$J$-pairing interaction, number of states, and nine-$j$ sum rules of four identical particles

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In this paper we study $J$-pairing Hamiltonian and find that the sum of eigenvalues of spin $I$ states equals sum of norm matrix elements within the pair basis for four identical particles such as four fermions in a single-$j$ shell or four bosons with spin $l$. We relate number of states to sum rules of $J$-$l$ coefficients. We obtained sum rules for nine-$j$ coefficients $\langle (jj) J, (jj) K : I (jj) J, (jj) K : I \rangle$ and $\langle (ll) J, (ll) K : I (ll) J, (ll) K : I \rangle$ summing over (1) even $J$ and even $K$, (2) even $J$ and odd $K$, (3) odd $J$ and odd $K$, and (4) both even values for $J$ and $K$, where $j$ is a half integer and $l$ is an integer.

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I. INTRODUCTION

The $J$-pairing Hamiltonian for a single-$j$ shell is an important topic to study nuclear structure theory and also to study general many-body systems. For the case of $J = 0$, i.e., the monopole pairing interaction, the famous seniority scheme provides exact solutions; for $J = J_{\text{max}}$, the “cluster” picture of Ref. presents an asymptotic classification of states. For other $J$ cases, it was found that pairs with spin $J$ are reasonable building blocks for low-lying states but little is known about exact eigenvalues. In this paper we shall go one step forward along this line by proving that for four identical particles the sum of eigenvalues for the $J$-pairing interaction is connected with a sum of nine-$j$ coefficients.

The enumeration of spin $I$ states (the number of spin $I$ states is denoted by $D_I$ in this paper) for fermions in a single-$j$ shell or bosons with spin $l$ (We use a convention that $j$ is a half integer and $l$ is an integer) is also a very common practice in nuclear structure theory. $D_I$ is usually obtained by subtracting the number of states with total angular momentum projection $M = I + 1$ from that with $M = I$. Because numbers of states with different $M$’s seem irregular, $D_I$ values are usually tabulated in text-books, for sake of convenience. Other methods include Racha’s method in terms of seniority scheme, generating function method proposed and studied by Katriel et al., Sunko and collaborators. All these works are interesting and important. However, the results are not algebraic. It is therefore desirable to obtain analytical formulas of $D_I$. For $n = 1$ and $2$, $D_I$ is known and is understood very well; but the situation becomes complicated when $n \geq 3$, except for a few cases with $I \sim I_{\text{max}}$. Historically, the first interesting formula of $D_I$ was given for the case with $I = 0$ and $n = 4$ by Ginocchio and Haxton in Ref. Their result was revisited by Zn nick and Escuderos in Ref. In Ref., authors of the present paper empirically constructed $D_I$ for $n = 3$ and 4, and some $D_I$’s for $n = 5$. Recently, Talmi suggested a recursion formula of $D_I$ in Ref., and used this formula to prove $D_I$ formulas obtained empirically for $n = 3$ in Ref. Talmi’s recursion formula is also readily applied to prove the empirical formulas of Ref. for $n = 4$. In Ref., we showed that $D_I$ of $n$ particle systems can be enumerated by using the reduction from $SU(n + 1)$ to $SO(3)$, and as an example, $D_4$ for $n = 4$ was obtained analytically.

The results of $D_I$ for three identical particles were applied to obtain a number of sum rules for six-$j$ symbols in Appendix of Ref. One can ask whether the results for $n = 4$ can be used similarly to obtain sum rules for nine-$j$ symbols. If the answer is yes, the application would be very interesting, because sum rules of angular momentum couplings is widely applied in many branches of physics, in particular in nuclear structure theory (angular momentum coupling-recoupling coefficients and sum rules were compiled in Ref. in 1988). This paper addresses the following question: can we obtain sum rules for nine-$j$ symbols based on studies of $J$-pairing Hamiltonian and number of spin $I$ states for four identical particles? Furthermore, how far can one go along this line?

This paper is organized as follows. In Sec. II, we study $J$-pairing Hamiltonian in order to obtain summation of all non-zero eigenvalues in the presence of only one $J$-pairing force. In Sec. III, we present sum rules of nine-$j$ symbols found by using these summations and number of states for $n = 4$ obtained in earlier works. In this paper we show that the $D_I$ formulas provide us with a bridge between the $J$-pairing interaction and sum rules of nine-$j$ symbols for identical particles. The summary and discussion are given in Sec. IV. Appendix A present...
formulas of nine-\( j \) symbols in some special cases. Appendix B discusses number of matrices involved in our sum rule calculations. 

II. J-PARING INTERACTION 

In this section we discuss the \( J \)-pairing interaction only for identical fermions in a single-\( j \) shell. Similar results are readily obtained for bosons with spin \( l \). Our \( J \)-pairing Hamiltonian \( H_J \) is defined as follows. 

\[
H_J = G_J \sum_{M=-J}^{J} A_M^{(j)^\dagger} A_M^{(j)} + A_M^{(j)^\dagger} A_M^{(j)} = \frac{1}{\sqrt{2}} \left[ a_j^\dagger x a_j^\dagger \right]^{(j)} , 
\]

\[
A_M^{(j)} = (-1)^M \frac{1}{\sqrt{2}} \left[ a_j x a_j^\dagger \right]_{-M} , \quad A_M^{(j)^\dagger} = -\frac{1}{\sqrt{2}} \left[ a_j x a_j^\dagger \right]_{(j)} , 
\]

where \( j \) means coupled to angular momentum \( J \) and projection \( M \). We take \( G_J = 1 \) in this paper. 

