Tensorial Spin-s Harmonics

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Abstract

We show how to define and go from the spin-s spherical harmonics to 
the tensorial spin-s harmonics. These quantities, which are functions on the 
sphere taking values as Euclidean tensors, turn out to be extremely useful 
for many calculations in General Relativity. In the calculations, products 
of these functions, with their needed decompositions which are given here, 
often arise naturally.
I. Introduction

The use of the ordinary spherical harmonics, \( Y_{lm}(\theta, \varphi) \) (and their associated vector and tensor harmonics) with all their properties, their eigenvalue/eigenvector behavior, their orthonormality properties, their use in solving various problems and equations in mathematical physics, their interpretations in terms of multipole expansion have been ubiquitous in theoretical physics. Several years ago a generalization of the ordinary harmonics was developed\[12\] and referred to as the spin-\( s \) spherical harmonics and denoted by

\[ (s)Y_{lm}(\theta, \varphi) \]

with \( s = 0 \) the ordinary spherical harmonics. They have proved to be very useful in problems involving spin-\( s \) fields and have become an almost essential tool in problems involving gravitational physics. Though the \( (s)Y_{lm} \) are closely related to the Wigner D-matrices, \( D^{(s)}_{mpn'} \) (and to generalized versions of vector and tensor spherical harmonics) and can be derived from them, nevertheless it is the particular form and specific properties of the \( (s)Y_{lm} \) that have been of great use.

Recently, it turns out, that there are quantities that have been appearing in many calculations that are closely related to the \( (s)Y_{lm} \), which we refer to as the tensor spin-\( s \) harmonics and are denoted by \( Y^{(s)}_{(l)i\ldots \ldots k} \). This note is devoted to a discussion of these new harmonics.

There is a one-to-one correspondence between \( Y^{(s)}_{(l)i\ldots \ldots k} \) and the \( (s)Y_{lm} \). The indices \( i\ldots k \) indicate symmetric and trace-free 3-dimensional Euclidean tensors; the number of tensor indices equals \( l \). The number of independent components, in both \( Y^{(s)}_{(l)i\ldots \ldots k} \) and \( (s)Y_{lm} \), is the same; \( N = 2l + 1 \) and, in fact, the quantities \( Y^{(s)}_{(l)i\ldots \ldots k} \) are just linear combinations of the \( (s)Y_{lm} \) and could in principle be written as

\[ Y^{(s)}_{(l)i\ldots \ldots k} = \sum_m K^{(s)(m)}_{(l)i\ldots \ldots k} \cdot (s)Y_{lm}. \]

It is however difficult and a bit unwieldy to find the \( K^{(s)(m)}_{(l)a\ldots \ldots b} \) and a more direct approach, namely to define them independently, is considerably easier.

The value in using the \( Y^{(s)}_{(l)i\ldots \ldots k} \) instead of the spin-\( s \) spherical harmonics is two-fold. (i) They simply appear as they are defined in many of the calculations and (ii) in non-linear theories very often within calculations one finds
that products of these tensor harmonics automatically are there. They, in turn, must be decomposed via a Clebsch-Gordon expansion. These expansions are most easily done with the $Y_{(ij)\ldots}^{(s)}$.

In section II we will describe how the $Y_{(ij)\ldots}^{(s)}$ arise in GR and precisely how they are defined. In addition for clarity we give many examples. In section III, we discuss how products [Clebsch-Gordon expansions] are found and give an example from GR in section IV. The main bulk of this note appears in appendices. Appendix A contains some useful miscellaneous relations while Appendix B contains a table of Clebsch-Gordon expansions of some of the most useful products.

II. The Tensor Spin-s Harmonics

A. Notation & Basics

We begin with either Minkowski space or at an arbitrary point in a Lorentzian space-time and take the metric as

$$\eta_{ab} = \text{diag}[1, -1, -1, -1].$$

(1)

The standard null tetrad, parametrized by the complex stereographic angles $(\zeta, \bar{\zeta})$, with $\zeta = e^{i\phi} \cot \frac{\theta}{2}$, can be given by

$$l^a = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} \left(1 + \zeta \bar{\zeta}, \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta}\right),$$

(2)

$$m^a = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} \left(0, 1 - \zeta^2, -i(1 + \zeta^2), 2\zeta\right),$$

(3)

$$\bar{m}^a = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} \left(0, 1 - \zeta^2, i(1 + \zeta^2), 2\zeta\right),$$

(4)

$$n^a = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}} \left(1 + \zeta \bar{\zeta}, -(\zeta + \bar{\zeta}), i(\zeta - \bar{\zeta}), 1 - \zeta \bar{\zeta}\right),$$

(5)

As $(\zeta, \bar{\zeta})$ sweeps out the sphere, the null vector $l^a$ sweeps out the light-cone and ‘drags along with it’ the remainder of the tetrad. From the tetrad we obtain the time-like and space-like vectors,
\[ t^a \equiv l^a + n^a = \sqrt{2} (1, 0, 0, 0) \quad (6) \]
\[ c^a \equiv l^a - n^a = \frac{\sqrt{2}}{(1 + \zeta \bar{\zeta})} \left( 0, \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta} \right) \quad (7) \]

with norms 2 and -2 respectively.

The main tool or ingredient in our work will be the 3-dimensional \{Euclidean\} vectors, \([i, j, \ldots. = 1, 2, 3]\), obtained by projections that are normal to \(t^a\), i.e.,

\[ l_i = \frac{-1}{\sqrt{2}(1 + \zeta \bar{\zeta})} \left( \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta} \right), \quad (8) \]
\[ m_i = \frac{-1}{\sqrt{2}(1 + \zeta \bar{\zeta})} \left( 1 - \zeta^2, -i(1 + \zeta^2), 2\zeta \right), \quad (9) \]
\[ \overline{m}_i = \frac{-1}{\sqrt{2}(1 + \zeta \bar{\zeta})} \left( 1 - \zeta^2, i(1 + \zeta^2), 2\zeta \right), \quad (10) \]
\[ n_i = -l_i = \frac{1}{\sqrt{2}(1 + \zeta \bar{\zeta})} \left( (\zeta + \bar{\zeta}), -i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta} \right), \quad (11) \]
\[ c_i = l_i - n_i = \frac{-\sqrt{2}}{1 + \zeta \bar{\zeta}} \left( \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta} \right). \quad (12) \]

In terms of \(\theta\) and \(\phi\),

\[ c_i = -\sqrt{2}(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta). \quad (13) \]

and hence \(c_i\) is just \(-\sqrt{2}\) times the unit Euclidean radial vector. Note that we have used the Minkowski metric, \([1]\) to raise and lower even the Euclidean indices in Eqs.\((\ref{eq:6}) - \(\ref{eq:12}\)) which gives rise to the minus sign.

