Entanglement swapping for generalized non-local correlations

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We consider an analogue of entanglement-swapping for a set of black boxes with the most general non-local correlations consistent with relativity (including correlations which are stronger than any attainable in quantum theory). In an attempt to incorporate this phenomenon, we consider expanding the space of objects to include not only correlated boxes, but ‘couplers’, which are an analogue for boxes of measurements with entangled eigenstates in quantum theory. Surprisingly, we find that no couplers exist for two binary-input/binary-output boxes, and hence that there is no analogue of entanglement-swapping for such boxes.

I. INTRODUCTION

One of the most surprising aspects of quantum theory is its ability to yield non-local correlations, which cannot be explained by any local hidden-variable model\textsuperscript{1, 2}. These correlations do not allow superluminal signalling, but are nevertheless useful in many information theoretic tasks\textsuperscript{2, 3}. An interesting question is whether quantum theory yields the maximal amount of non-local correlations consistent with causality. Perhaps surprisingly, it has been shown that this is not the case\textsuperscript{2}, and that it is possible to construct a theoretical system which does not allow superluminal signalling, yet which is more non-local than quantum theory. Such a system can achieve the maximal possible value of 4 for the Clauser-Horne-Shimony-Holt (CHSH)\textsuperscript{2} expression, compared to $2\sqrt{2}$ for quantum theory (the Cirel’son bound\textsuperscript{3}), or 2 for any local hidden variable model.

Recently, the idea of non-local correlations stronger than those attainable in quantum theory has received considerable interest: Van Dam\textsuperscript{6} has shown that they allow any bipartite communication complexity problem to be solved with only one bit of communication. Wolf and Wullschleger\textsuperscript{6}, Buhrman et al.\textsuperscript{6}, and Short et al.\textsuperscript{10} have considered whether they can be used for oblivious transfer and bit-commitment. Cerf et al.\textsuperscript{11} have shown that they can be used to simulate measurements on a quantum singlet state, and Barrett et al.\textsuperscript{12, 13} and Jones and Masanes\textsuperscript{14} have characterised and considered the inter-convertibility of different non-local correlations.

To investigate the properties of general non-local no-signalling correlations, we consider an abstract correlation-system composed of a number of black boxes (subsystems) held by different parties, each of which has an input (their measurement setting) and an output (their measurement result)\textsuperscript{17}. We represent the combined state of all of the boxes by the conditional probability distribution for their outputs given their inputs.

The non-local correlations achievable using correlated box-states are analogous to those achievable using entangled quantum states. It is therefore interesting to see what properties of entanglement have analogues for these more general non-local correlations. A property which does have such an analogue can be viewed as a general property of non-local correlations (and therefore not specifically quantum), while a property without such an analogue is specific to quantum theory, and therefore may reveal why quantum theory has the particular form that it does.

In this paper, we consider the analogue of entanglement-swapping for correlated box states, in which non-local correlations between Alice and Bob, and between Bob and Charlie, are used to generate non-local correlations between Alice and Charlie.

In an attempt to achieve this, we consider the possibility of introducing a new class of objects called ‘couplers’, which can perform an analogue for boxes of measurements with entangled eigenstates in quantum theory. We first consider a natural potential coupler in detail, showing how it fails to provide a consistent solution, then proceed to develop a general framework with which to explore other possibilities. Surprisingly, we find that no couplers exist for two binary-input/binary-output boxes, and hence that there is no analogue of entanglement-swapping for such boxes.

The structure of the paper is as follows: In section II we define general correlated box states, and in section III we briefly review quantum entanglement-swapping. In section IV, we attempt to achieve an analogue of entanglement-swapping for non-locally correlated boxes. However, we show this is impossible to achieve using a sequence of conditional measurements on individual boxes. In section V we introduce couplers, and consider both general and specific cases, and in section VI we present our conclusions.
II. CORRELATED NO-SIGNALLING BOXES

A. General case

Consider a general multi-partite system composed of $N$ correlated subsystems, each of which can be moved about freely. We represent each subsystem by a black-box, which has an input (corresponding to the choice of which measurement to perform on that subsystem), and an output (which is the result of the chosen measurement). We will assume that only one input can be made to each box, and that the corresponding output is obtained immediately (without having to wait for messages to travel between the boxes). Furthermore, we assume that the probability of obtaining a given set of outputs $O = \{O_1, \ldots, O_N\}$ from a system of boxes depends only on the inputs $I = \{I_1, \ldots, I_N\}$ which are made to those boxes, and not on the timings of those inputs (which would be reference-frame dependent). The state of the boxes can therefore be represented completely by a conditional probability distribution $P(O|I)$.

