Quasinormal modes of Unruh’s Acoustic Black Hole

Joel Saavedra

Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile.

We have studied the sound perturbation of Unruh’s acoustic geometry and we present an exact expression for the quasinormal modes of this geometry. We are obtain that the quasinormal frequencies are pure-imaginary, that give a purely damped modes.

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I. INTRODUCTION

General relativity has many successful predictions, some of them are passing experimental tests with high precision, for example mercury’s perihelion and gravitational redshift. On the other hand, the most spectacular and promising prediction is the existence of black holes. In this sense the theoretical area of black hole physics became an attractive and productive field of research in the past century. Those classical and quantum properties are well understood within the general relativity framework. From a classical point of view, anything can escape from the black holes; however, from the quantum point of view, black holes are not completely black and they can emit radiation with a temperature given by $\frac{\hbar}{8\pi k_B GM}$ [1]. This radiation is essentially thermal and they slowly evaporate by emitting quanta. Another property of interest that play an important role in black hole physics are the quasinormal modes (QNM’s). They determine the late-time evolution of fields in the exterior of the black hole, and numerical simulations of stellar collapse and black hole collisions have shown that in the final stage of such processes the quasinormal modes eventually dominate the black hole response to any kind of perturbation. QNM’s has received great attention in the last few years, since it is believed that these models shed light on the solution or understanding to fundamental problems in loop quantum gravity. In particular, from Hod’s proposal [2], modes with high damping have received great attention, specially their relevance in the quantization of the area of black holes and the possibilities of fixing the Immirzi parameter in loop quantum gravity [3, 4].

Obviously the experimental test are impossible to make at the level of terrestrial laboratories. This opens the interest in analog models that mimic the properties of black hole physics. In this sense, the field of analog models of gravity allows in principle that the most processes of black hole physics can be studied in a laboratory. In particular the use of supersonic acoustics flows as an analogy to gravitating systems was for first the time proposed by Unruh [5] and with the works of Visser [6, 7, 8, 9, 10] has received an exponentially growing attention.

The basis of the analogy between gravitational black holes and sonic black holes comes from considering the propagation of acoustic disturbances on a barotropic, inviscid, inhomogeneous and irrotational (at least locally) fluid flow. It is well known that the equation of motion for this acoustic disturbance (described by its velocity potential) is identical to the Klein-Gordon equation for a massless scalar field minimally coupled to gravity in a curved spacetime [8, 9, 10]. In this work we analytically compute the QNMs or acoustic disturbances in Unruh’s 3+1 dimensional sonic black hole (Laval nozzle fluid flow), and also we show their large damping limit.

The organization of the paper is as follows: In Sec. II we specify the Unruh’s Sonic Black Hole. In Sec. III we determine the QNMs, and the large damping limits. Finally, we conclude in Sec. V.

II. UNRUH’S SONIC BLACK HOLE

The idea of using supersonic acoustic flows as analog systems to mimic some properties of black hole physics was proposed for the first time by Unruh [5]. Essentially he showed the possibility to use sonic flow in order to explore properties like the Hawking temperature near the sonic horizon and, in principle to developed an experimental setting to study the fundamental problem of evaporation of real general relativity black holes. As we mentioned in the introduction, the basis of the analogy between gravitational black hole and sonic black holes comes from considering

*Electronic address: Joel.Saavedra@ucv.cl
the propagation of acoustic disturbances on a barotropic, inviscid, inhomogeneous and irrotational (at least locally) fluid flow. It is well known that the equation of motion for this acoustic disturbance (described by its velocity potential $\psi$) is identical to the Klein-Gordon equation for a massless scalar field minimally coupled to gravity in a curved space \[8, 9, 10\].

In Unruh's work, the acoustic geometry is described by the following sonic line element

$$ds^2 = \frac{\rho_0}{c} \left( -\left( \tilde{c}^2 - v_0^2 \right) d\tau^2 + \frac{\tilde{c} dr^2}{\tilde{c}^2 - v_0^2} + r^2 d\Omega^2 \right),$$

(1)

where $\rho_0$ is the density of the fluid, $\tilde{c}$ is the velocity of sound in the fluid (by simplicity we will assume these quantities constant) and $v_0^r$ represent the radial component of the flow velocity.