Some of the important algebraic properties of these vectors obtained by direct calculation from their definitions, are

\[ \delta_{ij} = \frac{1}{2} c_i c_j + \overline{m}_i m_j + m_i \overline{m}_j, \quad (14) \]
\[ c_k = -\sqrt{2} e_{kij} m_i \overline{m}_j, \quad (15) \]
\[ m_k = \frac{i}{\sqrt{2}} e_{kij} m_i c_j, \quad (16) \]
\[ m_j \bar{m}_k - m_j m_k = \frac{i}{\sqrt{2}} \epsilon_{jki} c_i, \]  
(17)

\[ m_k c_j - m_j c_k = -i \sqrt{2} \epsilon_{kji} m_i. \]  
(18)

Noting that \( c_k, m_k \) and \( \bar{m}_k \) have respectively spin-weights \((0,1,-1)\) and an \( l \)-value, \( l = 1 \), we have, either from the definitions of the differential operator \( \eta(s) \) acting on a spin-wt. \( s \) function \( \eta(s) \), i.e.,

\[ \delta \eta(s) = P_0^{1-s} \partial_s (P_0^s \eta(s)), \]  
(19)

\[ \bar{\delta} \eta(s) = P_0^{1+s} \partial_s (P_0^{-s} \eta(s)), \]  
(20)

\[ P_0 = 1 + \zeta \zeta, \]  
(21)

or from the general eigenvalue relations,

\[ \bar{\delta} \delta \eta(s) Y_{lm} = -(l-s)(l+s+1)(s) Y_{lm}, \]  
(22)

\[ \bar{\delta} \delta \eta(s) Y_{lm} = -(l+s)(l-s+1)(s) Y_{lm}. \]  
(23)

the differential relations

\[ \delta c_i = 2m_i, \]  
(24)

\[ \bar{\delta} c_i = 2\bar{m}_i, \]  
(25)

\[ \delta \bar{m}_i = -c_i, \]  
(26)

\[ \bar{\delta} m_i = -c_i, \]  
(27)

\[ \bar{\delta} c_i = -2c_i, \]  
(28)

\[ \bar{\delta} \bar{m}_i = -2\bar{m}_i, \]  
(29)

\[ \bar{\delta} m_i = -2m_i. \]  
(30)

**B. Definition of \( Y^{(s)}_{(l)i \ldots k} \)**

The essential idea to define the tensor harmonics \( Y^{(s)}_{(l)i \ldots k} \) is simply to use, in an appropriate way, tensor products of the three basic Euclidean vectors, \((c_i, m_i, \bar{m}_i)\). Since each of them have the value \( l = 1 \), in any product of \( n \) terms the \( l \)-value of the product is \( l = n \). The spin wt. of the product is given by the algebraic sum of the spin wts. of the constituent vectors. In addition we require the products to be symmetric trace-free Euclidean tensors.
The easiest way to do this is to first define the spin-wt. \( s = l \) tensor harmonic \( Y_{(l)i.....k}^{(s)} \), with \( l \) indices, \( i,...,k \) or \( l \) factors \( m_k \) and \( Y_{(s)i.....k}^{(-s)} \) by

\[
Y_{(l)i.....k}^{(l)} = m_im_j.....m_k \\
Y_{(l)i.....k}^{(-l)} = \bar{m}_im_j.....\bar{m}_k.
\]

(31)  
(32)

First note that both are obviously symmetric and trace-free [since \( m_im_j\delta^{ij} = 0 \)] and that any derivatives will also be symmetric and trace-free. Furthermore we recall that the edth operators

\[ \delta \text{ and } \delta \]

are stepping operators, i.e., they add or subtract one to the value of \( s \).

For positive values of \( s \), \([s = 0, 1, ..., l]\), the \( Y_{(l)i.....k}^{(s)} \) are defined by applying the operator

\[ \delta \]

\( l - s \) times to \( Y_{(s)i.....k}^{(s)} \), i.e.,

\[
Y_{(l)i.....k}^{(s)} = \delta^{l-s} \{ Y_{(l)i.....k}^{(l)} \}
\]

(33)

and for negative values of \( s \), i.e., for \([0, -1, ..., -l]\),

\[
Y_{(l)i.....k}^{(-|s|)} = \delta^{l+|s|} \{ Y_{(l)i.....k}^{(-l)} \}.
\]

(34)

Note that

\[ Y_{(l)i.....k}^{(-|s|)} = Y_{(l)i.....k}^{(s)} \]

It is easy to show that the tensor harmonics satisfy the eigenvalue equations

\[
\delta \delta Y_{(l)i.....k}^{(s)} = -(l-s)(l+s+1)Y_{(l)i.....k}^{(s)} \]

(35)

\[
\delta \delta Y_{(l)i.....k}^{(s)} = -(l+s)(l-s+1)Y_{(l)i.....k}^{(s)}.
\]

(36)

C. Examples

In order to clarify these definitions and for later use, using Eqs. 24, 27 and 14, we give several examples:
1 = 0:

\[ Y_0^0 = 1. \]  \hspace{1cm} (37)

1 = 1:

\[ Y_{1i}^1 = m_i, \tag{38} \]
\[ Y_{1i}^0 = \delta Y_{1i}^1 = \delta Y_{1i}^{-1} = -c_i \tag{39} \]
\[ Y_{1i}^{-1} = \overline{m}_i. \tag{40} \]