As the boxes can be moved to any point in space, and their inputs applied at any time, the ability to transmit information using them would allow superluminal signalling. We therefore require that the boxes obey the following no-signalling condition: For all partitions of the boxes into two disjoint sets held by a sender (with inputs $I_S$ and outputs $O_S$) and a receiver (with inputs $I_R$ and outputs $O_R$), there exists a probability distribution $P(O_R|I_R)$ such that

$$\sum_{O_S} P(O|I) = \sum_{O_S} P(O_SO_R|I_SI_R) = P(O_R|I_R) \quad (1)$$

for all $O_R, I_R, I_S$. This ensures that when the sender and receiver are separated (and therefore the receiver does not know $O_S$) the receiver can learn nothing about the sender’s inputs $I_S$. It is therefore impossible for the sender to transmit information to the receiver using the boxes.

The probability distribution $P(O_R|I_R)$ plays an analogous role to the reduced density matrix of the receiver’s subsystems in quantum theory\cite{18}, providing the probabilities for his boxes’ outputs when considered independently.

B. Two-box binary-input/binary-output states

We now consider in detail the simplest case admitting non-local correlations, which is that of two binary-input/binary-output boxes. Taking $I = \{x, y\}$ and $O = \{a, b\}$, the two box state is represented by the probability distribution $P(ab|xy)$, where all of the inputs and outputs are binary variables. We will assume that the first box is held by Alice and the second box is held by Bob (as shown in figure 1).

![FIG. 1: Two correlated boxes held by Alice and Bob, in the state $P(ab|xy)$. The dashed line between the two boxes represents that they are non-locally correlated.](image)

The no-signalling condition corresponds to the two requirements

$$\sum_a P(ab|xy) = P(b|y) \quad \forall x, y, b \quad (2)$$
$$\sum_b P(ab|xy) = P(a|x) \quad \forall x, y, a \quad (3)$$

for some pair of probability distributions $P(a|x)$ and $P(b|y)$. The meaning of (2) is that Alice cannot signal superluminally to Bob, while (3) means that Bob cannot signal superluminally to Alice.

The complete class of probability distributions (and hence 2-box-states) consistent with relations (2) and (3) has been investigated by Barrett et al.\cite{13}, and was found to form an 8-dimensional convex polytope with 24 vertices. Of these, 16 vertices represent the deterministic ‘local’ states, for which Alice and Bob’s outputs are a function of their inputs alone, and hence their boxes are uncorrelated. These are the analogues of quantum mechanical product states. The probability distributions for these extremal local states are:

$$P^{L}_{\alpha\beta\gamma\delta}(ab|xy) = \begin{cases} 1 & : \alpha = ax \oplus \beta \\ b = \gamma y \oplus \delta \\ 0 & : \text{otherwise} \end{cases} \quad (4)$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ parameterize the 16 states and $\oplus$ denotes addition modulo 2. A state will lie within the convex polytope $\mathcal{L}$ formed by these 16 vertices if and only if it can be simulated by local operations and shared randomness, and we will refer to any such state as a local state.

The remaining 8 vertices represent non-local states (lying outside $\mathcal{L}$), with probability distributions given by:

$$P^{N}_{\alpha\beta\gamma}(ab|xy) = \begin{cases} 1/2 & : a \oplus b = xy \oplus \alpha x \oplus \beta y \oplus \gamma \\ 0 & : \text{otherwise} \end{cases} \quad (5)$$

where $\alpha, \beta, \gamma \in \{0, 1\}$. Each non-local extremal state maximally violates a CHSH-type inequality (achieving a greater value than is attainable in quantum theory), and cannot be simulated with local operations and shared randomness. These states are the analogues of maximally-entangled states in quantum theory. Following\cite{13}, we will refer to all of the non-local extremal states as PR-states\cite{2}, although for simplicity we will usually consider the standard PR-state $P^{N}_{000}(ab|xy)$, for which $a \oplus b = xy$. 
III. ENTANGLEMENT-SWAPPING IN QUANTUM THEORY

In the quantum case, the simplest example of entanglement-swapping is as follows. Suppose that Alice shares a singlet state with Bob, and Bob shares a singlet state with Charlie, such that their combined state is

$$| \Psi \rangle = \frac{1}{2} ((|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)(|0\rangle_B |1\rangle_C - |1\rangle_B |0\rangle_C)$$

In this state, Alice’s and Charlie’s qubits are completely uncorrelated. However, expanding the bipartite states held by Bob, and by Alice and Charlie, in the Bell basis of maximally entangled states,

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$
$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

yields

$$|\Psi\rangle = \frac{1}{2} \left( |\psi^+\rangle_B |1\rangle_A |0\rangle_C - |\psi^+\rangle_B |0\rangle_A |1\rangle_C - |\psi^-\rangle_B |0\rangle_A |1\rangle_C + |\psi^-\rangle_B |1\rangle_A |0\rangle_C \right)$$

A measurement by Bob in the Bell-basis \{ |\psi^+\rangle_B |1\rangle_A |0\rangle_C, |\psi^-\rangle_B |0\rangle_A |1\rangle_C, |\phi^+\rangle_B |0\rangle_A |1\rangle_C, |\phi^-\rangle_B |1\rangle_A |0\rangle_C \} will therefore leave Alice and Charlie sharing the same maximally-entangled state as Bob, which contains strong non-local correlations. Alice and Charlie will know which entangled state they share as soon as Bob tells them his measurement result. If they wish, they can transform this state into a specific maximally entangled state (e.g. |\psi^-\rangle) by performing a local operation on one subsystem.

