An exact Lagrangian integral for the Newtonian gravitational field strength

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An exact expression for the gravitational field strength in a self–gravitating dust continuum is derived within the Lagrangian picture of continuum mechanics. From the Euler–Newton system a transport equation for the gravitational field strength is formulated and then integrated along trajectories of continuum elements. It is shown that the so–obtained integral solves one of the Lagrangian equations of the corresponding Lagrange–Newton system in general, whereas the remaining equations reduce to constraints on initial data. Relations to known exact solutions without symmetry in Newtonian gravity are discussed. The presented integral may be employed to access the non–perturbative regime of structure formation in Newtonian cosmology, and to define iterative Lagrangian schemes to solve the Lagrange–Newton system.

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I. THE PROBLEM

Let us characterize the state of a self–gravitating “dust continuum” (i.e., a continuum without pressure) at the initial time $t_i$ by a velocity and a density field $v(x,t_i)$ and $\rho(x,t_i)$. As usual, the fields are represented in a non–rotating Eulerian coordinate system $x$. We are interested in the evolution of the continuum governed by the Eulerian evolution equations for the velocity field $v(x,t)$ and the density field $\rho(x,t)$:

$$\frac{\partial}{\partial t} v = -(v \cdot \nabla) v + g ; \quad v(x,t_i) =: \mathbf{V} ;$$
$$\frac{\partial}{\partial t} \rho = -\nabla \cdot (\rho v) ; \quad \rho(x,t_i) =: \rho_i .$$

The evolution of the continuum is constrained by Newton’s field equations for the gravitational field strength $g(x,t)$, (which is equal to the acceleration field according to Einstein’s equivalence principle of inertial and gravitational mass):

$$\nabla \cdot g = \Lambda - 4\pi G \rho ;$$
$$\nabla \times g = 0 ,$$

where $\Lambda$ is the cosmological constant, and the density field obeys $\rho \geq 0$. (The introduction of a gravitational potential $\Phi$ defined by $g =: -\nabla \Phi$ (the existence of which is guaranteed by (4)) will not be needed.) Eqs. (1) - (4) form the Euler–Newton system of equations [17], [18], [19], [14].

Integration of this system is relevant in the context of Newtonian cosmology, where the “dust continuum” models the (together with the source $\Lambda$) dominant collisionless “dark matter” and “dark energy” in the Universe, and is mostly studied perturbatively, i.e. perturbative solutions at a homogeneous–isotropic solution of the Euler–Newton system are found in the Eulerian and Lagrangian representations of the system (see, e.g., [21], [22], [15], [3] and references therein). Besides these analytical schemes the problem is commonly treated by employing a variety of numerical integrators [4]. Exact solutions for inhomogeneous fields are known for special spatial symmetries like planar and spherical symmetry; three–dimensional solutions without symmetries are rare and will be put into perspective in Sect VI.

We proceed as follows. We start by deriving a transport equation for the gravitational field strength, working in the traditionally more emphasized Eulerian picture (Sect III). Then, we move to the Lagrangian picture and integrate the transport equation along the flow–lines of continuum elements, first by setting the cosmological constant $\Lambda$ equal to zero, in Sect III. The result is generalized to the case of a non–vanishing cosmological constant in Sect IV. Finally, Sect V provides relations to known exact solutions and a short discussion.
II. TRANSPORT EQUATION FOR THE GRAVITATIONAL FIELD STRENGTH

We first recall some equations that have been obtained earlier \[8\], and which provide an alternative formulation of the Euler–Newton system: we can combine the evolution equation \((2)\) with Eq. \((3)\) to obtain the following evolution equation for the gravitational field strength (\[8\] Eq. \((7c^*)\)):

\[
\frac{d}{dt} g - \Lambda v = (v \cdot \nabla) g - v(\nabla \cdot g) + \nabla \times \tau ,
\]

where \(\frac{d}{dt} := \frac{\partial}{\partial t} |_{x + v \cdot \nabla}\) is the total (Lagrangian) time–derivative, abbreviated sometimes by an overdot. (Hereafter, we shall drop the subscript \(|_{x}\) for notational ease.

