choose the background spacetime to be a C-metric \( \mathbb{C} \). The C-metric describes the spacetime corresponding to two charged black holes of opposite charge, uniformly accelerating away from each other along a symmetry axis, being pulled apart by a cosmic string. More precisely, the electrically charged C-metric is given by

\[
ds^2 = \frac{1}{A^2(x-y)^2} [G(y)dt^2 - G^{-1}(y)dy^2 + G^{-1}(x)dx^2 + \mu^2 G(x)d\phi^2],
\]

\[
A_t = \sqrt{r_+ r_- y}, \quad G(\xi) = (1 + r_- A\xi)[1 - \xi^2(1 + r_+ A\xi)].
\]

Here \( \phi \) ranges from 0 to \( 2\pi \).

The metric has two Killing vectors, \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \phi} \). The horizons occur where norm of \( \frac{\partial}{\partial t} \) vanishes i.e., at the zeroes of \( G(y) \). Let \( \xi_1 \equiv -\frac{1}{r_- A}, \xi_2, \xi_3, \) and \( \xi_4 \) be the four real roots of \( G(\xi) \), which exist for \( r_+ A < 2/3\sqrt{3} \). The surface \( y = \xi_2 \) is the compact black hole horizon and the surface \( y = \xi_3 \) is the noncompact acceleration horizon; they both are Killing horizons for \( \frac{\partial}{\partial t} \). For the range of coordinates \( \xi_2 \leq y \leq \xi_3 \), \( G(y) \) is negative and hence the Killing field \( \frac{\partial}{\partial t} \) is timelike. Thus the Killing field \( \frac{\partial}{\partial t} \) is timelike in a part of the spacetime which is bounded by the acceleration horizon, black hole horizon and a part of spatial infinity. Therefore, according to our definition in section 4.1 we identify the Killing field \( \frac{\partial}{\partial t} \) as a boost Killing field.

The coordinates \((x, \phi)\) are angular coordinates. \( \xi_3 < x < \xi_4 \), where \( G(x) \) is positive. The norm of the Killing vector \( \frac{\partial}{\partial \phi} \) vanishes on the axis, at \( x = \xi_3 \) and
$x = \xi_4$. The axis $x = \xi_3$ extends to spatial infinity. The axis $x = \xi_4$ points towards the other black hole. Spatial infinity is reached by fixing $t$ and letting both $y$ and $x$ approach $\xi_3$.

The C metric has conical singularities on the symmetry axis. The deficit angle $\delta_{in} = 2\pi[1 - (\frac{\xi_3}{2})\lambda'(\xi_4)]$ is on the inner part of the axis which is between the two black holes. On the outer part of the axis, extending from each black hole to infinity, the deficit angle is $\delta_{out} = 2\pi[1 - (\frac{\xi_3}{2})\lambda'(\xi_3)]$. We interpret the conical singularities as a model for a thin cosmic string along the symmetry axis. We take $\delta_{in} < \delta_{out}$, i.e., the mass per unit length of the string is greater on the outer axis than on the inner axis. The corresponding difference in string tension between the outer and inner part of the axis provides the force which accelerates the black holes.

The C-metric has four parameters $r_+, r_-, A, \mu$. Define $m$ and $q$ via $m = \frac{1}{2}(r_+ + r_-)$, $q = \sqrt{r_+r_-}$. Then in the limit $r_{\pm}A \ll 1, m, q, A$ and $\mu$ denote the mass, charge, acceleration of the black holes and the mass per unit length of the cosmic string respectively [19] [8].