1 = 2:

\[ Y_{2ij}^2 = m_im_j, \tag{41} \]
\[ Y_{2ij}^1 = \delta Y_{2ij}^2 = -(c_im_j + m_ic_j), \tag{42} \]
\[ Y_{2ij}^0 = \delta Y_{2ij}^1 = 3c_ic_j - 2\delta_{ij}, \tag{43} \]
\[ Y_{2ij}^{-1} = -(c_i\overline{m}_j + \overline{m}_ic_j), \tag{44} \]
\[ Y_{2ij}^{-2} = \overline{m}_i\overline{m}_j. \tag{45} \]

1 = 3:

\[ Y_{3ijk}^3 = m_im_jm_k \tag{46} \]
\[ Y_{3ijk}^2 = -(c_im_jm_k + m_ic_jm_k + m_im_jc_k) \]
\[ Y_{3ijk}^1 = -[\delta_{ij}m_k + \delta_{kj}m_i + \delta_{ik}m_j] + \frac{5}{2}[c_ic_jm_k + c_kc_jm_i + c_ic_km_j] \]
\[ Y_{3ijk}^0 = 6(\delta_{ij}c_k + \delta_{ik}c_j + \delta_{kj}c_i) - 15c_ic_jc_k \]

1 = 4:
\[ Y_{ijkl}^4 = m_i m_j m_k m_l \] (47)
\[ Y_{ijkl}^3 = -c_i m_j m_k m_l - m_i c_j m_k m_l - m_i m_j c_k m_l - m_i m_j m_k c_l \]
\[ Y_{ijkl}^2 = 2[c_i c_j m_k m_l - \overline{m}_j m_i m_j m_k m_l - \overline{m}_j m_i m_j m_k m_l - \overline{m}_j m_i m_j m_k m_l + c_j m_k (c_i m_i + c_j m_l) + c_k (c_i m_i + c_j m_j + (c_j m_i + c_j m_j) m_l)] \]
\[ Y_{ijkl}^1 = 6[\overline{m}_i c_k m_i m_j + \overline{m}_k c_i m_i m_j + \overline{m}_j c_i m_i m_k + \overline{m}_i c_i m_i m_k + \overline{m}_j c_i m_i m_l + \overline{m}_i c_i m_i m_l - c_k (c_i m_i + c_i m_l)] \]
\[ Y_{ijkl}^0 = 105 c_i c_j c_k c_l + 12[\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + \delta_{ij} \delta_{kl}] - 30[\delta_{ij} c_k c_l + \delta_{ik} c_j c_l + \delta_{ik} c_k c_l + \delta_{ij} c_k c_l + \delta_{kl} c_i c_j] \]

**III. Products of the \(Y_{l}^{(s)}(l)^{i} \ldots \ldots k\)**

Very often in doing detailed calculations in general relativity due to the non-linearity one has to deal with products of different \(Y_{l}^{(s)}(l)^{i} \ldots \ldots k\). Usually they involve only small values of both \(s\) and \(l\), most often from 0 to 3 or 4. In principle, by using the products of the Wigner \(D\)-functions, i.e., Clebsch-Gordon expansions, one could work out the \(Y_{l}^{(s)}(l)^{i} \ldots \ldots k\) products. In practice since the conversion of the \(Y_{l}^{(s)}(l)^{i} \ldots \ldots k\) to the \(D_{mmt'}^l\) is quite complicated - with various conventions - and we are most often interested in the low values of \(s\) and \(l\), it is easier to do the calculation of the expansion directly.

First, instead of using \((c_i, m_i, \overline{m}_i)\), we use

\[(Y_{1i}^0, Y_{1i}^1, Y_{1i}^{-1}) = (-c_i, m_i, \overline{m}_i).\]

The simplest and most important product relations are found most easily by direct calculations using the definitions, (9), (10) and (12), of \((c_i, m_i, \overline{m}_i)\) yielding

\[ Y_{1j}^1 Y_{1i}^{-1} - Y_{1j}^1 Y_{1i}^{-1} = -\frac{i}{\sqrt{2}} \epsilon_{ijk} Y_{1k}^0, \] (48)
\[ Y_{1j}^1 Y_{1k}^0 - Y_{1k}^1 Y_{1j}^0 = i\sqrt{2} \epsilon_{jkl} Y_{1l}^1. \] (49)
To find the products $Y_{ij}^0 Y_{1j}^0$, $Y_{1k}^0 Y_{ij}^1$ and $Y_{1k}^0 Y_{ij}^0$ one writes them out as a sum of tensor terms from $l = 0$ to $l = 2$ and applies the operator
\[ \delta \]
several times using the eigenvalue equations (55) and (56) to determine the coefficients.

For example $Y_{ij}^0 Y_{1k}^0$ could be written out as
\[ Y_{ij}^0 Y_{1k}^0 = A Y_{ij}^0 + B^j Y_{1j}^0 + C^{ij} Y_{2jk}^0 \]  \hspace{1cm} (50)
with $A$, $B^i$ and $C^{ij}$ to be determined, and then from the eigenfunction relations, we have
\[ \overline{\delta} | Y_{ij}^0 Y_{1k}^0 \rangle = -2 B^j Y_{1j}^0 - 6 C^{ij} Y_{2jk}^0 \]  \hspace{1cm} (51)
\[ \overline{\delta} \overline{\delta} | Y_{ij}^0 Y_{1k}^0 \rangle = 4 B^j Y_{1j}^0 + 36 C^{ij} Y_{2jk}^0. \]  \hspace{1cm} (52)

Since the left sides of (50), (51) and (52) are known we can evaluate the $A$, $B^i$ and $C^{ij}$ yielding
\[ Y_{ij}^0 Y_{1k}^0 = \frac{2}{3} \delta_{ij} + \frac{1}{3} Y_{2ij}^0. \]  \hspace{1cm} (53)

In a similar manner, with the help of (48) and [49], we obtain
\[ Y_{ij}^1 Y_{1k}^0 = \frac{i}{\sqrt{2}} \epsilon_{ijk} Y_{1k}^0 + \frac{1}{2} Y_{2ij}^1 \]  \hspace{1cm} (54)
\[ Y_{ij}^1 Y_{1j}^{-1} = \frac{1}{3} \delta_{ij} - \frac{i}{4} \epsilon_{ijk} Y_{1k}^0 - \frac{1}{12} Y_{2ij}^0. \]  \hspace{1cm} (55)

In principle all products expansions can be found in this manner. As the details can become quite tedious we simply list the most important ones in the Appendix B.