IV. GENERALISED NON-LOCALITY SWAPPING I

We now consider the analogue of entanglement swapping for generalised non-local correlations. Consider first the simplest case in which Bob shares a standard PR-state \(P_{00}(ab|xy)\) with Alice and an identical PR-state \(P_{00}(bc|yz)\) with Charlie.

The actions available to Bob are somewhat limited. The most general thing that he can do is to apply an input to one of his two boxes, and then use the output of this box in deciding which input to apply to his second box. This can be represented by a circuit diagram incorporating the two boxes (an example of which is shown in figure 2).

Can Bob introduce non-local correlations between Alice and Charlie? Without loss of generality, suppose that Bob first inputs \(y_1 = \lambda\) into his box which is correlated with Alice’s box, obtaining an output \(b_1\). He can then input a general function \(y_2 = \mu b_1 + \nu\) of \(b_1\) into his box which is correlated with Charlie’s box, obtaining the output \(b_2\) (where \(\lambda, \mu, \nu, b_1, b_2 \in \{0, 1\}\)). An example circuit

![Diagram showing the two PR-states held by Alice and Bob, and by Bob and Charlie, and a circuit applied by Bob to his two boxes.](image)

for \(\lambda = \mu = \nu = 1\) is shown in figure 2. Regardless of the values of \(\lambda, \mu\) and \(\nu\), the probability \(P(b)\) of Bob obtaining the two-bit output set \(b = \{b_1, b_2\}\) will be 1/4, as \(P(b_1|y_1) = P(b_2|y_2) = 1/2\).

After completing this procedure, Bob announces his outputs \(b_1\) and \(b_2\) (and his strategy choices \(\lambda, \mu\) and \(\nu\)) to Alice and Charlie. The probability distribution for their inputs and outputs is then given by

$$P'(ac|xz) = \begin{cases} 1 : a = \lambda x + b_1, c = (\mu b_1 + \nu)z + b_2 \hfill \\
0 : \text{otherwise} \end{cases}$$

which is the probability distribution corresponding to the local state \(P_{00}(\lambda b_1 + \mu b_2 | \nu z)\). By applying inputs to his boxes, Bob has therefore collapsed the state of the remaining two boxes to an extremal local state. This is analogous to the collapse of entangled quantum states after a measurement by one party.

By switching the ordering of his inputs, selecting his strategy choices \((\lambda, \mu, \nu)\) probabilistically, restricting the information he gives to Alice and Charlie, or sometimes announcing that the process has ‘failed’, Bob can cause the state shared by Alice and Charlie to be any probabilistic combination of the extremal local states (and hence any local state). However, there is no way that Bob can introduce non-local correlations between Alice and Charlie, since in each particular instance of the procedure Alice and Charlie share a local state.

The above result extends to the general case in which Bob shares any correlated box state with Alice, and any correlated box state with Charlie, but has no boxes which are correlated with both parties. In this case, the initial state of all of the boxes will be a product of two separate states:

$$P(ab_1 b_2 c | xy_1 y_2 z) = P(ab_1 | xy_1) P(b_2 c | y_2 z),$$

where \(x\) and \(a\), and \(z\) and \(c\), represent the inputs and outputs of Alice’s and Charlie’s boxes respectively, and Bob’s boxes are partitioned into two sets (with inputs \(y_1\) and \(y_2\), and outputs \(b_1\) and \(b_2\) respectively) depending on whether they are correlated with Alice or with Charlie.

The most general strategy Bob can adopt is to choose which of his boxes to apply an input to, and what input to apply to that box, dependent on all earlier outputs.
Using such an approach, he will generate a particular set of inputs and outputs with probability $P(b_1 b_2 y_1 y_2)$. As Bob applies all of his inputs first, $P(b_1 b_2 y_1 y_2)$ can be calculated without reference to Alice and Charlie, and will depend only on the reduced state of Bob's boxes $(P(b_1 | y_1) P(b_2 | y_2))$, and his particular choice of strategy.

Unfortunately, once all of Bob's inputs and outputs are known, the state $P(ab_1 | xy_1)$ collapses to $P(axb_1, y_1) = P(b_1, y_1) (a | x)$, which is a local probabilistic operation for Alice, and similarly $P(b_2 c | y_2 z)$ collapses to $P(c zb_2 y_2) = P(b_2 y_2) (c | z)$. For a given set of inputs and outputs for Bob, the final state of Alice's and Charlie's boxes will therefore take the form

$$P'(ac | xz) = P(b_1 | y_1) (a | x) P(b_2, y_2) (c | z)$$  \hspace{1cm} (12)

which is manifestly local between Alice and Charlie. Regardless of his strategy, it is therefore impossible for Bob to generate non-local correlations between Alice and Charlie, even with some small probability.