The vector field \(\tau\) specifies the freedom after formally integrating the divergence of equation \((5)\), and can be interpreted as the vector potential of the current density \(j = \rho v\) (\[8\] Eq. \((15a^*)\)):

\[
4\pi G j = -\nabla \frac{\partial}{\partial t} \Phi - \nabla \times \tau ,
\]

Eq. \((6)\) is equivalent to Eq. \((5)\) using the definition \(\frac{d}{dt} g = \frac{\partial}{\partial t} g + v \cdot \nabla g\) and the source equation for the gravitational field strength \((3)\).

In the next section we shall integrate the transport equation \((11)\) after introducing the Lagrangian form of the Euler–Newton system.
III. LAGRANGIAN INTEGRAL OF THE TRANSPORT EQUATION FOR $Λ = 0$

The Lagrangian description is based on integral curves $x = f(X, t)$ of the velocity field $v(x, t)$:

$$\frac{df}{dt} = v(f, t) \quad ; \quad f(X, t_i) =: X.$$

(12)

In other words, $f$ is the position vector field that locates a fluid element, indexed by its initial position vector (the Lagrangian coordinates $X_i$), in Eulerian space at a given time $t$. Introducing this family of trajectories, we can express all Eulerian fields, e.g., the velocity $v$, the gravitational field strength $g$, the density $\rho$, and the vorticity $\omega$ (cf. APPENDIX A) in terms of the field of trajectories $x = f(X, t)$ as follows:

$$v = \dot{f}(X, t) \quad ; \quad g = \ddot{f}(X, t) ;$$

(13)

$$\rho = \varrho_i(X) J^{-1} ;$$

(14)

$$\omega = \omega_i \cdot \nabla \varrho f \ J^{-1} ,$$

(15)

with the Jacobian of the transformation from Eulerian to Lagrangian coordinates $J := \det(f_{ik}(X, t)) > 0$, $\omega_i := \omega(X, t_i)$, and the nabla–operator with respect to Lagrangian coordinates denoted by $\nabla$. (Throughout this paper a vertical slash $|$ denotes partial differentiation with respect to Lagrangian coordinates, which commutes with the Lagrangian time–derivative, while Eulerian spatial differentiation is abbreviated by a comma.)

Eqs. (13) – (15) are the known Lagrangian integrals of the Euler–Newton system, i.e., they represent an Eulerian field as a functional of $f$. As a result of the definitions (13) and the density integral (14) both Eulerian evolution equations are solved exactly for any given trajectory field in the Lagrangian picture. Transforming those fields back to Eulerian space we need that the transformation $f$ is invertible, i.e. $J > 0$, defining regular solutions (for more details the reader may consult the review [15]).

While the Eulerian evolution equations are represented by definitions and Lagrangian integrals, the relevant equations in the Lagrangian picture are the Eulerian field equations (3) and (4), which are transformed into a system of Lagrangian evolution equations by virtue of the following transformation of the field strength gradient $g_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$:

$$g_{ij} = g_{ijk} h_{k,j} = \frac{1}{2J} \varepsilon_{ikm} \varepsilon_{jpq} g_{ijk} f_{p} f_{q} |f|^{m} ;$$

(16)

the gradient of the inverse transformation from Lagrangian to Eulerian coordinates, $h = f^{-1}$ (which we require to exist), was expressed in terms of $f$ through the algebraic relationship

$$h_{ij} = J^{-1}_{ij} = \frac{\varepsilon}{\varepsilon} \left( J^{-1} \right) = \frac{1}{2J} \varepsilon_{ikm} \varepsilon_{jpq} f_{m} f_{n} |f|^{l} .$$

(17)