For \( n = 3 \), it was shown in Ref. 3 that there is only one non-zero eigenvalue for \( H_J \) when \( I \geq J - 1 \), and all eigenvalues are zero when \( I < J - 1 \). For \( n = 4 \) the situation is more complicated, because there can be many non-zero eigenvalues of spin \( I \) states for \( H_J \), and most of these eigenvalues are not known except \( I = 0 \) and \( I \approx I_{\text{max}} \). However, their summation is the trace of \( H_J \) matrix with total spin \( I \), and is a constant with respect to any linear transformation. This trace can be obtained by summing the diagonal matrix elements 

\[
\langle 0 | \left[ A^{(j)^\dagger} x A^{(K)^\dagger} \right]_{M}^{(j)^\dagger} \left[ A^{(j)} x A^{(K)} \right]_{M}^{(j)} | 0 \rangle = 1 + (-)^I \delta_{JK} - 4(2J + 1)(2K + 1) \left\{ \begin{array}{c} j j J \\ j j J \\ J K I \end{array} \right\} 
\]

over \( K \). Here \( J \) and \( K \) take only even values. This fact can be proved by using two-body coefficients of fractional parentages which are defined by 

\[
\langle j^4 \alpha IM \rangle \left[ j^2(J), j^2(K)I \right] = \frac{1}{\sqrt{6}} (-)^I \langle j^4 \alpha IM \rangle \left[ A^{(j)^\dagger} x A^{(K)^\dagger} \right]_{M}^{(j)^\dagger} \left[ A^{(j)} x A^{(K)} \right]_{M}^{(j)} | 0 \rangle . 
\]

The trace can be calculated as follows, 

\[
\sum_{\alpha} \langle j^4 \alpha I | H_J | j^4 \alpha I \rangle = \sum_{K} \langle j^2(J), j^2(K)I | j^4 \alpha I \rangle^2 \]

\[
= \sum_{K} \langle j^2(J), j^2(K)I | \left[ A^{(j)^\dagger} x A^{(K)^\dagger} \right]_{M}^{(j)^\dagger} \left[ A^{(j)} x A^{(K)} \right]_{M}^{(j)} | 0 \rangle \]

\[
= \sum_{K} \left[ \langle j^2(J), j^2(K)I | \left[ A^{(j)^\dagger} x A^{(K)^\dagger} \right]_{M}^{(j)^\dagger} \left[ A^{(j)} x A^{(K)} \right]_{M}^{(j)} | 0 \rangle \right]^{(j)^\dagger} \left[ \langle j^2(J), j^2(K)I | \left[ A^{(j)^\dagger} x A^{(K)^\dagger} \right]_{M}^{(j)^\dagger} \left[ A^{(j)} x A^{(K)} \right]_{M}^{(j)} | 0 \rangle \right]^{(j)} \]

\[
= \frac{1}{\sqrt{2}} \left[ a_j^\dagger x a_j^\dagger \right]^{(j)^\dagger} \left[ a_j x a_j^\dagger \right]^{(j)} , 
\]

where \( G_J = 1 \) is used. This is just the summation of Eq. 2 over even \( K \). One can also regard Eq. 2 as a simple generalization of the result in Ref. 13, where it was shown that the non-zero eigenvalue of \( H_J \) for spin \( I \) states of three particles is given by the norm \( \langle j^2(J)I : I | j^2(J)I : I \rangle \). Note that similar results are applicable to bosons with spin \( l \).

Let us look at nine-\( j \) symbols of identical particles under certain conditions. Based on Eq. 2 we easily find the following well-known fact 

\[
\left\{ \begin{array}{c} j j J \\ j j J \end{array} \right\} = 0 
\]

for odd \( I \) when \( K \neq J \) or \( K' \neq J \), based on the fact that two identical pairs produce only even values of \( I \) and thus the norm in Eq. 2 equals zero. Here \( K \) and \( K' \) take even values, or odd values simultaneously. This can be also seen from the permutation symmetry of the nine-\( j \) symbol, which requires the left hand side of Eq. 2 equals zero unless \( K + K' \) is even. We note without details that this formula is also applicable to four bosons with spin \( l \), i.e., one can replace \( j \) by \( l \) in formula 2.