At spatial infinity, the C-metric is asymptotic to Rindler spacetime minus a wedge. To see this we consider a particular C-metric and expand the function $G(\xi)$ around the spatial infinity, $x, y \to x_3$, i.e $G(\xi) = G(\xi_3) + \lambda(\xi - \xi_3) + \beta(\xi - \xi_3)^2$, where $\lambda = \frac{dG}{d\xi}\big|_{\xi=\xi_3}$ and $\beta = \frac{1}{2}\frac{d^2G}{d\xi^2}\big|_{\xi=\xi_3}$. Make the transformations $(y - \xi_3) = -\frac{4}{A^2\kappa}\sin^2\alpha$ and $(x - \xi_3) = \frac{4}{A^2\kappa}\sin^2\alpha$ where $0 < \alpha \leq \frac{\pi}{2}$. Rescaling the time coordinate so that it has the dimensions of time $\tilde{t} = \frac{\tilde{t}}{\kappa}$, we get the asymptotic form of the metric at spatial infinity $\tilde{r} \to \infty$ as

$$
\begin{align*}
\det s^2 & \to -\kappa_{acc}^2\tilde{r}^2\cos^2\alpha(1 - \frac{\beta}{\kappa_{acc}^2\tilde{r}^2}\cos^2\alpha)\tilde{d}t^2 + (1 + \frac{\beta}{\kappa_{acc}^2\tilde{r}^2}\cos^2\alpha)\tilde{d}r^2 + \tilde{r}^2\tilde{d}\alpha^2 \\
& \quad + \frac{\beta}{2\kappa_{acc}^2\tilde{r}^2}\sin2\kappa_{acc}\tilde{d}\alpha + \tilde{r}^2\nu^2\sin^2\alpha(1 + \frac{\beta}{\kappa_{acc}^2\tilde{r}^2}\sin^2\alpha)d\phi^2. \\
\end{align*}
$$

(35)

Here $\kappa_{acc} = \frac{1}{2}A\lambda$ and $\nu = \frac{1}{2}\mu\lambda$.

For $r_{\pm} = 0$, then $\xi_3 = -1$, $\lambda = 2$, and $\beta = -1$. Hence equation (35) becomes

$$
\det s^2 \to -\kappa_{acc}^2\tilde{r}^2\cos^2\alpha(1 + \frac{1}{\kappa_{acc}^2\tilde{r}^2}\cos^2\alpha)\tilde{d}t^2 + (1 - \frac{1}{\kappa_{acc}^2\tilde{r}^2}\cos^2\alpha)\tilde{d}r^2 + \tilde{r}^2\tilde{d}\alpha^2 \\
- \frac{1}{2\kappa_{acc}^2\tilde{r}^2}\sin2\alpha\tilde{d}\alpha + \tilde{r}^2\nu^2\sin^2\alpha(1 - \frac{1}{\kappa_{acc}^2\tilde{r}^2}\sin^2\alpha)d\phi^2. \\
$$

(36)

where $\kappa_{acc} = A$ and $\nu = \mu$. Equation (36) is our reference spacetime, but it is not of the same form as Rindler equation (24). This just means that the $(R, \theta)$ coordinates of equation (24) are not the same as the $(\tilde{r}, \alpha)$ coordinates of equation (36). For $r_{\pm} = 0,$
Kinnersley and Walker [8] have given a sequence of coordinate transformations that take the C-metric to flat spacetime minus a wedge of equation (23). However, it is nontrivial to work in their coordinates when \( r_\pm \neq 0 \). To calculate \( \delta M_{\text{boost}} \) we will need equation (35), in which \( \beta \) is general.

To summarize, equation (36) is Rindler spacetime with a missing angle \( \eta^{(w)}_{ab} \), which is our reference metric. The asymptotic form of a general C-metric is given by equation (35). The reference spacetime is fixed by specifying \( \kappa_{\text{acc}} \) and \( \nu \). Physically this means fixing surface gravity of the acceleration horizon and mass per unit length of the cosmic string, both at infinity. The initial C-metric has four parameters, \( r_+, r_-, A, \mu \) and when we perturb this metric to another close by C-metric (section 5.3), \( \kappa_{\text{acc}} \) and \( \nu \) are kept fixed. This leaves a 2-parameter family of solutions to the linearized equations.