IV. Application: The Robinson-Trautman Equation

Though there are many applications of these results that will be given elsewhere, here we will show how the tensor harmonics can be used to approximate solutions to the Robinson-Trautman equation (43), the final equation that determines the type II, non-twisting metrics. Our purpose, in this example, is simply to illustrate how the tensor harmonics and their products
enter into GR calculations and so we will not be concerned with the certain
details.

I. Robinson and A. Trautman in their investigation of algebraically special
vacuum metrics with non-twisting principle null vector found that the
algebraically special type II metrics could be reduced to the single partial
differential equation for a ‘mass parameter’, \( \chi(\tau) \), as a function of
the ‘time’ parameter and a function

\[
P(\tau, \zeta, \bar{\zeta}) = V(\tau, \zeta, \bar{\zeta})(1 + \zeta \bar{\zeta}) \equiv V(\tau, \zeta, \bar{\zeta})P_0,
\]
a time dependent conformal factor for a two-surface metric.

**Remark 1** The function \( \chi(\tau) \) though not the Bondi mass is a close relative.
The Bondi mass is given, up to a numerical factor, by \( M_B = \chi(\tau)W(\tau) \),
where

\[
W(\tau) = \int \frac{d\zeta d\bar{\zeta}}{V^3(1 + \zeta \bar{\zeta})^2}
\]  
(56)

One can easily show\(^4\) from Eq. (56) that \( M_B' < 0 \), i.e., the Bondi mass
loss theorem.

The Robinson/Trautman equation could be rewritten\(^3\) using the edth
notation, \(^5\), as

\[
\chi' - 3 \frac{V'}{V} \chi = V^3[\delta \delta \delta V + 2 \delta \delta V] - V^2(\delta^2 V)(\delta^2 V)
\]  
(57a)

with (') meaning the \( \tau \)-derivative. They showed that by choosing a different
time parameter \( \tau \rightarrow \tau^* = F(\tau) \) and rescaling the \( V \) one could then make

\[
\chi' = 0
\]

thereby simplifying the equation. We however will use the reparametrizeaion
freedom in a different way: we choose it to make the leading term in \( V \) to be one
and thereby allow \( \chi \) to be time dependent. In particular we assume
that \( V \) takes the form of the tensor harmonic expansion with the \( l = 0 \) term
unity:

\[
V = 1 - \frac{1}{2} \eta^{ij} Y^{0}_{1i} + \eta^{ij} Y^{0}_{2ij} + \ldots
\]  
(58)

with \( \eta^{ij} \) symmetric and trace-free.
Remark 2: In other publications, and for specific reasons, we have chosen the \( \eta \)'s as \( \tau \)-derivatives of other functions, i.e., \( \eta = \zeta' \). In the present work this is not necessary.

Our object is to try to find approximate solutions to (57a) by assuming that, in an expansion near the Schwarzschild solution, \( \chi \) is zero order while \( \eta^i \) and \( \eta^{ij} \) are respectively first and second order. Note that the order of the \( \tau \)-derivatives is at this stage not known but be determined by the differential equation, (57a). Furthermore we will truncate the harmonic series at two, the quadrupole term. The expression for \( V \), (58), will be substituted into (57a) and expanded up to the \( l = 2 \). This yields three different evolution equations, \([l = 0, 1, 2]\), for the \( \tau \)-derivatives of \( \chi \), \( \eta^i \) and \( \eta^{ij} \). For a reason made clear later we will keep terms up to fourth order, even though most will eventually be discarded.

As a preliminary to the substitution of \( V \) into (57a) we calculate from the eigenvalue equations, (55) and (56), the following relations:

\[
\delta \delta V = -\frac{1}{2} \eta^i \delta \delta Y^0_{1i} + \eta^{ij} \delta \delta Y^0_{2ij} = \eta^i Y^0_{1i} - 6 \eta^{ij} Y^0_{2ij},
\]

(59)

\[
\delta \delta \delta \delta V = -2 \eta^i Y^0_{1i} + 36 \eta^{ij} Y^0_{2ij},
\]

(60)

\[
\delta \delta \delta \delta V + 2 \delta \delta V = 24 \eta^{ij} Y^0_{2ij},
\]

(61)

\[
\delta^2 V = \delta^2[1 - \frac{1}{2} \eta^i Y^0_{1i} + \eta^{ij} Y^0_{2ij}] = \eta^{ij} \delta^2 Y^0_{2ij} = 24 \eta^{ij} Y^2_{2ij},
\]

(62)

\[
\delta^2 V = 24 \eta^{ij} Y^{-2}_{2ij},
\]

(63)

\[
(\delta^2 V)(\delta^2 V) = (24)^2 \eta^{ij} \eta^{kl} Y^2_{2ij} Y^{-2}_{2kl},
\]

(64)

Using these relations, Eq. (57a) can be written as

\[
V \chi' - 3 V' \chi - V^4[24 \eta^{ij} Y^0_{2ij} + ...] + V^3[(24)^2 \eta^{ij} \eta^{kl} Y^2_{2ij} Y^{-2}_{2kl} + ...] = 0
\]

(65a)

By expanding \( V^4 \) as

11
\[ V^4 = \left[ 1 - \frac{1}{2} \eta^i Y^0_{1i} + \eta^{ij} Y^0_{2ij} \right]^4 \]

\[ = 1 - 2\eta^i Y^0_{1i} + \frac{3}{2} \eta^i \eta^j Y^0_{1i} Y^0_{1j} + 4\eta^{ij} Y^0_{2ij} + \ldots \]

and using the product

\[ Y^0_{1i} Y^0_{2ij} = \frac{2}{3} \delta_{ij} + \frac{1}{3} Y^0_{2ij} \]

we have

\[ V^4 = 1 - 2\eta^i Y^0_{1i} + (\frac{1}{2} \eta^i \eta^j + 4\eta^{ij}) Y^0_{2ij} + \ldots \] (68)