V. GENERALISED NON-LOCALITY SWAPPING II: COUPLERS

A. A potential coupler

As we have seen above, it seems impossible for Bob to generate non-local correlations between Alice and Charlie. However, we know from quantum theory that such 'entanglement-swapping' is possible. Why then, can generalised non-locality not be swapped? Pondering this question, one soon realises that the set of actions considered above is far too restrictive. If we were to consider analogous procedures in the case of quantum entanglement-swapping (as introduced in section III), they would correspond to Bob making an individual measurement on one of his two qubits, and then performing a measurement on his second qubit in a basis determined by his first measurement result. Whatever Bob's results, such a procedure would collapse the state of Alice's and Charlie's qubits into a local product state, and hence cannot achieve entanglement-swapping.

In quantum theory, entanglement swapping is achieved by a joint measurement on both of the subsystems held by Bob (i.e. a measurement with entangled eigenstates). It is this coupling of two subsystems which we need to incorporate into our box model. We therefore need to introduce a new device called a 'coupler', which is connected to the input and output channels of the two boxes held by Bob, and produces a single output $b'$ (as shown in figure 3). This output can be interpreted as the result of a joint measurement on Bob's two subsystems.

Consider first the simple situation in which Alice and Bob share a standard PR-state with probability distribution $P_{000} (ab_1 | x y_1)$, and Bob and Charlie share a standard PR-state with probability distribution $P_{000} (b_2 c | y_2 z)$. We would like our coupler to generate, with some probability, a standard PR-state $P_{000} (ac | xz)$ between Alice and Charlie.

As in the quantum case, such a process cannot be achieved with certainty, as otherwise if Alice and Charlie brought their boxes together Bob could signal superluminally to them by applying the coupler (thereby changing Alice and Charlie's joint probability distribution from $P(ac | xz) = 1/4$ to $P'(ac | xz) = P_{000} (ac | xz)$). However, if the coupler has binary output $b'$, we can consider the case in which the probability distribution for two coupled standard PR-states is

$$P'(ab' c | xz) = \begin{cases} 1/4 & a \oplus b' \oplus c = xz \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (13)

In this case, it is easy to see that each party's output is random, and the outcomes of any two parties are uncorrelated (in particular $P'(ac | xz) = P(ac | xz) = 1/4$ as required). Only by learning all three outputs is any information about the inputs obtained, and the coupler cannot therefore be used for signalling.

However, after Bob has applied the coupler and obtained an output $b'$ (with probability $P'(b') = 1/2$), Alice and Charlie will share the maximally non-local state $P_{000} (ac | xz)$. If Bob then announces his measurement result, and one party performs the local operation $a \rightarrow a \oplus b'$ on their output (i.e. applies a NOT-gate if $b' = 1$), Alice and Charlie will be left with the standard PR-state $P_{000} (ac | xz)$ as desired. This procedure, in which Alice or Charlie must perform a local correction conditional on Bob's measurement result in order to obtain a desired final state, is strongly analogous to the quantum case.

It therefore seems that, by enlarging the class of generalised non-local objects to include couplers in addition to boxes, we achieve generalised non-locality swapping. This is in complete analogy with quantum mechanics where in addition to entangled states, we consider measurements with entangled eigenstates. However, as we will show below, the above coupler actually cannot exist.
B. Difficulties with the potential coupler

Before allowing the coupler defined in the last section in our model, it is important to check that it gives consistent results when applied to all possible states. To investigate this, we consider the case in which Alice and Charlie apply inputs to their boxes before Bob applies the coupler to his boxes. If they initially share standard PR-states as before, their inputs and outputs will obey the relations $a \oplus b_1 = xy_1$ and $b_2 \oplus c = y_2z$. After Alice and Charlie have applied inputs to and obtained outputs from their boxes, the outputs of Bob’s two boxes will be given by $b_1 = xy_1 \oplus a$ and $b_2 = y_2z \oplus c$. The probability distribution for Bob’s two boxes has therefore collapsed to the extremal local state $P^{L}_{xy} (b_1 b_2 | y_1 y_2)$.