In view of (13) and (14) we obtain with (16) the following set of four Lagrangian equations (3) ($\Lambda = 0$), and (5) ($\Lambda \neq 0$); $i, j, k = 1, 2, 3$ with cyclic ordering; summation over repeated indices is understood):

$$\frac{1}{2} \varepsilon_{abc} \frac{\partial (g_{a}, f_{b}, f_{c})}{\partial (X_{1}, X_{2}, X_{3})} - \Lambda J = -4\pi G \varrho_i(X) ;$$

(18)

$$\varepsilon_{pqj} \frac{\partial (g_{p}, f_{q}, f_{j})}{\partial (X_{1}, X_{2}, X_{3})} = 0 , \quad i \neq j .$$

(19)

Note that these equations only involve the dynamical variable $f$ by virtue of $g = \ddot{f}$. Inserting this definition into the equations above, we obtain a set of four evolution equations furnishing the Lagrange–Newton system. (Alternative forms of these equations may be found in [11] and [12].)

We now integrate the transport equation (13) along trajectories $f$ of continuum elements indexed by $X$. The system (5), (11) is a set of nonlinear partial differential equations coupled in a complicated way, and certainly more involved than their Eulerian counterparts that are linear field equations. Therefore, it is surprising that one can find – as we shall see – a relatively simple expression for $g$ as a functional of $f$, extending the set of known integrals Eqs. (3) – (5). With the condition

$$\nabla \times \tilde{\tau} = 0$$

(20)

we single out a class of motions that admits a quasi–local Lagrangian integral. The term ‘quasi–local’ is to be understood as follows: if we expect $g = \ddot{f}$ to be a functional of $f$, then the trajectory field is represented locally,
while the initial data are still constructed non–locally according to the structure of the theory. However, we also expect a general integral, if it exists, to contain non–local parts that arise when the imposed restriction \((20)\) is relaxed. Requiring \((20)\) to hold is sufficient to enable us to integrate Eq. \((11)\), but it is not necessary for the following Proposition 1.

Setting first \(\Lambda = 0\) we make the following ansatz:

\[
\frac{g}{\varrho} = (s \cdot \nabla_0)f ; \quad s = s(X, t) .
\]

(21)

Applying the total time derivative to this ansatz, and using the identity

\[
(s \cdot \nabla_0)\dot{f} = ((s \cdot \nabla_0)f) \cdot \nabla v ,
\]

(22)

we compare the resulting terms with the transport equation \((11)\) to obtain:

\[
(s \cdot \nabla_0)f = 0 \quad \text{i.e., in general : } s = \text{const.} = \frac{G}{\varrho_i} ; \quad G := g(x, t_i) .
\]

(23)

Using the integral for the density field, Eq. \((14)\), we arrive at an exact integral for the gravitational field strength (this result has been first given in \([10]\), however, without providing the derivation):

**Proposition 1 (Lagrangian Integral for \(\Lambda = 0\))**

The Lagrangian integral

\[
\frac{g}{J} = \frac{(G \cdot \nabla_0)f}{J} ; \quad J = \det(f_{ij})
\]

(24)

solves the Lagrangian equation \((12)\) for \(\Lambda = 0\) exactly, while the remaining equations \((13)\) are reduced to constraint equations restricting \(f\).

(The proof follows by explicitly inserting the integral into the Lagrangian equations, either manually or using an algebraic manipulation system, e.g. REDUCE. We outline an analytical proof in APPENDIX B.) Note that this integral has its counterpart in the classical integral \([15]\) for the vorticity as reviewed in APPENDIX A.