This is a generalization of the result in Ref. 14, where it was found 

\[
\left\{ \begin{array}{c} j j J \\ j j J \end{array} \right\} = 0 , 
\]

The norm of Eq. 2 equals zero when \( I = 4J - 7, 4J - 5, 4J - 4 \), because there are no such states. We find thus 

\[
\left\{ \begin{array}{c} j j J \\ j j J \end{array} \right\} = \frac{1}{4(2J + 1)(2K + 1)} 
\]

for \( I = 4J - 7, 4J - 5, 4J - 4 \) and \( J \neq K \) (\( J, K \) are even). This is also a generalization of a formula in Ref. 13: 

\[
\left\{ \begin{array}{c} j j J \\ j j J \end{array} \right\} = \frac{1}{4(4J - 5)(4J - 1)} 
\]

for \( I = 4J - 7, 4J - 5, 4J - 4 \). Similarly, we have 

\[
\left\{ \begin{array}{c} j j J \\ j j J \end{array} \right\} = \frac{1}{2(4J - 1)^2} 
\]

for \( I = 4J - 2, 4J - 4 \). This formula was also obtained in Ref. 13 and holds for both integer and half-integer value of \( j \). In Appendix A, we present some explicit formulas of nine-\( j \) symbols with \( J = K = 2j \) or \( J = K = 2j - 1 \). For completeness, we also refer Refs. 9, 13, 14, 15, concerning formulas of six-\( j \) and nine-\( j \) symbols for identical particles.

Now we enumerate the number of matrices of Eq. 2 with different \( J \). This is related to the number of non-zero two-body coefficients of fractional parentage which...
was obtained for specific examples in studying regularities of energy centroids in the presence of random interactions \[12\]. Without going to details we present the results of the number of matrices involved in Eq. 2 with different \(J\) as below.

For \(I \geq 2j\), the number of matrices with \(K = J\) is given by

\[
[(4j + 2 - I)/4]
\]

and the number of matrices with \(K \neq J\) is

\[
[(4j - I)/2][(4j - I)/2 + 1)/2 - [(4j + 2 - I)/4].
\]

The [ ] in this paper means to take the largest integer not exceeding the value inside.

For \(I \leq 2j - 1\), the number of matrices with \(J = K\) is always 1 for even \(I\), the case with \(J \neq K\) is more complicated and the number of such matrices is given in the Appendix B. It is noted that \(J\) and \(K\) take only even values in this Section.

## III. SUM RULES FOR NINE-J SYMBOLS

The procedure to obtain sum rules of nine-j symbols in this paper is straightforward. In Sec. II we obtain summation of eigenvalues for \(H = H_f\). From the sum rule of two-body coefficients of fractional parentage, one obtains \(\frac{n(n-1)}{2}\) multiplied by the number of spin \(I\) states, \(D_I\), if one sums Eq. 2 over even \(J\) and even \(K\), namely:

\[
\sum_j \sum_\alpha (j^4 \alpha_I |H_f| j^4 \alpha_I)
= \sum_{\text{even } J \text{ even } K} \langle 0 | A^{(J)} \times A^{(K)} | M \rangle \left[A^{(J)^{I}} \times A^{(K)^{I}}\right]_M | 0 \rangle
= 6D_I.
\]

where \(D_I\) formulas were given in Refs. 11,12. New sum rules of nine-j symbols now can be obtained by using \(D_I\) formulas, Eq. 2 and Eq. 6.

For realistic systems both \(J\) and \(K\) are even, as in Eq. 2 of Sec. II and Eq. 6. In this paper we also discuss sum rules of nine-j symbols under other conditions for \(J\) and \(K\), such as odd \(J\) and odd \(K\), etc. We denote

\[
S_I(j^4, \text{condition } X \text{ on } J \text{ and } K)
= \sum_X 4(2J + 1)(2K + 1) \begin{pmatrix} j & j & J \\ j & j & K \\ J & K & I \end{pmatrix}
\]

for sake of simplicity. The condition \(X\) of the sum rules for \(J\) and \(K\) will be one of the following: (1) even \(J\) and even \(K\) (realistic); (2) even \(J\) and odd \(K\) or odd \(J\) and even \(K\); (3) odd \(J\) and odd \(K\); and (4) both even and odd values for \(J\) and \(K\). Conditions (2-4) are not physical for identical particles in quantum mechanics. We similarly define \(S_I(j^4, \text{condition } X \text{ on } J \text{ and } K)\) for \(I\).

First we present our results of \(S_I(j^4, \text{even } J \text{ even } K)\), which provides us with rich sum rules of nine-j symbols.

For \(I \geq 2j\), one obtains

\[
S_I(j^4, \text{even } J \text{ even } K)
= \frac{1}{2} \left[\frac{4(j - I)}{2} \times \frac{4(j - I + 2)}{2}\right]
\]

\[
(-)^I \frac{4j + 2 - I}{4} - 6D_I,
\]

based on Eqs. 2, 6, 8 and 9. Let us introduce \(I_0\) by the relation \(I = I_{\text{max}} - 2I_0\) for even \(I\) and \(I = I_{\text{max}} - 2I_0 - 3\) for odd \(I\), where \(I_{\text{max}} = 4j - 6\). Using the \(I_0\) we can rewrite \(D_I = D_{I_{\text{max}} - 2I_0}\) for even \(I\) and \(D_I = D_{I_{\text{max}} - 2I_0 - 3}\) for odd \(I\). According to Ref. 12,

\[
D_I = 3 \left[\frac{I_0}{6}\right] \left[\frac{I_0}{6} + 1\right] - \left[\frac{I_0}{6}\right]
+ (\left[\frac{I_0}{6}\right] + 1)(\text{mod } 6) + 1) + \delta(I_0, \text{mod } 6), 0 - 1.
\]