Up to fourth-order, Eq. (63) reduces to

\[-\chi' + \frac{1}{2} (\chi' \eta^i - 3\chi \eta^{ij}) Y^0_{1i} + [-\chi' \eta^{ij} + 3\chi' \eta^{ji} + 24(1 + \eta^k \eta^k) \eta^{ij}] Y^0_{2ij} \]

\[-48 \eta^i \eta^{kl} Y^0_{1i} Y^0_{2kl} + (12 \eta^i \eta^j + 96 \eta^{ij}) \eta^{kl} Y^0_{2ij} Y^0_{2kl} \]

\[-(24)^2 \eta^i \eta^{kl} Y^2_{2ij} Y^{-2} \]

From the product relations

\[ Y^0_{1i} Y^0_{2jk} = -\frac{4}{3} \delta_{kj} Y^0_{1i} + \frac{6}{5} (\delta_{ij} Y^0_{1k} + \delta_{ik} Y^0_{1j}) + \frac{1}{3} Y^0_{3ijk} \]

and those for \( Y^0_{2ij} Y^0_{2kl} \) and \( Y^2_{2ij} Y^{-2} \), given in the appendix B, (and using the symmetry and trace-free properties of \( \eta^{ij} \)), we have after a lengthy calculation that Eq. (63) becomes, to fourth-order and to the \( l = 2 \) harmonic,

\[-\chi' + \frac{1}{2} \left\{ \chi' \eta^i - 3\chi \eta^{ij} \right\} Y^0_{1i} + \left\{ -\chi' \eta^{ij} + 3\chi' \eta^{ji} + 24(1 + \eta^k \eta^k) \eta^{ij} \right\} Y^0_{2ij} \]

\[+ 48\{ \frac{6}{5} (\eta^i \eta^j + 8 \eta^{ij}) \eta^{ij} - \frac{48}{7} \left\{ \frac{1}{24} \eta^k \eta^k \eta^{ij} - \frac{1}{8} \eta^i \eta^k + \eta^{ik} \eta^{kj} \right\} Y^0_{2ij} \} \]

\[-48\{ \frac{12}{5} \eta^i \eta^{ji} Y^0_{1i} \} - (24)^2 \{ \frac{1}{5} \eta^{ij} \eta^{ij} - \frac{1}{7} \eta^{ij} \eta^{jk} Y^0_{2ij} \} = 0. \]

From the \( l = 0, 1, 2 \) harmonics this is equivalent to
\[-\chi' + \frac{(24)^2}{5} (\frac{1}{2} \eta^i \eta^j + 3 \eta^{ij}) \eta^{ij} = 0, \quad (72)\]

\[3\chi \eta^{i'} - \chi' \eta^i + \frac{2(24)^2}{5} \eta^j \eta^{ji} = 0, \quad (73)\]

\[-\chi' \eta^{ij} + 3 \chi \eta^{ij'} + 24 \eta^{ij} + \frac{12(6)}{7} \eta^k \eta^k \eta^{ij} \]
\[+ \text{SymTrFr}(ij) \left( \frac{(24)^2}{7} \left( \frac{1}{2} \eta^i \eta^k + 5 \eta^{ik} \eta^{kj} \right) \right) = 0, \quad (74)\]

where \(\text{SymTrFr}(ij)\) means symmetrize and remove the trace.

These three equations will be used in two different ways: (i) They will first be used to determine the order of the \(\tau\)-derivatives and consequently simplify the equations and (ii) then integrate them.

From the first we see that \(\chi'\) is fourth order

\[\chi' = \frac{(24)^2}{5} \left( \frac{1}{2} \eta^i \eta^j + 3 \eta^{ij} \right) \eta^{ij} = 0. \quad (75)\]

Using this in the second equation and keeping only the leading term, we have that \(\eta^{i'}\) is third order,

\[\chi \eta^{i'} = -\frac{(16)(24)}{5} \eta^j \eta^{ji}. \quad (76)\]

while the third equation for \(\eta^{ij'}\), is second order,

\[\chi \eta^{ij'} = -8 \eta^{ij}. \quad (77)\]

These equations, by taking them in the reverse order and treating \(\chi = \text{constant}\) in the 2\(\text{nd}\) and 3\(\text{rd}\) equations, can be integrated as

\[\eta^{ij} = \eta_0^{ij} e^{-8\chi^{-1}\tau}. \quad (78)\]

The second equation can be integrated exactly with the solutions depending on an exponential of the exponential \(e^{-8\chi^{-1}\tau}\), i.e., it involves functions of the form

\[e^{\exp[-8\chi^{-1}\tau]}\]
which goes to one rapidly so here we simply take \( \eta^i \) as constant,

\[
\eta^i = \eta^i_0.
\]

[The same result could be obtained by neglecting the third order term.] The first equation then simply integrates to the form

\[
\chi = \chi_0 + Ke^{-16\chi^{-1}r}.
\]  \hspace{1cm} (79)

Since here we are not interested in the details of the Robinson-Trautman metric, but only to illustrate the use of the tensor harmonics, we will not give the detailed expressions for these quantities. We simply mention that using Eqs. (58) and (79), we can obtain the evolution of the Bondi mass.
References