**IV. THE CASE OF A NON–VANISHING COSMOLOGICAL CONSTANT**

The introduction of a cosmological constant in the Euler–Newton system corresponds to the introduction of a background field. In Newtonian cosmology such a background is commonly installed in terms of a “Hubble flow”, i.e., a homogeneous–isotropic solution of the Euler–Newton system that is furnished by the deformation field:

\[
f_H = a(t)X \quad \Rightarrow \quad v_H = \frac{\dot{a}}{a}x ; \quad \varrho_H = \varrho_H(t_i)a^{-3} ,
\]

(25)

with the “scale–factor” \(a(t)\), the linear “Hubble velocity” \(v_H\), and the homogeneous background density \(\varrho_H(t)\). Inserting \(f_H\) and \(g_H := \dot{f}_H\) into the Lagrange–Newton system \((18), (19)\), we obtain the well–known cosmological equation:

\[
3\frac{\ddot{a}}{a} = \Lambda - 4\pi G \varrho_H .
\]

(26)

Together with \(\varrho_H\) as given in terms of \(a(t)\) in \((25)\), this equation determines the solutions \(a(t)\). Deviation fields from the “Hubble flow” are then introduced. (Such a description is the topic of a forthcoming work, where the results of the present work are exploited in favour of cosmological structure formation models \([12]\).) In this line, the cosmological constant plays the role of a static background, which introduces qualitatively new features into the solutions.

Relaxing the restriction to a vanishing cosmological constant, we have to generalize the ansatz \((21)\): since \(\Lambda\) is constant in space and time, we expect a deviation from the field strength in \((21)\) that is proportional to the displacement defined by \(f\):

\[
\frac{g - Df}{\varrho} = (s \cdot \nabla_0)f ,
\]

(27)
with $D = \text{const}$. Applying the total time derivative to the new ansatz (27), using again the identity (22) and comparing the resulting terms with the transport equation (11) we first obtain:

$$\frac{D}{\partial t} \left[ f(\nabla \cdot \mathbf{v}) - (f \cdot \nabla)\mathbf{v} + \frac{\mathbf{v}}{\partial} \right] = \Lambda \frac{\nabla}{\partial} .$$

(28)

With the following identity we are able to evaluate the constant $D$:

$$\boldsymbol{\nabla} \cdot [ f(\nabla \cdot \mathbf{v}) - (f \cdot \nabla)\mathbf{v} ] = \boldsymbol{\nabla} \cdot [ \mathbf{v}(\nabla \cdot f) - (\mathbf{v} \cdot \nabla)f ] = \boldsymbol{\nabla} \cdot (d-1)\mathbf{v} ,$$

where the dimension $d$ of the continuum has to be explicitly taken into account.

Since all equations are thus far understood up to a vector potential that we neglected, we obtain in view of (25) and (29): $D = \Lambda / d$. Using the Lagrangian integral for the density (14), we finally arrive at a more general exact integral for the gravitational field strength:

**Proposition 2** (Lagrangian Integral for $\Lambda \neq 0$)

In $d$ dimensions of the continuum the Lagrangian integral

$$g^{IA} = \frac{(C \cdot \nabla_0)f}{J} + \frac{\Lambda}{d} f ; \quad C := \mathbf{G} - \frac{\Lambda}{d} \mathbf{X} ; \quad J = \det(f_{ik}) (30)$$

solves the Lagrangian equation (18) exactly, while the remaining equations (19) are reduced to constraint equations restricting $f$.

(Again, the proof follows by explicitly inserting the integral into the Lagrangian equations, either manually or using an algebraic manipulation system, e.g. REDUCE, see APPENDIX B for an analytical proof.)

The explicit dependence of the integral (30) on the dimension $d$ of the continuum can be easily understood by noting that from $\varrho = 0$, $\nabla \cdot \mathbf{g} = \Lambda$, we have the integral $\mathbf{g} = \frac{\Lambda}{d} f$ which is confirmed by calculating the divergence of this integral with respect to $x = f(X, t)$.

V. RELATION TO EXACT SOLUTIONS AND DISCUSSION

Let us first consider the integral (24) for $\Lambda = 0$. It provides the (with respect to (18) without any restriction) exact field strength along any given family of trajectories. Inserting the integral into the remaining Lagrangian equations (19) delivers constraints that have to be imposed on the given family of trajectories yielding in general restrictions on the spatial coefficient functions. (It is possible to find a corresponding exact integral for the Lagrangian equations corresponding to $\nabla \times \mathbf{g} = \mathbf{0}$ which, again, requires constraints by inserting it into (18); this and also a formally general integral will be given elsewhere within a more general context; the integral of (18) is favoured with regard to applications (12).)