On the other hand, if then we assume that at some value of $r = r_+$ we have the background fluid smoothly exceeding the velocity of sound,

$$v_0^r = -\tilde{c} + \tilde{a}(r-r_+) + \vartheta(r-r_+)^2,$$

(2)

the above metric assumes just the form it has for a Schwarzschild metric near the horizon. In this limit metric our (1) reads as follows

$$ds^2 = \rho_0 \frac{\tilde{c} dr^2}{2\tilde{a}\tilde{c}(r-r_+)} + r^2 d\Omega^2,$$

(3)

where $\tilde{a}$ is a parameter associated with the velocity of the fluid defined as $(\nabla \cdot \nabla)|_{r=r_+}$. Note that this geometry was study in Ref. [11] where the author studied the low energy dynamics and obtained the greybody factors for the sonic horizon from the absorption and the reflection coefficients.

In the quantum version of this system we can hope that the acoustic black hole emits ”acoustic Hawking radiation”. This effect coming from the horizon of events is a pure kinematical effect that occurs in any Lorenzian geometry independent of its dynamical content [3]. It is well known that the acoustic metric does not satisfied the Einstein equations, due to the fact that the background fluid motion is governed by the continuity and the Euler equations. As a consequence of this fact, one should expect that the thermodynamic description of the acoustic black hole is ill defined. However, this powerful analogy between black hole physics and acoustic geometry admit to extend the study of many physical quantities associated to black holes, such as quasinormal modes which we consider in the next section.

### III. Quasinormal Modes of Unruh’s Sonic Black Hole

The basis of the analogy between Einstein black holes and sonic black holes comes from considering the propagation of acoustic disturbances on a barotropic, inviscid, inhomogeneous and irrotational fluid flow. It is well known that the equation of motion for these acoustic disturbances (described by its velocity potential $\Phi$) is identical to the Klein-Gordon (KG) equation for a massless scalar field minimally coupled to gravity in a curved spacetime \[8, 9, 10\],

$$\frac{1}{\sqrt{g}} \partial_{\mu} \left( \sqrt{g} g^{\mu\nu} \partial_{\nu} \Phi \right) = 0.$$

(4)

In order to compute the QNM’s we apply the standard procedure described in Refs. [11] [12] [13], and begin to rewrite the metric (1) in the following form

$$ds^2 = -f(r) d\tau^2 + \frac{\tilde{c} dr^2}{f(r)} + r^2 d\Omega^2,$$

(5)

where $f(r) = 2\tilde{a}\tilde{c}(r-r_+)$. In order to compute the QNM’S we need to solve the Klein-Gordon equation (4) in curve space described by the metric (5) and, by virtue of symmetries of the metric we use the following Ansatz for the scalar field

$$\Phi = e^{-i\omega t} Y_l^m(\theta, \varphi) R(r),$$
with this ansatz the KG equation can be separated as follows

\[ \frac{1}{\sin \theta} \left( \partial_\theta (\sin \theta \partial_\theta Y^m_l(\theta, \varphi)) + \frac{1}{\sin \theta} \partial_\varphi^2 Y^m_l(\theta, \varphi) \right) = l(l + 1)Y^m_l(\theta, \varphi), \tag{6} \]
\[ \frac{1}{c^2 r^2} \frac{d}{dr} \left( r^2 f(r) \frac{d}{dr} R(r) \right) + \left( \frac{\omega^2}{f(r)} + \frac{l(l + 1)}{r^2} \right) R(r) = 0. \]

Now, if then we consider the change of variables \( z = 1 - r_+/r \), that transform the radial equation to

\[ z(1 - z) \frac{d^2 R}{dz^2} + \frac{dR}{dz} + P(z)R = 0, \tag{7} \]

here,

\[ P(z) = \frac{B}{z(1 - z)} - A, \tag{8} \]

where,

\[ A = \frac{l(l + 1)\tilde{c}}{2\alpha r_+}, \tag{9} \]
\[ B = \left( \frac{\omega}{2\alpha} \right)^2. \]

Note that in the new coordinate system, \( z = 0 \) correspond to the horizon and \( z = 1 \) corresponds to infinity, and if we consider the definition

\[ R = z^\alpha(1 - z)^\beta F(z), \tag{10} \]

the radial equation satisfies then the standard hypergeometric form \[14\]

\[ z(1 - z) \frac{d^2 F}{dz^2} + (c - (1 + a + b)z) \frac{dF}{dz} + abF = 0, \tag{11} \]

where

\[ c = 2\alpha + 1, \]
\[ ab = A + (\alpha + \beta)(\alpha + \beta - 1), \]
\[ a + b = 2(\alpha + \beta) - 1 \]

and

\[ \alpha^2 = -B, \tag{12} \]
\[ \beta = 1 \pm \sqrt{1 - B}. \]