### 5.2 \( \delta M_{\text{boost}} \) and \( M_{\text{ADM}} \) for C-metric

Let \( g^{(w)}_{ab} \) be a C-metric with particular values of \( \kappa_{\text{acc}} \) and \( \nu \) as defined in equation (23). The C-metric has both time translation and boost symmetry asymptotically, though only the boost is a Killing vector throughout the spacetime. Using these asymptotic Killing vectors, we can compute \( M_{\text{boost}} \) [see equation (26)] and \( M_{\text{ADM}} \) [see equation (18), with \( h_{ab} \) replaced by \( \gamma_{ab} \)]. To do this, we need the far field \( \gamma_{ab} \) of the metric near spatial infinity, where \( \gamma_{ab} = g^{(w)}_{ab} - \eta_{ab}^{(w)} \). The components in an ortho-normal frame are

\[
\begin{align*}
\gamma^i_t &= -\frac{\Delta \beta}{\kappa_{\text{acc}}^2 r^2 \cos^2 \alpha}; & \quad \gamma^\phi_t &= \frac{\Delta \beta}{\kappa_{\text{acc}}^2 r^2 \cos^2 \alpha}; & \quad \gamma^\phi_\alpha &= 0; \\
\gamma^\phi_\phi &= \frac{\Delta \beta}{\kappa_{\text{acc}}^2 r^2 \sin^2 \alpha}; & \quad \gamma^\phi_\alpha &= \frac{\Delta \beta}{2\kappa_{\text{acc}}^2 r^2 \sin 2\alpha}; & \quad \gamma^\phi_\phi &= \frac{\Delta \beta}{2\kappa_{\text{acc}}^2 r^2 \sin 2\alpha},
\end{align*}
\]

(37)

where \( \Delta \beta = \beta^{(w)}_0 - (1) \). Note that all the perturbations are proportional to \( \Delta \beta \).

Now, we can compute \( M_{\text{boost}} \) with \( r_\pm \neq 0 \). On a constant time slice the lapse function is \( F = -\hat{n} \cdot \xi = \hat{r} \cos \alpha \) and the area element \( da^a \sim \hat{r}^2 \). Plugging the perturbations in equation (26), we get \( M_{\text{boost}} = (\frac{\mu}{8\kappa_{\text{acc}}}) \Delta \beta \), a finite and nonzero result. In the next section we will solve for \( \Delta \beta \) in terms of \( m \) and \( q \) for \( Am, Ag << 1 \).

In a similar way we can compute \( M_{\text{ADM}} \). For simplicity, evaluate the integral on a constant time \( t = 0 \) slice. Then \( F = -\hat{T}_i \hat{n} = 1 \). The perturbations fall off as \( \gamma^i_j \sim \frac{1}{r^2} \), and therefore following the general definition of gravitational charge, we find \( M_{\text{ADM}} = 0 \). This is simply because fixing \( \kappa_{\text{acc}} \) and \( \nu \) at infinity is equivalent to fixing monopole moment of the system. The C-metric is a sort of dipole rearranging of the reference spacetime - some of the string mass goes into black hole mass or vice
versa. To summarize, the charge of interest for the C-metric is the boost mass, which measures a dipole rearrangement of the background stress-energy.

5.3 $\delta M_{\text{boost}}$ for the C-metric

In this section we will compare two nearby C-metrics to find the perturbative charge defined in equation (17). More specifically, we calculate $\delta M_{\text{boost}}$ between two C-metric spacetimes. Take a particular C-metric as the background spacetime $g_{ab}^{(0)}$. Fixing $\nu$ and $\kappa_{\text{acc}}$, choose another nearby C-metric $g_{ab}$. Let $^{(4)}h_{ab} = g_{ab} - g_{ab}^{(0)}$ and $\delta A_{b}$ be the perturbations to linear order in $\delta A, \delta \mu, \delta r_{\pm}$. So $^{(4)}h_{ab}, \delta A_{b}$ is a solution to the linearized Einstein equation, with no sources. Note that, at infinity $h_{ab}$ is same as $\gamma_{ab}$ with $\Delta \beta \equiv \beta - (-1)$ replaced by $\delta \beta \equiv \beta[g] - \beta[g^{(0)}]$. Thus $\delta M_{\text{boost}} = \left(\frac{\nu}{8\kappa_{\text{acc}}}ight)\delta \beta$.