One thus has

\[
S_I(j^4, \text{even } J \text{ even } K)
= (-)^I \frac{4j + 2 - I}{4} + \frac{1}{2} \frac{4(j - I)}{2} \times \frac{4(j - I + 2)}{2}
- 1 \left[\frac{I_0}{6}\right] \left[\frac{I_0}{6} + 1\right] + 6 \left[\frac{I_0}{6}\right] + 1
- 6 \left[\frac{I_0}{6}\right] + 1)(\text{mod } 6) + 1) - 6\delta(I_0, \text{mod } 6), 0.
\]

The behavior of the right hand side is not easy to see due to those \((I_0 \text{ mod } 6), \delta\) and \([\frac{I_0}{6}\]), and so on. The situation becomes much more transparent when one writes \(S_I(j^4, \text{even } J \text{ even } K)\) values explicitly. For \(I = \text{even}\), we find the following formulas:

\[
S_I(j^4, \text{even } J \text{ even } K) = \begin{cases} 2 & \text{for } I = I_{\text{max}}, \\ 6 & \text{for } I = I_{\text{max}} - 2, \\ 6 & \text{for } I = I_{\text{max}} - 4, \\ 6 & \text{for } I = I_{\text{max}} - 6, \\ 8 & \text{for } I = I_{\text{max}} - 8, \\ 10 & \text{for } I = I_{\text{max}} - 10, \\ \vdots & \vdots \end{cases}
\]

(12)

for \(I = \text{odd}\) we can use Eq. 12 to obtain the sum rules: \(S_I(j^4, \text{even } J \text{ even } K) = S_{I+3}(j^4, \text{even } J \text{ even } K)\). We find that \(S_I(j^4, \text{even } J \text{ even } K)\) has a modular behavior:

\[
S_I(j^4, \text{even } J \text{ even } K)
= S_{(I_{\text{max}} - I) \text{ mod } 12}(j^4, \text{even } J \text{ even } K) + 6 \left[\frac{I_{\text{max}} - I}{12}\right].
\]

(13)
For \( I = I_{\text{max}} - 1 \), one obtains \( S_I(j^4, \text{even } JK) = 4 \) based on the right hand side of Eqs. (2) and (3).

For \( I \leq 2j - 1 \), Eq. (8) is less transparent to be simplified, due to the complexity for \( D_I \) formula (See Eq. (3) of Ref. [10]) and number of \( J = K \) and \( J \neq K \) matrices of Eq. (2). However, by using Eq. (8) of this paper, Eq. (3) of Ref. [10], and results in Appendix B, one is able to obtain explicitly the sum rules for \( I \leq 2j - 1 \):

\[
S_I(j^4, \text{even } J \text{ even } K) = \begin{cases} 
2m - 2 & \text{for } I = 0, \\
0 & \text{for } I = 1, \\
4 - 2m & \text{for } I = 2, \\
2m & \text{for } I = 3, \\
2 & \text{for } I = 4, \\
4 - 2m & \text{for } I = 5, \\
2 + 2m & \text{for } I = 6, \\
4 & \text{for } I = 7, \\
6 - 2m & \text{for } I = 8, \\
2 + 2m & \text{for } I = 9, \\
6 & \text{for } I = 10, \\
8 - 2m & \text{for } I = 11, \\
\vdots & \vdots 
\end{cases}
\]

(14)

has a modular behavior:

\[
S_I(j^4, \text{even } J \text{ even } K) = \begin{cases} 
2 - 2m & \text{for } I = 0, \\
0 & \text{for } I = 1, \\
2m & \text{for } I = 2, \\
4 - 2m & \text{for } I = 3, \\
2 & \text{for } I = 4, \\
2m & \text{for } I = 5, \\
6 - 2m & \text{for } I = 6, \\
4 & \text{for } I = 7, \\
2 + 2m & \text{for } I = 8, \\
6 - 2m & \text{for } I = 9, \\
6 & \text{for } I = 10, \\
4 + 2m & \text{for } I = 11, \\
\vdots & \vdots 
\end{cases}
\]

(17)

In Eq. (14), \( m = (j + 3/2) \mod 3 \).

In Eqs. (8) and (10), \( J \) and \( K \) take only even values. It is interesting to discuss whether there are simple sum rules in which \( J \) and \( K \) can be both even and odd. For this case \( 0 \leq I \leq I_{\text{max}} = 4j \). Starting from Eq. (9.29) of Ref. [2] for \( J_1 = J_2 = J_3 = J_4, J_{12} = J_{13} = J, J_{34} = J_{24} = K, J = I \), we multiply \( 4(2J + 1)/(2K + 1) \) and sum over all \( JK \) (i.e., \( J \) and \( K \) take both even and odd non-negative integers). Using Eq. (9.28) of Ref. [2], we find

\[
S_I(j^4, \text{both even and odd values for } J \text{ and } K) = \sum_{J,K=0}^{2j} (-)^{J+1} \Delta(JKI)
\]

\[
= \begin{cases} 
4[(1 + I)/2] & \text{for } I \leq 2j + 1, \\
4 + 4[(4j - I)/2] & \text{for } I \geq 2j,
\end{cases}
\]

(16)

where \( \Delta(JKI) \) means that \( J, K \) and \( I \) satisfy the triangle relation of angular momentum coupling.