Suppose that Bob then applies a coupler to his two boxes. As is the case for the original box inputs, we assume that the final probability distribution given by [13] will be the same regardless of the timings of Alice and Charlie’s inputs, and of Bob’s application of the coupler. In order to satisfy [13], it is therefore necessary that the probability distribution $P'(b')$ for the coupler output when it is applied to two boxes in the state $P^{L}_{\alpha \beta \gamma \delta} (b_1 b_2 | y_1 y_2)$ be given by

$$P'(b') = \begin{cases} 1 & : b' = \alpha \gamma \oplus \beta \oplus \delta \\ 0 & : \text{otherwise} \end{cases} \quad (14)$$

Similarly, in the case in which Alice and Charlie measure first, all possible pairs of initial bipartite extremal-states shared by Alice-Bob and Bob-Charlie (e.g. $P^N_{001}(ab_1 | xy_1)$ and $P^L_{0110}(b_2 | y_2 z)$) will collapse to a local extremal state for Bob’s two boxes (e.g. $P^L_{x(a \oplus b)01}(b_1 b_2 | y_1 y_2)$ for the example given). If we assume that the coupler always acts in the same way when applied to the same state of Bob’s two boxes, then equation (14) can be used deduce the action of the coupler on all initial states of this type.

Furthermore, if we assume linearity in the initial probability distributions (which is necessary to ensure no-signalling - as shown in section VC), then we can deduce the action of the coupler when applied between any two general states by expanding them as convex mixtures of extremal states and using the above results. It appears that such a coupler is consistent.

However, suppose that in addition to Bob applying a coupler to his two boxes, Alice and Charlie bring their two boxes together and apply a coupler between them (as shown in figure 4). If Alice and Charlie’s coupler is applied first and they obtain output $a'$, Bob’s two boxes will collapse to the PR-state $P^N_{00a'}(b_1 b_2 | y_1 y_2)$ (from equation 13). In order to obtain Bob’s output, we must therefore determine the coupler output when it is applied to two boxes in an PR-state, rather than to two boxes in a local state as we have considered so far.

To investigate this case, we consider a particular non-extremal state $P^B(b_1 b_2 | y_1 y_2)$. As for mixed states in quantum theory, every non-extremal box state can be obtained in an infinite number of different ways by taking probabilistic mixtures of other box states [13]. We require that the coupler act in the same way regardless of how the probability distribution $P^B(b_1 b_2 | y_1 y_2)$ was prepared (i.e. it should be decomposition invariant). The state $P^B(b_1 b_2 | y_1 y_2)$ has two particular decompositions of interest. The first is obtained by adding random noise to the standard PR-state until it becomes local, and the second of which is its explicit decomposition in terms of local extremal states:

$$P^B(b_1 b_2 | y_1 y_2) = \frac{1}{2} P^N_{000}(b_1 b_2 | y_1 y_2) + \frac{1}{8} \sum_{\beta \delta} P^L_{010\beta}(b_1 b_2 | y_1 y_2) \quad (15)$$

Using (14), it is easy to see that the eight local states in the decomposition (13) of $P^B(b_1 b_2 | y_1 y_2)$ all give $b' = 0$ when the coupler is applied to them. Applying linearity, the coupler must therefore give $b' = 0$ with certainty when applied to the state $P^B(b_1 b_2 | y_1 y_2)$.

However, it is also evident from (14) that the coupler will give $b' = 1$ when applied to the local states $P^L_{0001}(b_1 b_2 | y_1 y_2)$ and $P^L_{0100}(b_1 b_2 | y_1 y_2)$, which appear in the decomposition (13) with probability 1/8 each. Given any non-negative probability distribution $P^N(b')$ for the coupler output when applied to a standard PR-state, the probability of obtaining $b' = 1$ from the state $P^B(b_1 b_2 | y_1 y_2)$ cannot therefore be less than 1/4.

The results obtained from the two decompositions of $P^B(b_1 b_2 | y_1 y_2)$ are therefore inconsistent, and cannot be reconciled by any physical probability distribution $P^N(b')$. Only the non-physical distribution

$$P^N(b') = \frac{3}{2} - 2b', \quad (17)$$

which is not a valid probability distribution as it does not satisfy $0 \leq P^N(b') \leq 1$, could recover the desired decomposition invariance.
Because of this inconsistency, the ‘naive’ coupler defined by (12) is not an allowable object within the correlated-box model. However, this raises the question of whether other any coupler exists which can achieve an analogue of entanglement swapping.

C. General couplers

To generalise the approach of the previous section, we will consider a coupler as any device which acts on \( n \) boxes with a given range of inputs and outputs and produces a single output \( b' \) (also with a given range), and which cannot be implemented by applying a sequence of individual inputs to the coupled boxes (as in section IV). In this context, the coupler considered in the last section, were it to have proved consistent, would have been an \( n = 2 \) coupler, with all inputs and outputs being binary.

We consider a general set of \( N \) boxes divided between the person who is going to apply the coupler (who we call Bob) and the rest of the world (which we call Alice). In the most general case, Bob has the \( n \) boxes to which the coupler is to be applied (with inputs \( y \) and outputs \( b \)), and an additional set of \( m \) boxes (with inputs \( y \) and outputs \( b \)). Alice has the remaining \( (N - n - m) \) boxes (with inputs \( x \) and outputs \( a \)). In the last section, for example: \( x = \{ x_1, x_2 \}, a = \{ a, c \}, y = \{ y_1, y_2 \}, b = \{ b_1, b_2 \}, \tilde{b} = \tilde{y} = \{ \} \).