Without loss of generality, we put \( \alpha = -i\sqrt{B} \) and \( \beta = 1 - \sqrt{1 - B} \). It is well known that the hypergeometric equation has three regular singular point at \( z = 0, z = 1 \) and \( z = \infty \), and it has two independent solutions in the neighborhoods of each point \[14\]. The solutions of the radial equation reads as follows

\[ R(z) = C_1 z^\alpha(1 - z)^\beta F(a, b, c, z) + C_2 z^{-\alpha}(1 - z)^\beta F(a, b, c, z). \tag{13} \]

In the gravitational case quasinormal modes of a scalar classical perturbation of black holes are defined as the solution the Klein-Gordon equation characterized by purely ingoing waves at the horizon \( \Phi \sim e^{-i\omega(t+r)} \), since at least
classically outgoing flux is not allowed at the horizon. In addition, one has to impose boundary conditions on the solutions at the asymptotic region (infinity). It is crucial using the asymptotic geometry of the space time under study. In the case of asymptotically flat space time, the condition we need to imposes to the wave functions is a purely outgoing waves $\Phi \sim e^{-i\omega(t-r)}$ at the infinity. For non asymptotically flat space time (AdS space time for example), several boundary conditions have been discussed in the literature. In the two dimensional cases of BTZ black holes, the quasinormal modes for scalar perturbations were found in Refs. [12][16] by imposing the vanishing Dirichlet condition at infinity: In Ref. [17] these modes were computed with the condition of a vanishing energy momentum flux density at asymptotia. In this paper we consider the vanishing flux condition.

In the neighborhood of the horizon ($z = 0$) using the property $F(a,b,d,0) = 1$ the radial solution is given by

$$R(z) = C_1 \exp(-\frac{i\omega}{2a} \ln(1 - \frac{r^+}{r})) + C_2 \exp(-\frac{i\omega}{2a} \ln(1 + \frac{r^+}{r})).$$

while the boundary conditions of purely ingoing waves at the horizon demands $C_2 = 0$.

In order to implement boundary condition at infinity ($z = 1$), we use the linear transformation $z \rightarrow 1 - z$ formula for the hypergeometric function and we obtain,

$$R = C_1 z^\alpha(1 - z)^\beta \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, c - a - b - c + 1, 1 - z) + C_1 z^\alpha(1 - z)^c - a - b + \beta \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1, 1 - z).$$

Using the condition that the flux is given by

$$F = \sqrt{g} \frac{\partial R}{\partial \mu} R - R \partial_\mu R^*,$$

is vanished at infinity. Due to, $B > 0$, the asymptotic flux has a set of divergent terms, with the leading term of order $(1 - z)^{1 - 2\beta}$. Then according to Eq. (15) each of these terms are proportional to

$$\left| \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \right|^2,$$

and hence for a vanishing flux at $z = 1$ the following restrictions have to be applied

$$a = -n \text{ or } b = -n,$$

where $n = 0, 1, 2, \ldots$. These conditions lead directly to an exact determination of the quasinormal modes as follows

$$\omega = i \frac{(n - 1)(n + 3)\tilde{\alpha}}{n + 1}.$$

Note that the quasinormal modes have the properties being purely imaginary and independent of the radius of the horizon of the sonic black holes. Besides this, it shows the stabilities of this kind of analog black hole under sonic perturbations. For the infinite limit damping (i.e. the $n \rightarrow \infty$ limit)

$$\omega_{\infty} = i(n + \frac{1}{2})\tilde{\alpha}.$$

IV. CONCLUSIONS AND REMARKS

In this paper we have computed the exact values of the quasinormal modes of Unruh’s sonic black holes and according to Ref. [5] this QNM’s should be behaved similarly to QNM’s of near horizon Schwarzschild black holes. The QNM’s are pure imaginary, this kind of QNM’s was reported in Refs. [18][13]. This result shows the large stabilities of the dumb hole for overtone with $n > 1$ and for all overtones in higher damping limit. We also want to note that the frequencies of the QNM’s do not depend of radius of the horizon and on the angular momentum of the perturbation. They only have a proportional dependence the control parameter ($\tilde{a}$) which describes the velocity of the sonic flow, in according to Ref. [11] where the decay rate for this geometry and the thermal emission was only proportional to the control parameter.
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