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VI. Appendices

A. miscellaneous relations

\[ i = 1: \]

\[
\begin{align*}
Y_{i_1}^0 &= \delta Y_{i_1}^1 = -c_i \\
Y_{i_1}^{-1} &= \delta Y_{i_1}^{-1} = -c_i \\
\delta Y_{i_1}^0 &= -2m_i = -2Y_{i_1}^1 \\
\delta Y_{i_1}^{-1} &= -2\bar{m}_i = -2Y_{i_1}^{-1} \\
\bar{\delta} Y_{i_1}^0 &= -2Y_{i_1}^0 \\
\delta Y_{i_1}^1 &= 0 \Rightarrow \bar{\delta} Y_{i_1}^1 = 0 \\
\delta Y_{i_1}^{-1} &= 0 \Rightarrow \bar{\delta} Y_{i_1}^{-1} = 0,
\end{align*}
\]
\[ \delta Y_{1i}^{-1} = -2Y_{1i}^{-1} \]
\[ \delta Y_{1i}^{1} = -2Y_{1i}^{1} \]  \hspace{1cm} (87)

\[ \delta Y_{1i}^{1} = \delta Y_{1i}^{2} = -(c_{i}m_{j} + m_{i}c_{j}) \]  \hspace{1cm} (89)

\[ \delta Y_{1i}^{0} = \delta Y_{1i}^{2} = -4m_{i}m_{j} = -4Y_{2ij}^{2} \]  \hspace{1cm} (90)

\[ Y_{2ij}^{0} = \delta Y_{2ij}^{2} = 3c_{i}c_{j} - 2\delta_{ij} \]  \hspace{1cm} (91)

\[ \delta Y_{2ij}^{0} = 6(m_{i}c_{j} + c_{i}m_{j}) = -6Y_{2ij}^{1} \]  \hspace{1cm} (92)

\[ \delta Y_{2ij}^{0} = -6Y_{2ij}^{0} \]  \hspace{1cm} (93)

\[ Y_{3ijk}^{2} = \delta Y_{3ijk}^{3} = -(c_{i}m_{j}m_{k} + m_{i}c_{j}m_{k} + m_{i}m_{j}c_{k}) \]  \hspace{1cm} (94)

\[ Y_{3ijk}^{1} = \delta Y_{3ijk}^{2} = 2[c_{i}c_{j}m_{k} + c_{i}m_{j}c_{k} + m_{i}c_{j}c_{k} \]  
\[ -m_{i}m_{j}m_{k} - m_{i}m_{j}m_{k} - m_{i}m_{j}m_{k}] \]  \hspace{1cm} (95)

\[ Y_{3ijk}^{1} = 4[\delta_{ij}m_{k} + \delta_{kj}m_{i} + \delta_{ik}m_{j}] \]  
\[ -10[m_{i}m_{j}m_{k} + m_{i}m_{j}m_{k} + m_{i}m_{j}m_{k}] \]  \hspace{1cm} (96)

\[ Y_{3ijk}^{1} = -[\delta_{ij}m_{k} + \delta_{kj}m_{i} + \delta_{ik}m_{j}] + \frac{5}{2}(c_{i}c_{j}m_{k} + c_{k}c_{j}m_{i} + c_{k}c_{i}m_{j}) \]  \hspace{1cm} (97)

\[ Y_{3ijk}^{0} = \delta Y_{3ijk}^{1} = 6(\delta_{ij}c_{j} + \delta_{ik}c_{j} + \delta_{kj}c_{i}) - 15c_{i}c_{j}c_{k} \]  \hspace{1cm} (98)

\[ \delta Y_{3ijk}^{2} = \delta Y_{3ijk}^{3} = -6m_{i}m_{j}m_{k} = -6Y_{3ijk}^{3} \]  \hspace{1cm} (99)

\[ \delta Y_{3ijk}^{0} = \delta Y_{3ijk}^{2} = -10Y_{3ijk}^{2} \]  \hspace{1cm} (100)

\[ \delta Y_{3ijk}^{0} = -12Y_{3ijk}^{1} \]  \hspace{1cm} (101)

\[ \delta Y_{3ijk}^{2} = -10Y_{3ijk}^{2} \]  \hspace{1cm} (102)

\[ \delta Y_{3ijk}^{1} = -10Y_{3ijk}^{1} \]  \hspace{1cm} (103)

\[ \delta Y_{3ijk}^{0} = -12Y_{3ijk}^{0} \]  \hspace{1cm} (104)
B. Clebsch-Gordon Expansions

We present a table of products involving functions with \( s = (2, 1, 0, -1, -2) \) and \( l = (0, 1, 2) \).

1. Products of \( l = 1 \) with \( l = 1 \):

\[
Y^1_{ij}Y^0_{ij} = \frac{i}{\sqrt{2}} \epsilon_{ijk} Y^1_{1k} + \frac{1}{2} Y^1_{2ij} \tag{105}
\]

\[
Y^1_{ij}Y^{-1}_{ij} = \frac{1}{3} \delta_{ij} - \frac{i\sqrt{2}}{4} \epsilon_{ijk} Y^0_{1k} - \frac{1}{12} Y^0_{2ij} \tag{106}
\]

\[
Y^0_{ij}Y^{-1}_{ij} = \frac{2}{3} \delta_{ij} + \frac{1}{3} Y^0_{2ij} \tag{107}
\]

2. Products of \( l = 1 \) with \( l = 2 \):

\[
Y^1_{ij}Y^2_{2jk} = Y^3_{3ijk} \tag{108}
\]

\[
Y^0_{ij}Y^0_{2jk} = -\frac{4}{5} \delta_{kj} Y^0_{1i} + \frac{6}{5} (\delta_{ij} Y^0_{1k} + \delta_{ik} Y^0_{1j}) + \frac{1}{5} Y^0_{3ijk} \tag{109}
\]

\[
Y^1_{ij}Y^0_{2jk} = \frac{2}{5} Y^1_{1i} \delta_{kj} - \frac{3}{5} Y^1_{1j} \delta_{ik} - \frac{3}{5} Y^1_{1k} \delta_{ij}
+ \frac{i}{\sqrt{2}} (\epsilon_{ikl} Y^1_{2jl} + \epsilon_{ijl} Y^1_{2kl}) + \frac{2}{5} Y^1_{3ijk} \tag{110}
\]

\[
Y^1_{ij}Y^1_{2jk} = -\frac{1}{6} \delta_{ij} Y^1_{1k} Y^1_{2jk} \tag{111}
\]