Next we rewrite this result in a more meaningful way. Take the $g_{ab}^{(0)}$ to be just a string, with no black holes, $g_{ab}^{(0)} = \eta_{ab}^{(s)}$ and let $g_{ab}$ be a C-metric with small black holes and the same $\kappa_{\text{acc}}, \nu$. The expressions can be simplified for $Ar_{\pm} \ll 1$. To leading order we find that $\nu = \mu(1 - 2Am)$ and $\kappa_{\text{acc}} = A(1 - 2Am)$. The missing angle parameter on the inner axis is given by $\nu_{in} = \mu(1 + 2Am)$, which is always greater than $\nu$, i.e., the deficit angle on the inner axis is less than the deficit angle on the outer axis. This also means that the metric parameter $\mu$ is constrained by $\mu \leq (1 - 2Am)$. Lastly, $\delta \beta = 6Am$ and therefore

$$\delta M_{\text{boost}} = \frac{3}{4}\nu m. \quad (38)$$

This result is proportional to the black hole mass parameter times the angle deficit parameter. So for fixed $m$, the boost mass decreases as the outer deficit angle increases, using $\nu = (1 - \frac{\Delta \alpha}{2\pi})$. The proportionality to $\nu$ is due to the modification in the area of two-spheres from the missing angle. One can see this by computing the ADM mass for Schwarzschild with a missing angle. In the usual Schwarzschild metric, replace $d\phi^2$ by $\nu^2 d\phi^2$. This is still a vacuum solution to the Einstein equation. Then the only change in the integral for the ADM mass is that the area element has a factor of $\nu$, just as in computing $\delta M_{\text{boost}}$.

Rewriting equation (38) in terms of the boost charge ($\delta Q_{\text{boost}} = \frac{1}{\kappa_{\text{acc}}}\delta M_{\text{boost}}$ and $\xi$ is dimensionless) highlights the difference between the boost charge and the ADM mass. We have

$$\delta Q_{\text{boost}} = \frac{1}{\kappa_{\text{acc}}}\delta M_{\text{boost}} = \frac{1}{\kappa_{\text{acc}}}\frac{3}{4}\nu m. \quad (39)$$

Suppose we were computing the boost mass for a Rindler particle of mass $m$, instead of a black hole. The Rindler particle moves on the hyperbola $-(t')^2 + (z')^2 = \kappa_{\text{acc}}^{-2}$ in the coordinates of equation (22), for which $t'$ is the particle’s proper time. The
surface $t' = 0$ coincides with the surface of constant Rindler time $t = 0$, and we can approximate the source integral for $\delta Q_{\text{boost}}$ on that slice: $\int dv \delta T^a_b \xi^b n_a = \int dv' \delta \rho \sim \kappa^{-1} \nu m$. That is, $\delta Q_{\text{boost}}$ is the dipole moment of the Rindler particle, where the length of the moment arm $\kappa^{-1} \nu m$ is the semi-major axis of the hyperbola.

Note that although it is tempting to compare this result to the analogous result for Schwarzschild, one cannot take the limit where the background goes to flat space in equation (38). Taking both $\nu \to 1$ and $\nu_m \to 1$ requires that $A m = 0$. Further, there is no reason to expect that the answers would be the same, since the Killing vectors are different vector fields.

It would be nice to check these results by computing $\delta Q_{\text{boost}}$, $\delta A_{\text{bh}}$, and $\delta A_{\text{acc}}$, and substituting in the first law, equation (32). However, in the coordinates of equation (34) the C-metric is badly behaved on the horizons, and one finds that in particular the metric perturbation $h_{ab}$ is badly behaved. One would need to first find better, Kruskal-type coordinates near the horizons, and then expand to find $h_{ab}$. We leave this exercise to future work.

We close this section with some comments about the variation of the ADM mass for perturbations about a C-metric. Since the metric is asymptotically Rindler spacetime with a time translation Killing vector $T^a$, the ADM mass is defined. However, since $T^a$ is not a Killing vector throughout the spacetime, we do not have a theorem which relates $\delta M_{\text{ADM}}$ to the variations in the horizon areas. Still, one can compute $\delta M_{\text{ADM}}$. If we compare the ADM mass for two perturbatively close C-metrics with the same value of $\kappa_{\text{acc}}$ and $\nu$ then $\delta M_{\text{ADM}} = 0$. The perturbations are given in (37), and since $F = -T^a n_a$ goes to a constant, the boundary term vanishes. On the other hand, if two C-metrics are compared that have different values of $\nu$, then $\delta M_{\text{ADM}} = \infty$. This is essentially because of the fact that a change in the mass per unit length over an infinite length is infinite. By contrast, if one adds a small mass source to the C-metric—say a planet orbiting a black hole—then one expects the change in the ADM mass to be finite, and the change in the boost mass to be infinite.