Alternatively, the integral (24) may be regarded as a set of three partial differential equations for the components of the trajectory field by virtue of the definition of $\mathbf{g}$ (13). Integrating the integral twice with respect to the time and fixing the two integration constants by the initial data: $f(x,t_i) = X, v(x,t_i) = V(X)$, we obtain an integral equation for the trajectory field:

$$f = X + V(X)(t - t_i) + \int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' (G(X) \cdot \nabla_0)f(X,t')J(X,t')^{-1} .$$

(31)

This equation suggests, as a manifest structural property of the integral (24), the following iterative definition of the trajectory field $f$:

$$f^{(n+1)} = X + V(X)(t - t_i) + \int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' (G(X) \cdot \nabla_0)f^{(n)}(X,t')J^{(n)}(X,t')^{-1} ; \quad J^{(n)} := \det(f_{ik}^{(n)}(X,t')^{-1} ,$$

(32)

where $f^{(n)}$ is the $n^{th}$ iterate of the integral. A trajectory field that is inserted on the r.-h.-s. of (32) does not in general yield the same trajectory field $f$ on the l.-h.-s. of (32). This way we only consider $f$ as a solution of the Lagrangian equation (18), if $f$ is, in the language of dynamical systems theory, a fix-point of this iteration, i.e.,

$$f^{(n+1)} = f^{(n)} ,$$

(33)
for all $n$ larger than some $n^\ast$.

Let us now consider some known cases of exact solutions in Newtonian gravity. Among the simplest of exact solutions we have already introduced the homogeneous–isotropic deformation fields $f_H$ and $g_H$. Inserting this solution for $\Lambda = 0$ into the r.h.s. of (42), we recover the same solution on the l.h.s. of (42) (with careful handling of the integration constants). Thus, in this case, the iterated solution is identical to the solution fed into the integral. We face the same situation in the case of the exact general one–dimensional solution $24$, $F$:

$$f_1 = X_1 + V_1(X_1)(t - t_i) + G_1(X_1)\frac{(t - t_i)^2}{2}.$$  

(34)

Also, the three–dimensional generalization of this solution $\tilde{\xi}$, admitting locally (at each $X$ different) one–dimensional motions and imposing no global symmetry restrictions, forms a fix–point of this iteration:

$$f = X + V(X)(t - t_i) + G(X)\frac{(t - t_i)^2}{2}.$$  

(35)

Here, we have to make explicit use of the constraints given in $\tilde{\xi}$ and the identity $10$ to verify this. (The solution class is defined by three–dimensional initial data that are composed of 2–surfaces with vanishing Gaussian curvature; the plane–symmetric case is contained in a subclass formed by cylindrical 2–surfaces.) The basic assumption that was used to derive this class of exact solutions was the constancy of the gravitational field strength along the solution curves: $g(X, t) = G(X)$. The integral $23$ can be viewed as a generalization of this assumption, where $g$ changes along solution curves according to their directional derivative with respect to the initial field $G$, weighted in addition by the local density. It is interesting that we obtain the solution curves (35) as the first iterate of the integral $24$, if we start with the trivial trajectories $f^{(0)} = X$. Any further iterate is identical to $f^{(1)}$, if the constraints quoted above are respected.

Of course, it is possible to employ the trajectory field $\xi$ for generic initial data defining an approximation to the Lagrange–Newton system (this idea is followed in the context of building cosmological models of structure formation, see $12$). This approximation, if iterated, will produce another approximation that will contain, at each iteration step, higher spatial derivatives of the initial data. At this stage it is premature to expect that such an iteration scheme may converge in the sense of producing more refined models which approach a solution of the Lagrange–Newton system.