\[
Y^2_{2ij}Y^1_{1k} = \frac{3}{10} Y^0_{1i} \delta_{jk} + \frac{3}{10} Y^0_{1j} \delta_{ik} - \frac{2}{10} Y^0_{1k} \delta_{ij}
+ \frac{i\sqrt{2}}{12} (\epsilon_{ikl} Y^0_{2jl} + \epsilon_{ijl} Y^0_{2kl}) - \frac{1}{30} Y^0_{3ijk} \tag{112}
\]

\[
Y^0_{ij}Y^1_{2jk} = -\frac{2}{5} Y^0_{1i} \delta_{jk} + \frac{3}{5} Y^1_{1j} \delta_{ik} + \frac{3}{5} Y^1_{1k} \delta_{ij}
- \frac{i}{3\sqrt{2}} (\epsilon_{ikl} Y^1_{2jl} + \epsilon_{ijl} Y^1_{2kl}) + \frac{4}{15} Y^1_{3ijk} \tag{113}
\]
\[ -\frac{i\sqrt{2}}{12} [\epsilon_{ijkl} Y^1_{2jl} + \epsilon_{jkl} Y^1_{2il}] - \frac{1}{30} Y^1_{3ijk} \]  
(114)

\[ Y^2_{2ij} Y^0_{1k} = \delta[Y^2_{2ij} Y^{-1}_{1k}] \]  
(115)

3. **Products of** \( l = 2 \) **with** \( l = 2 \):

**total** \( s = 4 \)

\[ Y^4_{4ijkl} = Y^2_{2ij} Y^2_{2kl} \]  
(116)

**total** \( s = 3 \)

\[ Y^2_{2kl} Y^1_{2ij} = -\frac{i}{\sqrt{2}} [\epsilon_{ile} Y^3_{3jke} + \epsilon_{jke} Y^3_{3ile}] + \frac{1}{2} Y^3_{4ijkl} \]  
(117)

**total** \( s = 2 \)

The numbers in parentheses as superscripts give the eigenvalues of the associated quantities:

\[ Y^1_{2kl} Y^1_{2ij} = \frac{3}{7} K^{2(0)}_{ijkl} + \frac{4}{7} K^{2(14)}_{ijkl}, \]  
(118)

\[ Y^0_{2kl} Y^2_{2ij} = -\frac{3}{7} K^{2(0)}_{ijkl} + \frac{1}{2} K^{2(6)}_{ijkl} + \frac{3}{7} K^{2(14)}_{ijkl}, \]  
(119)

where the \( K' \)s are eigenfunctions with \( l = (2, 3, 4) \)

\[ K^{2(0)}_{ijkl} = Y^1_{2kl} Y^1_{2ij} - \frac{2}{3} (Y^0_{2kl} Y^2_{2ij} + Y^2_{2kl} Y^0_{2ij}), \quad l = 2 \]  
(120)

\[ K^{2(6)}_{ijkl} = Y^0_{2kl} Y^2_{2ij} - Y^2_{2kl} Y^0_{2ij}, \quad l = 3 \]  
(121)

\[ K^{2(14)}_{ijkl} = Y^1_{2kl} Y^1_{2ij} + \frac{1}{2} (Y^0_{2kl} Y^2_{2ij} + Y^2_{2kl} Y^0_{2ij}), \quad l = 4 \]  
(122)

and explicitly decomposed as

\[ K^{2(0)}_{ijkl} = -\frac{8}{3} (\delta_{kl} Y^2_{2ij} + \delta_{ij} Y^2_{2kl}) + 2[\delta_{ij} Y^2_{2ik} + \delta_{ik} Y^2_{2lj} + \delta_{kj} Y^2_{2il} + \delta_{il} Y^2_{2kj}], \] 

\[ K^{2(6)}_{ijkl} = \frac{i}{\sqrt{2}} [\epsilon_{ike} Y^2_{3ejl} + \epsilon_{jke} Y^2_{3eik} + \epsilon_{ile} Y^2_{3eik} + \epsilon_{jle} Y^2_{3eik}], \] 

\[ K^{2(14)}_{ijkl} = \frac{1}{2} Y^2_{4ijkl}. \]  
(123)
\textbf{total} \( s = 1 \)

\[
Y_{2ij}^0 Y_{2kl}^1 = \frac{3}{10} J_{4ijkl}^{1(0)} + \frac{1}{14} J_{4ijkl}^{1(4)} + \frac{1}{5} J_{4ijkl}^{1(10)} + \frac{3}{7} \bar{J}_{4ijkl}^{1(18)} \tag{124}
\]

\[
Y_{2ij}^{-1} Y_{2kl}^2 = \frac{1}{20} J_{4ijkl}^{1(0)} + \frac{1}{28} J_{4ijkl}^{1(4)} - \frac{1}{20} J_{4ijkl}^{1(10)} - \frac{1}{28} \bar{J}_{4ijkl}^{1(18)} \tag{125}
\]

where the \( J \)'s are eigenfunctions with \( l = (1, 2, 3, 4) \)

\[
J_{4ijkl}^{1(0)} = C_{4ijkl}^{1} - D_{4ijkl}^{1} + 4E_{4ijkl}^{1} - 4F_{4ijkl}^{1}, \quad l = 1 \tag{126}
\]

\[
J_{4ijkl}^{1(4)} = C_{4ijkl}^{1} + D_{4ijkl}^{1} + 12E_{4ijkl}^{1} + 12F_{4ijkl}^{1}, \quad l = 2 \tag{127}
\]

\[
J_{4ijkl}^{1(10)} = C_{4ijkl}^{1} - D_{4ijkl}^{1} - 6E_{4ijkl}^{1} + 6F_{4ijkl}^{1}, \quad l = 3 \tag{128}
\]

\[
J_{4ijkl}^{1(18)} = C_{4ijkl}^{1} + D_{4ijkl}^{1} - 2E_{4ijkl}^{1} - 2F_{4ijkl}^{1}, \quad l = 4 \tag{129}
\]

with

\[
C_{4ijkl}^{1} \equiv Y_{2ij}^0 Y_{2kl}^1 \tag{130}
\]

\[
D_{4ijkl}^{1} \equiv Y_{2ij}^{-1} Y_{2kl}^2 \tag{131}
\]