If both \( J \) and \( K \) are odd values, \( 0 \leq I \leq I_{\text{max}} = 4j \). In this case we consider fictitious (not realistic for identical particles) “bosons” with a half integer spin \( j \). According to Ref. [12], the number of states \( D_I \) for four bosons with spin \( j \) equals that for four fermions in a single-\( l \) shell with \( 2l = 2j + 3 \). As \( D_I \) for four fermions in a single-\( l \) shell was given in Ref. [4], we can derive \( S_I(j^4, \text{odd } J \text{ odd } K) \) by using Eqs. (2) and (3), together with Eqs. (3-5) in Ref. [4]. Similar to Eqs. (10), (11) and (13), we obtain that when \( I \leq 2j \),

\[
S_I(j^4, \text{odd } J \text{ odd } K) = \begin{cases} 
2 - 2m & \text{for } I = 0, \\
0 & \text{for } I = 1, \\
2m & \text{for } I = 2, \\
4 - 2m & \text{for } I = 3, \\
2 & \text{for } I = 4, \\
2m & \text{for } I = 5, \\
6 - 2m & \text{for } I = 6, \\
4 & \text{for } I = 7, \\
2 + 2m & \text{for } I = 8, \\
6 - 2m & \text{for } I = 9, \\
6 & \text{for } I = 10, \\
4 + 2m & \text{for } I = 11, \\
\vdots & \vdots 
\end{cases}
\]

(17)

has a modular behavior:

\[
S_I(j^4, \text{odd } J \text{ odd } K) = S_{(I \mod 12)}(j^4, \text{odd } J \text{ odd } K) + 6 \left[ \frac{I}{12} \right].
\]

(18)

where \( m = (j - 3/2) \mod 3 \) in Eq. (17); and when \( I \geq 2j \)

\[
S_I(j^4, \text{odd } J \text{ odd } K) = \begin{cases} 
4 & \text{for } I = 4j, \\
2 & \text{for } I = 4j - 2, \\
4 & \text{for } I = 4j - 4, \\
6 & \text{for } I = 4j - 6, \\
6 & \text{for } I = 4j - 8, \\
6 & \text{for } I = 4j - 10, \\
\vdots & \vdots 
\end{cases}
\]

(19)

has a modular behavior:

\[
S_I(j^4, \text{odd } J \text{ odd } K) = S_{(I \mod 12)}(j^4, \text{odd } J \text{ odd } K) + 6 \left[ \frac{4j - I}{12} \right]
\]

(20)

for even \( I \), and \( S_I(j^4, \text{odd } J \text{ odd } K) = S_{I+3}(j^4, \text{odd } J \text{ odd } K) \) for odd \( I \). For \( I = 4j - 1 \) (odd \( I \)), \( S_I(j^4, \text{odd } J \text{ odd } K) = 0 \). For even \( J \) and odd \( K \) or for odd \( J \) and even \( K \), \( 0 \leq I \leq I_{\text{max}} = 4j - 1 \). For this case

\[
S_I(j^4, \text{even } J \text{ odd } K) = S_I(j^4, \text{odd } J \text{ even } K)
\]

\[
= (S_I(j^4, \text{both even and odd values for } J \text{ and } K) - S_I(j^4, \text{even } J \text{ even } K) - S_I(j^4, \text{odd } J \text{ odd } K))/2.
\]

Using this relation and above results we find that when \( I \leq 2j \),

\[
S_I(j^4, \text{even } J \text{ odd } K) = S_I(j^4, \text{odd } J \text{ even } K)
\]
has a modular behavior:

\[
S_I(l^4, \text{even } J \text{ even } K) = 4 + \frac{1}{2} \sum_{l, l}^{J, K} \left( \frac{l}{4} \right)
\]

and when \( I \geq 2j \),

\[
S_I(j^4, \text{even } J \text{ odd } K) \equiv S_I(j^4, \text{odd } J \text{ even } K)
\]

\[
= 2 + \left( \frac{I_{\text{max}} - I}{4} \right)
\]

Similarly, we obtain sum rules by replacing the half integer \( j \) to the integer \( l \). First, let us study the case for even values of \( J \) and \( K \). We find that when \( I \leq 2l \),

\[
S_I(l^4, \text{even } J \text{ even } K)
\]

\[
= \sum_{l, l}^{J, K} 4(2J + 1)(2K + 1) \left\{ \frac{l}{l} \right\}_{J, K, I}
\]

\[
= \begin{cases}
4 - 2m & \text{for } I = 0, \\
0 & \text{for } I = 1, \\
2m & \text{for } I = 2, \\
2 - 2m & \text{for } I = 3, \\
4 & \text{for } I = 4, \\
2m & \text{for } I = 5, \\
6 - 2m & \text{for } I = 6, \\
2 & \text{for } I = 7, \\
4 + 2m & \text{for } I = 8, \\
6 - 2m & \text{for } I = 9, \\
6 & \text{for } I = 10, \\
2 + 2m & \text{for } I = 11, \\
\vdots & \vdots \\
\end{cases}
\]

has a modular behavior:

\[
S_I(l^4, \text{even } J \text{ even } K)
\]

\[
= S_{(m \text{ mod } 12)}(l^4, \text{even } J \text{ even } K) + 6 \left( \frac{I_{\text{max}} - I}{12} \right)
\]

where \( I_{\text{max}} = 4l \).