6 The Ernst Spacetime and Conclusions

We have seen that the boost mass is a relevant charge for a spacetime which has stress energy at infinity, and of course, has an asymptotic boost Killing vector. If a metric with black holes has an exact boost symmetry, then we have shown that perturbations about the metric obey the first law of black hole mechanics. This work was motivated by studying the C-metric and Ernst metric, so we briefly mention the
The Ernst spacetime [11] is another analytic solution to the Einstein-Maxwell theory, which has a boost Killing vector. This spacetime represents two oppositely charged black holes, undergoing uniform acceleration by a background magnetic field.

The Ernst metric has two Killing vectors: $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. Killing field $\frac{\partial}{\partial t}$ becomes null on the compact black hole horizon, and on the noncompact acceleration horizon. The Killing field $\frac{\partial}{\partial \phi}$ is timelike in a region of the spacetime which is bounded by the black hole horizon, the acceleration horizon and a part of spatial infinity. Thus according to the definition in section 4.1, $\frac{\partial}{\partial t}$ is a boost Killing vector. At large spatial distances the Ernst metric reduces to the Melvin metric [10], which contains a magnetic field throughout the spacetime. We consider Melvin spacetime as our reference spacetime $g_{ab}^{(ref)}$. Since the reference spacetime has nonzero stress energy in the asymptotic region, the Ernst metric is not asymptotically flat. Following the general definition of equation (9) we can define the charge $M_{\text{boost}}$ for the Ernst spacetime, corresponding to the asymptotic boost Killing field of Melvin spacetime. Similar to the C-metric, the Ernst spacetime has a finite, nonzero boost mass. In addition to the black hole horizon, the Ernst metric has a spatially noncompact acceleration horizon both of which are generated by $\frac{\partial}{\partial t}$.

Now consider a background black hole spacetime spacetime $g_{ab}^{(0)}$ that has a boost Killing field, and which is asymptotic to Melvin, such as the Ernst metric. Fix the value of the surface gravity of the acceleration horizon and the magnetic field at spatial infinity. Following the general derivation in section 4.3 we can prove the first law of equation (32) for perturbations about accelerated black holes in an asymptotically Melvin Universe.

Of course, there are many open issues. It would be interesting to see if an analogous first law holds for higher dimensional black objects, for example, black strings being pulled apart by two-branes, or charged strings being accelerated apart by an external field. Another situation of interest would be to study the boost mass constraints in the context of brane world scenarios, with a boost symmetric background brane. Here one appreciates the importance of which boundary conditions are appropriate. In a brane-cosmology, depending on how cosmological perturbations are generated / present in initial conditions, $\delta M_{\text{boost}}$ could be infinite, finite, or zero.

One would like to see if the accelerated black hole mechanics discussed here is actually part of a thermodynamic structure. Three further key elements are needed; first whether, or not, there is an area increase theorem for black holes in asymptotically cosmic string / Melvin spacetimes. Second, if there is an area increase theorem...
for acceleration horizons in these cases. And third, a calculation of Hawing radiation in these spacetimes.

Lastly, it would be interesting to have a definition of acceleration horizon and accelerating black holes in the absence of a boost Killing vector. It may be that the best definition of black holes with constant acceleration is that the generator of the black hole horizon is a boost Killing vector of the spacetime. What about the case of non-constant acceleration? In a test particle limit, one can talk about the acceleration of the particles, and if these are black holes instead of particles, presumably one can talk about the acceleration of the the black holes. What is the description if the black holes have significant mass, but there is no spacetime symmetry? Is there an acceleration horizon in some meaningful sense, or is this notion special to the case of a boost Killing vector?

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References