\[
E_{4ijkl}^{1} \equiv Y_{2ij}^{-1} Y_{2kl}^2 \tag{132}
\]

\[
F_{4ijkl}^{1} \equiv Y_{2ij}^{-1} Y_{2kl}^2 \tag{133}
\]

and decomposed as

\[
J_{4ijkl}^{1(0)} = i2\sqrt{2}[(\delta_{ik}\epsilon_{ljf} + \delta_{jk}\epsilon_{lij} + \delta_{il}\epsilon_{kjf} + \delta_{jl}\epsilon_{kij})Y_{1f}^1],
\]

\[
J_{4ijkl}^{1(4)} = 6[\delta_{li} Y_{2kj}^1 + \delta_{ik} Y_{2lj}^1 + \delta_{jl} Y_{2ik}^1 + \delta_{ij} Y_{2lk}^1] - 8[\delta_{ij} Y_{2lk}^1 + \delta_{i} Y_{2lj}^1],
\]

\[
J_{4ijkl}^{1(10)} = -\frac{i}{\sqrt{2}}[\epsilon_{ike} Y_{3ejl}^1 + \epsilon_{jke} Y_{3ei,l}^1 + \epsilon_{ile} Y_{3ejk}^1 + \epsilon_{jle} Y_{3eik}^1],
\]

\[
J_{4ijkl}^{1(18)} = \frac{1}{3} Y_{4ijkl}^1, \tag{134}
\]

\textbf{total} \( s = 0 \)

\[
Y_{2kl}^2 Y_{2ij}^{-2} = \frac{1}{5} F_{ijkl}^{0(0)} + \frac{2}{5} F_{ijkl}^{0(2)} + \frac{2}{7} F_{ijkl}^{0(6)} + \frac{1}{10} F_{ijkl}^{0(12)} + \frac{1}{70} F_{ijkl}^{0(20)}, \tag{135}
\]
\[ Y_{2kl}^1 Y_{2ij}^{-1} = \frac{4}{5} F_{ijkl}^{0(0)} + \frac{4}{5} F_{ijkl}^{0(2)} - \frac{4}{7} F_{ijkl}^{0(6)} - \frac{4}{5} F_{ijkl}^{0(12)} - \frac{8}{35} F_{ijkl}^{0(20)}, \]  
\[ Y_{2kl}^0 Y_{2ij}^0 = \frac{24}{5} F_{ijkl}^{0(0)} - \frac{48}{7} F_{ijkl}^{0(6)} + \frac{72}{35} F_{ijkl}^{0(20)}, \]  
(136)
(137)
with the eigenfunctions \( F \) with \( l = (0, 1, 2, 3, 4) \)

\[ l = 0 : F_{ijkl}^{0(0)} = (Y_{2kl}^2 Y_{2ij}^{-2} + Y_{2ij}^2 Y_{2kl}^{-2}) + \frac{1}{4} (Y_{2kl}^1 Y_{2ij}^{-1} - Y_{2ij}^{-1} Y_{2kl}^1) + \frac{1}{24} Y_{2kl}^0 Y_{2ij}^0 Y_{2kl}^0. \]  
(138)
\[ l = 1 : F_{ijkl}^{0(2)} = (Y_{2kl}^2 Y_{2ij}^{-2} - Y_{2ij}^2 Y_{2kl}^{-2}) + \frac{1}{8} (Y_{2kl}^1 Y_{2ij}^{-1} + Y_{2kl}^1 Y_{2ij}^{-1}), \]  
(139)
\[ l = 2 : F_{ijkl}^{0(6)} = (Y_{2kl}^2 Y_{2ij}^{-2} + Y_{2ij}^2 Y_{2kl}^{-2}) - \frac{1}{8} (Y_{2kl}^1 Y_{2ij}^{-1} + Y_{2kl}^1 Y_{2ij}^{-1}) - \frac{1}{24} Y_{2kl}^0 Y_{2ij}^0 Y_{2kl}^0. \]  
(140)
\[ l = 3 : F_{ijkl}^{0(12)} = (Y_{2kl}^2 Y_{2ij}^{-2} - Y_{2ij}^2 Y_{2kl}^{-2}) - \frac{1}{2} (Y_{2kl}^1 Y_{2ij}^{-1} - Y_{2ij}^{-1} Y_{2kl}^1), \]  
(141)
\[ l = 4 : F_{ijkl}^{0(20)} = (Y_{2kl}^2 Y_{2ij}^{-2} + Y_{2ij}^2 Y_{2kl}^{-2}) - (Y_{2kl}^1 Y_{2ij}^{-1} + Y_{2kl}^1 Y_{2ij}^{-1}) + \frac{1}{4} Y_{2kl}^0 Y_{2ij}^0 Y_{2kl}^0. \]  
(142)
and decomposed into

\[ F_{ijkl}^{0(0)} = \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{kj} - \frac{1}{3} \delta_{ij} \delta_{kl}, \]  
\[ F_{ijkl}^{0(2)} = \frac{i}{8} \left[ (\delta_{ij} \epsilon_{ike} + \delta_{ki} \epsilon_{jle} + \epsilon_{jke} \delta_{il} + \delta_{kj} \epsilon_{ile}) Y_{1e}^0 \right], \]  
\[ F_{ijkl}^{0(6)} = \frac{1}{6} \left( \delta_{ij} Y_{2kl}^0 + \delta_{kl} Y_{2ij}^0 \right) - \frac{1}{8} \left( \delta_{ij} Y_{2ik}^0 + \delta_{kl} Y_{2il}^0 + \delta_{ki} Y_{2j}^0 + \delta_{kj} Y_{2il}^0 \right), \]  
\[ F_{ijkl}^{0(12)} = -\frac{i}{24 \sqrt{2}} \left[ \epsilon_{ike} Y_{2eji}^0 + \epsilon_{jke} Y_{3eij}^0 + \epsilon_{ile} Y_{3ejk}^0 + \epsilon_{jle} Y_{3eik}^0 \right], \]  
\[ F_{ijkl}^{0(20)} = \frac{1}{24} Y_{4ijkl}^0. \]  
(143)