If \( J \) and \( K \) take both even and odd values, similar to the process of obtaining Eq. (16), we find for \( I \leq 2l \)

\[
S_I(l^4, \text{both even and odd values for } J \text{ and } K)
\]

\[
= \begin{cases}
4 + 4 \left( \frac{4}{2} \right) & \text{for even } I, \\
4 \left( \frac{4}{2} \right) & \text{for odd } I
\end{cases}
\]

for \( I \geq 2l \),

\[
S_I(l^4, \text{both even and odd values for } J \text{ and } K)
\]

\[
= 4 + 4 \left( \frac{4l - I}{2} \right)
\]

We note this sum rule has the same form as Eq. (16) for \( I \geq 2j \).

For odd \( J \) and odd \( K \) values, \( 0 \leq I \leq I_{\text{max}} = 4l - 2 \). We find that when \( I \leq 2l \),

\[
S_I(l^4, \text{odd } J \text{ odd } K)
\]

\[
= \begin{cases}
2m & \text{for } I = 0, \\
0 & \text{for } I = 1, \\
4 - 2m & \text{for } I = 2, \\
-2 + 2m & \text{for } I = 3, \\
4 & \text{for } I = 4, \\
4 - 2m & \text{for } I = 5, \\
2 + 2m & \text{for } I = 6, \\
2 & \text{for } I = 7, \\
8 - 2m & \text{for } I = 8, \\
2 + 2m & \text{for } I = 9, \\
6 & \text{for } I = 10, \\
6 - 2m & \text{for } I = 11, \\
\vdots & \vdots \\
\end{cases}
\]
has a modular behavior:

\[ S_I(l^4, \text{odd } J \text{ odd } K) = S_{(l \mod 12)^4, \text{odd } J \text{ odd } K} + 6 \left[ \frac{J}{12} \right] \]

where \( m = l \mod 3 \) in Eq. \( \text{Eq. (30)} \); and when \( I \geq 2l \),

\[ S_I(l^4, \text{odd } J \text{ odd } K) = \begin{cases} 
2 \text{ for } I_{\text{max}} - I = 0, \\
4 \text{ for } I_{\text{max}} - I = 2, \\
2 \text{ for } I_{\text{max}} - I = 4, \\
6 \text{ for } I_{\text{max}} - I = 6, \\
6 \text{ for } I_{\text{max}} - I = 8, \\
6 \text{ for } I_{\text{max}} - I = 10, \\
\vdots \quad \vdots 
\end{cases} \]

has a modular behavior

\[ S_I(l^4, \text{odd } J \text{ odd } K) = S_{(I_{\text{max}}-l) \mod 12}^4(l^4, \text{odd } J \text{ odd } K) + 6 \left[ \frac{I_{\text{max}}-I}{12} \right] \]

For odd \( I \geq 2l \), \( S_I = S_{I+3} \) with \( I_{\text{max}}-1 = 0 \).

If we take odd \( J \) and even \( K \) values or we take even \( J \) and odd \( K \) values, \( 0 \leq I \leq I_{\text{max}} = 4l-1 \). For this case

\[
\begin{align*}
S_I(l^4, \text{even } J \text{ odd } K) & \equiv S_I(l^4, \text{odd } J \text{ even } K) \\
& = (S_I(l^4, \text{both even and odd values for } J \text{ and } K)) \\
& - S_I(l^4, \text{even } J \text{ even } K) - S_I(l^4, \text{odd } J \text{ odd } K))/2 \\
& = \begin{cases} 
2 \left[ \frac{l+2}{2} \right] & \text{for } I \leq 2l, \\
2 + \left[ \frac{I-I_{\text{max}}}{2} \right] & \text{for } I \geq 2l.
\end{cases}
\end{align*}
\]

IV. SUMMARY AND DISCUSSION

To summarize, in this paper we first show that the sum of eigenvalues of spin \( I \) states for \( J \)-pairing interaction is given by

\[
\sum_K \left( (1 + (-)^I \delta_{JK}) - 4 \sum_K (2J+1)(2K+1) \begin{pmatrix} J \\ J \\ J \\ J \\ J \\ J \end{pmatrix} \right) 
\]

for fermions, and

\[
\sum_K \left( (1 + (-)^I \delta_{JK}) - 4 \sum_K (2J+1)(2K+1) \begin{pmatrix} l \\ l \\ l \\ l \\ l \\ l \end{pmatrix} \right) 
\]

for bosons. Then we relate them with number of spin \( I \) states to obtain nine-\( j \) sum rules. We study

\[
4(2J+1)(2K+1) \begin{pmatrix} j \\ j \\ J \\ J \\ J \\ J \end{pmatrix}
\]

and

\[
4(2J+1)(2K+1) \begin{pmatrix} l \\ l \\ J \\ J \\ J \\ J \end{pmatrix}
\]

summing over \( J \) and \( K \) under following situations: (1) all \( J \) and \( K \) are even; (2) \( J \) and \( K \) can be both even and odd; (3) all \( J \) and \( K \) are odd; (4) \( J \) is even and \( K \) is odd. We also obtain formulas for special \( J, K \) and \( I \) values, based on the physical meaning of the norm in Eq. \( \text{Eq. (30)} \).