Relaxing the restriction $\Lambda = 0$, we accordingly obtain the more general trajectory field, which we may write as follows:

$$f = X + V(X)(t - t_i) + \int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' \left[ \left( G(X) - \frac{\Lambda}{d} X \right) \cdot \nabla_0 f^{-1} + \frac{\Lambda}{d} f \right].$$  

(36)

Viewing the above equation as an iteration scheme, then one possible choice is to write:

$$\tilde{f}^{(n+1)} - \frac{\Lambda}{d} f^{(n+1)} = \frac{\left( G(X) - \frac{\Lambda}{d} X \right) \cdot \nabla_0 f^{(n)}}{J(f^{(n)})}.$$  

(37)

If we start with a homogeneous–isotropic deformation, $f^{(0)} = a(t)X$, then we find a fix–point corresponding to all solutions of Eqs. (25), (26) for $d = 3$.

On the other hand, if we start with the trivial trajectories $f^{(0)} = X$, then the first iterate obeys:

$$\tilde{f}^{(1)} - \frac{\Lambda}{d} f^{(1)} = G(X) - \frac{\Lambda}{d} X,$$  

(38)

admitting the general integral–curves:

$$f^{(1)} = X + \frac{d}{2\Lambda} \left( G(X) + \sqrt{\frac{\Lambda}{d}} V(X) \right) \left[ e^{\sqrt{\frac{\Lambda}{d}} (t - t_i)} - 1 \right] + \frac{d}{2\Lambda} \left( G(X) - \sqrt{\frac{\Lambda}{d}} V(X) \right) \left[ e^{-\sqrt{\frac{\Lambda}{d}} (t - t_i)} - 1 \right].$$  

(39)

For the case $d = 3$ we propose a more refined choice of the iteration scheme $\xi$, motivated by the idea that the first iterate provides an exact solution of the Lagrange–Newton system $18$, $19$. We write:

$$\tilde{f}^{(n+1)} - \Lambda f^{(n+1)} = \frac{\left( G(X) - \frac{\Lambda}{d} X \right) \cdot \nabla_0 f^{(n)}}{J(f^{(n)})} - \frac{2\Lambda}{3} f^{(n)}.$$  

(40)
With this choice the Friedmannian solutions still form a fix-point corresponding to the iteration of a homogeneous–isotropic deformation (as above). The advantage of this latter choice becomes obvious by iteration of the trivial trajectories \( f^{(0)} = X \), since the first iterate then obeys:

\[
\ddot{f}^{(1)} - \Lambda f^{(1)} = G(X) - \Lambda X ,
\]

(41)

admitting the general integral–curves:

\[
f^{(1)} = X + \frac{1}{2\Lambda} \left( G(X) + \sqrt{\Lambda} V(X) \right) \left[ e^{\sqrt{\Lambda}(t-t_i)} - 1 \right] + \frac{1}{2\Lambda} \left( G(X) - \sqrt{\Lambda} V(X) \right) \left[ e^{-\sqrt{\Lambda}(t-t_i)} - 1 \right] .
\]

(42)

The trajectory field (42) provides a class of three–dimensional exact solutions without global symmetry restrictions that is contained in a subclass of solutions investigated in [8], which will not be further discussed here (see also [3]; for solutions including a cosmological constant see [6, 13], and for related rotational solutions [24]). Explicitly, this solution class has been discussed by Barrow & Götz [2] in the context of “no–hair” theorems.

Although the basic equations tend continuously, in the limit \( \Lambda \to 0 \), to their restricted set for \( \Lambda = 0 \), the solution (35) given above can only be obtained from (42) as an approximate limit by expanding the exponentials. This shows that the presence of a background entails a qualitative difference.

A final word on the range of validity of the presented integrals is in order. Although we discussed all equations and results in the framework of a dust continuum, Equations (11), (18), (19) and their integrals for \( g \) are valid for a much wider range of continua, e.g. including pressure forces. In those more general cases the representation of \( g \) in terms of the trajectory field changes, and is no longer given by the simple relationship \( g = \ddot{f} \) as in the dust continuum (see, e.g., [1] for the case of an isotropic pressure).