The coupler then performs the transformation:

\[
P(a\tilde{b}b|x\tilde{y}y) \rightarrow P'(a\tilde{b}b'|x\tilde{y})
\]  

(18)

Following the discussion in the previous section, we impose four natural constraints on the coupler’s action:

i) Universal: The coupler must be applicable to any set of \( n \) boxes with the appropriate range of inputs/outputs, that are part of any no-signalling correlated box state. Note that this is the condition for which the coupler proposed in the last section fails, as it cannot be applied to two boxes in a PR-state.

ii) Completeness: A correlated box state is completely specified by the conditional probability distribution for its outputs given its inputs. To respect this completeness, we require that the probability distribution \( P'(a\tilde{b}b'|x\tilde{y}) \) obtained by applying the coupler to a set of boxes depends only on the probability distribution \( P(abb|x\tilde{y}y) \) of those boxes to which it is applied.

The same probability distribution \( P(abb|x\tilde{y}y) \) can be obtained from many different mixtures of extremal states (as in [15] and [16]), or by collapsing a larger state by applying inputs to some of the boxes. This requirement ensures that the coupler gives the same outcome in all of these cases. It also ensures that the results do not depend on the time at which the coupler is applied, just as the timings of standard inputs do not affect the boxes’ outputs.

iii) No signalling: The coupler must not allow signalling. Note that the most powerful situations for sending and receiving information are when all of the boxes that are not held by Bob (i.e. Alice’s boxes) are gathered in the same place, and hence all of their inputs and outputs are immediately accessible. We must rule out two possibilities:

(a) Signalling from Alice to Bob: We require that Bob cannot learn anything about Alice’s inputs from his coupler and box outputs. We therefore require that:

\[
\sum_a P'(a\tilde{b}b'|x\tilde{y}) = P'(\tilde{b}b'|\tilde{y})
\]

(19)

for some probability distribution \( P'(\tilde{b}b'|\tilde{y}) \) which is independent of \( x \).

(b) Signalling from Bob to Alice: We require that Alice cannot learn anything about Bob’s box inputs, or about whether he has (or has not) applied his coupler. We therefore require that:

\[
\sum_{b,b'} P'(a\tilde{b}b'|x\tilde{y}) = \sum_{b,b} P(abb|x\tilde{y}y) = P(a|x).
\]

(20)

iv) Non-triviality: The coupler must represent something that was not previously possible. As discussed in section IV the most general strategy that Bob can adopt without a coupler is to apply a sequence of individual inputs to his boxes, where later inputs may depend on earlier outputs. A coupler cannot be simulated using such a procedure (with \( b' \) given by some function of the outputs).

We will now show that constraints (i)-(iii) allow only those couplers which act as linear maps on the reduced state of Bob’s boxes.

Let us consider the case in which Alice applies her inputs \( x \), and Bob applies his inputs \( \tilde{y} \) before he applies the coupler. They will obtain outputs \( a \) and \( \tilde{b} \) respectively with probability \( P(ab|x\tilde{y}) \), and the state of Bob’s boxes will collapse to a specific state \( P_{(x\tilde{y}b)}(b|y) = P(b|x\tilde{y}b) \), which depends on the inputs and outputs obtained.

From the completeness constraint (ii), the coupler will then act on Bob’s \( n \)-box reduced state exactly as it would have if that \( n \)-box state were prepared directly (without collapsing a larger \( N \) box system), giving the output probability distribution:

\[
P'_{(x\tilde{y}b)}(b') = C \left[ P_{(x\tilde{y}b)}(b|y) \right]
\]

(21)

where \( C \) is some function characteristic of the coupler. The final probability distribution for Alice’s and Bob’s
outputs must therefore given by
\[ P'(ab'|xy) = P(ab|xy)P'(x_\lambda y_\lambda)(b'). \] (22)

Note that as long as \( P'(x_\lambda y_\lambda)(b') \) is a valid probability distribution (satisfying \( \sum b' P'(x_\lambda y_\lambda)(b') = 1 \)), equation (22) will always satisfy the no-signalling constraint (iiib).

To investigate the class of allowed coupler functions \( C \), and to incorporate the no-signalling constraints (iia) and (iiia), it is helpful to consider a particular class of \((n+1)\)-box states, for which Alice has a single box with input \( x \) and output \( a \) and Bob has only those \( n \) boxes to which he will apply the coupler. The states we will consider are given by
\[
P(a|bxy) = \begin{cases} 
\lambda_a P_a(b|y) & : x = 0 \\
\sum a' \lambda_{a'} P_{a'}(b|y) & : x = 1, a = 0 \\
0 & : \text{otherwise}
\end{cases}
\] (23)

where \( P_a(b|y) \) are a set of no-signalling \( n \)-box states labelled by \( a \), and \( \lambda_a \) is the probability for Alice to obtain output \( a \) when \( x \) is chosen (satisfying \( \lambda_a > 0 \) and \( \sum a \lambda_a = 1 \)). It is easy to check that \( P(ab|xy) \) is a valid no-signalling state. A state of this type corresponds to the case in which Bob is given an \( n \)-box state selected randomly from some set, and Alice can either discover which state Bob has been given (by inputting \( x = 0 \) into her box) or not (by inputting \( x = 1 \)). In the latter case, the collapsed state of Bob's boxes will be a probabilistic mixture of all of the boxes in the set.