Sum rules in Eqs. (A1-A2) of Ref. \( \text{Ref. [3]} \) can be obtained as a special case of the results in this paper: \( I = 0 \) for Eqs. (14) and (23). This work is therefore a generalization of some of our earlier results. We use \( J \) pairing interaction as a tool to obtain the sum rules but these results are independent of the interaction.

In Ref. \( \text{Ref. [12]} \), it was found that number of spin \( I \) states \( D_I \) for four bosons with spin \( l \) and that for four fermions in a single-\( j \) shell are the same when \( 2l = 2j - 3 \). This produce the same value of the right hand side in Eq. \( \text{Eq. (32)} \) for fermions and bosons. Unfortunately, number of \( J \) and \( K \) for these two cases are different (number of \( J \) for bosons is \( l+1 = j-1/2 \) while that for fermions is \( j+1/2 \)), which present different sum rules of the case with even values for both \( J \) and \( K \).

One may ask how far one can go along this line, i.e., to construct sum rules of angular momentum coupling by using formulas of \( D_I \). As \( n \) increases, \( D_I \) formulas and sum of eigenvalues of spin \( I \) states become more and more complicated. The situation is already complicated for \( n = 4 \). For \( n = 5 \) there are \( D_I \) formulas for only \( I \sim 0 \) or \( I \sim I_{\text{max}} \). Therefore, it is difficult to obtain \( D_I \) formulas and new sum rules of angular momentum couplings in which more particles (\( n \geq 5 \)) are involved, except for a few cases with \( I \sim I_{\text{max}} \) where the \( D_I \) is given by a fixed number series \( \text{Ref. [10, 11]} \).

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[4] Y. M. Zhao, A. Arima, J. N. Ginocchio, and N. Yoshi-
Appendix A Formulas of special nine-j symbols

In this Appendix we present formulas for nine-j symbol

\[
\begin{aligned}
\{ j \quad j \quad J \\
j \quad j \quad J \\
J \quad J \quad I
\end{aligned}
\]

where \( J = 2j \) or \( 2j - 1 \), based on its expansion in terms of six-j symbols. Value of \( j \) in this Appendix can be either a half integer or an integer. One sees that the nine-j symbol of Eq. (34) vanishes unless \( I \) is even. Below we discuss nine-j symbols of Eq. (34), with \( I \) being even and \( J = 2j \) or \( 2j - 1 \).

We define

\[
f'_m = \begin{cases} 
j \quad j \quad 2j \\
j \quad j \quad 2j \\
2j \quad 2j \quad 4j - m
\end{cases}
\]

and obtain following formulas:

\[
\begin{aligned}
f'_0 &= \frac{1}{(4j + 1)^2}, \\
f'_2 &= \frac{-1}{2(4j + 1)^2(4j - 1)}, \\
f'_5 &= \frac{3(2j - 1)}{2(16j^2 - 1)^2(4j - 3)}, \\
f'_6 &= \frac{3 \times 5(2j - 2)}{4(4j - 5)(4j - 3)(4j - 1)^2(4j + 1)^2}, \\
f'_8 &= \frac{3 \times 5 \times 7(2j - 2)(2j - 3)}{4(4j - 7)(4j - 5)(4j - 3)(4j - 1)^2(4j + 1)^2}, \\
f'_{10} &= \frac{-3 \times 5 \times 7 \times 9}{8(4j - 3)^2(4j - 1)^2(4j + 1)^2} \times \frac{(2j - 4)(2j - 3)}{(4j - 9)(4j - 7)(4j - 5)}, \\
f'_{12} &= \frac{3 \times 5 \times 7 \times 9 \times 11}{8(4j - 5)^2(4j - 3)^2(4j - 1)^2(4j + 1)^2} \times \frac{(2j - 3)(2j - 4)(2j - 5)}{(4j - 11)(4j - 9)(4j - 7)}.
\end{aligned}
\]

We define

\[
f_I = \begin{cases} 
j \quad j \quad 2j \\
j \quad j \quad 2j \\
2j \quad 2j \quad I
\end{cases}
\]

and obtain following formulas:

\[
\begin{aligned}
f_0 &= (-2j)^2 \frac{([2j - 1]!)^2}{(4j + 1)^2(4j - 1)!^2} (2j), \\
f_2 &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \frac{1}{(2j)(2j + 1)} (4j - 1), \\
f_4 &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \frac{3}{(2j)(2j + 1)(2j + 2)} (4j - 3)(4j - 1), \\
f_6 &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \times \frac{5}{(2j)(2j + 1)(2j + 2)(2j + 3)} (4j - 5)(4j - 3)(4j - 1), \\
f_8 &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \times \frac{7 \times 5}{(2j)(2j + 1)(2j + 2)(2j + 3)} (4j - 7)(4j - 5)(4j - 3)(4j - 1), \\
f_{10} &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \times \frac{9 \times 7}{(2j)(2j + 1)^2(2j + 2)(2j + 3)(2j + 4)(4j - 9)(4j - 7)(4j - 5)(4j - 3)(4j - 1)}, \\
f_{12} &= (-2j)^2 \frac{[(2j - 1)!]^2}{(4j + 1)^2(4j - 1)!^2} \times \frac{11 \times 9 \times 7}{32} \times \frac{1}{(2j)(2j + 1)^2(2j + 2)(2j + 3)(2j + 4)(4j - 9)(4j - 7)(4j - 5)(4j - 3)(4j - 1)}, \\
f_{14} &= (-2j)^2 \frac{[(2j - 1)]!^2}{(4j + 1)^2(4j - 1)!^2} \times \frac{13 \times 11 \times 9}{32} \times \frac{1}{(2j)(2j + 1)^2(2j + 2)(2j + 3)(2j + 4)(4j - 9)(4j - 7)(4j - 5)(4j - 3)(4j - 1)},
\end{aligned}
\]
We define
\[ g_m' = \begin{cases} j & j \text{ or } j \text{ or } 2j-1 \text{ or } 2j-1 \text{ or } 4j-m \end{cases} \]  

\[ g_2' = g_1' = \frac{1}{2(4j-1)!}, \text{ see Eq.}(5) \text{ of Sec. } II. \text{ For } g_m' \text{ with larger } m \text{ we obtain} \]
\[ g_6' = -\frac{3(2j-2)(16j-15)}{2(4j-5)(4j-3)(4j-1)} \]
\[ g_8' = \frac{15(2j-3)(6j-7)}{7 \times 3} \]
\[ g_{10}' = -\frac{(4j-5)(4j-3)(4j-1)}{4(4j-9)(4j-7) \cdots (4j-1)} \]
\[ g_{12}' = \frac{(4j-5)(4j-3)(4j-1)}{4(4j-11)(4j-9) \cdots (4j-1)} \]

We define
\[ g_I = \begin{cases} j & j \text{ or } 2j-1 \text{ or } 2j-1 \text{ or } 2j-1 \text{ or } I \end{cases} \]

and obtain
\[ g_0 = (-)^{2j} \frac{(4j-3)(2j-1)!}{(4j-1)(4j-1)!} \]
\[ g_2 = (-)^{2j} \frac{(8j^2 - 6j - 3)(2j-1)!}{(4j-3)(4j-1)(4j-1)!} \]
\[ g_4 = (-)^{2j} \frac{3j(2j+1)(4j^2 - 3j - 5)(2j-1)!}{(4j-5)(4j-3)(4j-1)(4j-1)!} \]
\[ g_6 = (-)^{2j} \frac{(j+1)(2j+1)(2j-1)!}{(4j-1)!} \]
\[ g_8 = (-)^{2j} \frac{j(j+1)(2j+1)(2j-1)(2j-3)(2j-5)}{(4j-1)!} \]
\[ g_{10} = (-)^{2j} \frac{j(j+1)(j+2)(2j+1)(2j+3)}{(4j-1)!} \]
\[ g_{12} = (-)^{2j} \frac{j(j+1)(j+2)(2j+1)(2j+3)(2j+5)}{(4j-1)!} \]
\[ g_{14} = \frac{35(4j^2 - 3j - 18)}{2(4j-9)(4j-7)(4j-5)(4j-3)(4j-1)} \]
\[ g_{16} = \frac{63(8j^2 - 6j - 55)(2j-1)!}{2(4j-11)(4j-9) \cdots (4j-1)} \]
\[ g_{18} = \frac{231(4j^2 - 3j - 39)(2j-1)!}{2(4j-13)(4j-11) \cdots (4j-1)} \]

Some of above \( g_m' \) were also obtained for fermions in a single-\( j \) shell in Ref. [4] where \( j \) is a half integer, while here \( j \) can be either an integer or a half integer.

### Appendix B Number of matrices with \( K \neq J \)

for \( I \leq 2j \)

Number of matrices with \( J = K \) is always 1 for an even value of \( I \), which contribute \( 2 \times (j + \frac{1}{2}) \) on the left hand side of Eq. (5), while that (denoted by \( F_j \) here) with \( J \neq K \) is rather complicated.

For \( I \leq |J| \) with \( J > 2[|I-1|/2] \) and \( J < 2j-1-2[|J|/2] \), \( F_j = 2 \times [J/2] \):

For \( I \leq |J| \) with \( J > 2[|I-1|/2] \), \( F_j = J; \)

For \( I \leq |J| \) with \( J \geq 2j-1-2[|J|/2] \), \( F_j = \frac{1}{2}[J+1-|J|-2|J|/2]; \)

For \( |J| \leq I \leq 2j \) with \( J < 2j-1-2[|J|/2] \) and \( J < 2[|J|/2] \), \( F_j = 2j-1-2[|J|/2] \):

For \( |J| \leq I \leq 2j \) with \( J > 2j-1-2[|J|/2] \), \( F_j = (2j-1-J)/2 \).

Because of complexity in the above classification, it is tedious to show \( \sum_{J<K}(1-(-)^J) \) by one formula, because one must simplify many terms such as \( \binom{J}{k} \) which means to take the largest integer not exceeding the value inside. However, one can obtain explicit sum rules by writing down their value and studying their individual modular behavior, as shown in this paper.