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APPENDIX A: Lagrangian integral for the vorticity

We here review a classical result that displays formal similarities to the integration procedures used for the derivation of the transport equation for the gravitational field strength. The transport equation (11) is formally similar to Beltrami’s transport equation for the vorticity \( \omega := \frac{1}{2} \nabla \times v \) (see, e.g., [23] and [9]):

\[
\frac{d}{dt} \left( \frac{\omega}{\varrho} \right) = \left( \frac{\omega}{\varrho} \cdot \nabla \right) v + \frac{1}{\varrho} \nabla \times g .
\]

(43)

For curl–free forces, which is the case for Newtonian flows, this reduces to:

\[
\frac{d}{dt} \left( \frac{\omega}{\varrho} \right) = \left( \frac{\omega}{\varrho} \cdot \nabla \right) v .
\]

(44)

There is also a classical integral of Beltrami’s transport equation due to Cauchy in the case of curl–free forces (see [23] and [9]):

\[
\omega = \frac{\omega_i \cdot \nabla \varrho}{J} ; \quad \omega_i := \omega(X, t_i) .
\]

(45)

APPENDIX B: Proof of Propositions 1 and 2

A simple way to prove Proposition 2 and, being a subcase, Proposition 1 is to write the divergence of \( g \) as

\( g_{i,i} = g_{i|k} h_{k,i} \) (first without explicitly writing out \( h \)). Computing \( g_{i|k} \) from the integral (40), \( g_i = C_s f_{i|s} J^{-1} + \frac{\Lambda}{d} f_i \), we first have:

\[
g_{i|k} = \frac{1}{J} \left[ C_s f_{i|s} + C_s f_{i|sk} - \frac{J_{i|k} f_{i|s}}{J} \right] + \frac{\Lambda}{d} f_{i|k} .
\]

(46)
Multiplying by $h_{k,i}$ and using $f_{i,j}h_{k,i} = \delta_{sk}$, we obtain:

$$g_{i,i} = \frac{1}{J} \left[ C_{k|k} + C_{s}f_{i|sk}h_{k,i} - \frac{J_{k|k}}{J} C_{k} \right] + \frac{\Lambda}{d} \delta_{kk} .$$

(47)

We have to prove that $g_{i,i} = \Lambda - 4\pi G\rho_i J^{-1}$. Since $C_k = G_k - \frac{4}{3} X_k$, $G_{k|k} = \Lambda - 4\pi G\rho_i$, $X_{k|k} = \delta_{kk} = d$, the first term in Eq. (47) together with $\frac{4}{3}\delta_{kk} = \Lambda$, already provides the whole equation, $g_{i,i} = \Lambda - 4\pi G\rho_i$. It remains to prove that, for $J \neq 0$, the other two terms in the brackets of Eq. (47) cancel out:

$$C_{s}f_{i|sk}h_{k,i} = \frac{J_{k|k}}{J} C_{k} .$$

(48)

We now write the inverse matrix explicitly, $h_{s,i} = \frac{1}{2J}\varepsilon_{ipq}\varepsilon_{sro} f_{p|r}f_{q|o}$, and also the Jacobian, $J = \frac{1}{6}\varepsilon_{abc}\varepsilon_{def} f_{a|d}f_{b|e}f_{c|f}$ to obtain the equivalent requirement:

$$C_{s}\varepsilon_{ipq}\varepsilon_{sro} f_{i|sk}f_{p|r}f_{q|o} = C_{k}\frac{1}{3}\varepsilon_{abc}\varepsilon_{def} \left( f_{a|d}f_{b|e}f_{c|f} \right)_{|k} .$$

(49)

We appreciate that, by evaluating the r.-h.-s., we get three identical expressions, so that by relabelling the summation indices we conclude that the above equation holds. \[■\]

The REDUCE codes to compute the Lagrangian evolution equations and the presented integrals can be obtained from the author.