Applying a coupler to Bob's boxes (which must be possible due to the universality constraint (i)) yields the state:
\[ P'(ab'|x) = \begin{cases} 
\lambda_a C[P_a(b|y)] & : x = 0 \\
C[\sum a' \lambda_{a'} P_{a'}(b|y)] & : x = 1, a = 0 \\
0 & : \text{otherwise}
\end{cases}
\] (24)

which is no-signalling from Alice to Bob, as required by constraint (iia), only when
\[ \sum a \lambda_a C[P_a(b|y)] = C[\sum a' \lambda_{a'} P_{a'}(b|y)] = P'(b'). \] (25)

In order for this relation to be satisfied for all choices of \( \lambda_a \) and \( P_a(b|y) \), \( C \) must be a linear function of Bob's \( n \)-box probability distribution, of which the most general form is given by
\[ P'(b') = \sum_{by} \chi(b', by)P(b|y) + \xi(b'). \] (26)

As \( \sum b P(b|y) = 1 \), we can always eliminate \( \xi(b') \) by adding it to each of the coefficients \( \chi(b', by_0) \) for a particular \( y_0 \), hence (26) can be simplified to give the final coupler function
\[ P'(b') = \sum_{by} \chi(b', by)P(b|y). \] (27)

Combining (22) and (27), and using the fact that
\[ P(ab|x_\lambda y_\lambda) = P(ab|xy)P(x_\lambda y_\lambda)(b|y), \] (28)

we find that the effect of the coupler on a general box state is given by
\[ P'(ab'|x_\lambda y_\lambda) = \sum_{by} \chi(b', by)P(ab|xy) \] (29)

Although we have so far only considered the no-signalling constraint (iia) for a specific set of initial boxes given by (23), it is easy to see that (29) will obey (iia) for any initial state. As the initial distribution \( P(ab|xy) \) is no-signalling we have
\[ \sum a P'(ab'|x_\lambda y_\lambda) = \sum_{by} \chi(b', by)P(bb|y\lambda) \] (30)

\[ = P'(bb'|y\lambda) \] (31)
as required by (19). Hence any coupler obeying (29) cannot be used for signalling.

The only remaining constraints on \( \chi(b', by) \) are universality (i) and non-triviality (iv). The former requires that \( P'(b') \) must be a valid probability distribution (satisfying \( P'(b') > 0 \) and \( \sum b' P'(b') = 1 \)) for all initial \( n \)-box states \( P(by) \) to which the coupler can be applied, and the latter requires that the action of the coupler cannot be simulated using Bob's standard box inputs.

D. Two box binary-input/binary-output couplers

As a specific case of the general couplers introduced in the last section, we will investigate the class of couplers which act on two binary-input/binary-output boxes and generate a binary output \( b' \). Following equation (27), if a coupler characterised by \( \chi(b', b_1b_2y_1y_2) \) is applied to the two-box state \( P(b_1b_2|y_1y_2) \), the final probability distribution will be
\[ P'(b') = \sum_{b_1b_2|y_1y_2} \chi(b', b_1b_2y_1y_2)P(b_1b_2|y_1y_2) \] (32)

The only constraints on \( \chi(b', b_1b_2y_1y_2) \) are that it satisfies the universality and non-triviality conditions introduced in the previous section. We first consider universality, which requires that \( P'(b') \) is a valid probability distribution for all initial two-box binary-input/binary-output no-signalling states \( P(b_1b_2|y_1y_2) \). I.e.
\[ 0 \leq P'(b') \leq 1 \] (33)
\[ \sum_{b'} P'(b') = 1 \] (34)

It is easy to see that these properties are preserved under convex combination. It is therefore only necessary to ensure that (33) and (34) hold for the 24 extremal states
introduced in section [11]. All other states can be represented as convex combinations of the extremal states, and will therefore satisfy (33) and (34) automatically.

Note that once \( P'(0) \) is determined for each extremal state, \( P'(1) \) is fixed by equation (33) to be \( P'(1) = 1 - P'(0) \). Given any \( \chi(0, b_1 b_2 y_1 y_2) \) satisfying (33), it is always possible to find a \( \chi(1, b_1 b_2 y_1 y_2) \) satisfying (33) and (34) by defining \( \chi(1, b_1 b_2 y_1 y_2) = 1/4 - \chi(0, b_1 b_2 y_1 y_2) \), as this gives

\[
P'(1) = \sum_{b_1 b_2 y_1 y_2} \left( \frac{1}{4} - \chi(0, b_1 b_2 y_1 y_2) \right) P(b_1 b_2 | y_1 y_2) = 1 - P'(0).
\]

As all other choices of \( \chi(1, b_1 b_2 y_1 y_2) \) must give the same values for \( P'(1) \), they are all equivalent. Hence a coupler is completely specified by the 16 parameters \( \chi(0, b_1 b_2 y_1 y_2) \).

The class \( \chi \) of \( \chi(0, b_1 b_2 y_1 y_2) \) distributions satisfying the universality constraint are those which are consistent with the 48 linear inequalities that result from applying equation (33) to the 16 extremal local states \( P_{\alpha, \beta, \gamma, \delta}^{b_1 b_2} |y_1 y_2\rangle \) and 8 extremal non-local states \( P_{\alpha, \beta, \gamma, \delta}^{b_1 b_2} |y_1 y_2\rangle \). This approach yields a convex polytope for \( \chi \). It has 9 dimensions and 82 vertices, as well as 7 linearities which have no effect on the final probability distributions (and therefore define an equivalence class of \( \chi(0, b_1 b_2 y_1 y_2) \) which would correspond to the same coupler).

Each point in the polytope \( \chi \) corresponds to a different potential coupler, with the only remaining constraint on them being that of non-triviality. The 82 extremal points of \( \chi \) can all be expressed by \( \chi(b, b_1 b_2 y_1 y_2) \in \{0, 1\} \), and can therefore be characterised by the values of \( b, b_1 b_2 y_1 y_2 \) for which \( \chi(b, b_1 b_2 y_1 y_2) = 1 \) (with all other coefficients being zero). Table I gives a representation of each extremal \( \chi(b, b_1 b_2 y_1 y_2) \) distribution in this way. For simplicity, these extremal points have been divided into 5 classes, each of which are parameterised by a subset of the binary coefficients \( \{\alpha, \beta, \gamma, \delta, \epsilon\} \in \{0, 1\} \).

Perhaps surprisingly, all 82 extremal points of \( \chi \) fail to satisfy the non-triviality constraint (iv). Each of these potential couplers can be generated by applying wiring and logic gates to Bob's two boxes (indeed, they represent every inequivalent wiring of this type). A circuit diagram for the standard potential coupler \( \alpha = \beta = \gamma = \delta = \epsilon = 0 \) in each class is shown in figure 5. The remaining potential couplers in each class can be obtained by adding NOT-gates to some or all of the wires in the standard circuit, and/or by swapping the positions of Bob's two boxes. As every potential coupler in \( \chi \) can be realised by a probabilistic mixture of these extremal potential couplers (and hence a probabilistic wiring strategy for Bob), all of them fail to satisfy the non-triviality constraint. There are therefore no binary-output couplers for two binary-input/binary-output boxes.

Although we have so far only considered couplers with a binary output \( b' \), this result will hold for couplers with any range of outputs \( b \in \mathbb{N} \). If for some output \( b = b_0 \) a coupler characterised by \( \chi(b, b_1 b_2 y_1 y_2) \) existed, we could always construct a binary-output coupler characterised by \( \chi(b', b_1 b_2 y_1 y_2) \) which output \( b' = 0 \) if \( b = b_0 \) and \( b' = 1 \) when \( b \neq b_0 \), by taking

\[
\chi'(b' = 0, b_1 b_2 y_1 y_2) = \chi(b = b_0, b_1 b_2 y_1 y_2) \quad \text{(36)}
\]

\[
\chi'(b' = 1, b_1 b_2 y_1 y_2) = \sum_{b \neq b_0} \chi(b, b_1 b_2 y_1 y_2) \quad \text{(37)}
\]

As we have shown that there are no binary-output couplers, there must also be no couplers with a larger output range. We can therefore conclude that, in the case of two binary-input/binary-output boxes, no couplers exist. As

<table>
<thead>
<tr>
<th>Potential coupler classes</th>
<th>Number in class</th>
<th>Entries with ( \chi(b', b_1 b_2 y_1 y_2) = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic (( \chi^D ))</td>
<td>2</td>
<td>( y_1 = y_2 = 0, b' = \alpha )</td>
</tr>
<tr>
<td>One-sided (( \chi^O_{\alpha, \beta, \gamma} ))</td>
<td>8</td>
<td>( y_0 = y_1 = \alpha, b' = b_1 \oplus \gamma )</td>
</tr>
<tr>
<td>XOR-gated (( \chi^X_{\alpha, \beta, \gamma} ))</td>
<td>8</td>
<td>( y_1 = \alpha, y_2 = \beta, b' = b_1 \oplus b_2 \oplus \gamma )</td>
</tr>
<tr>
<td>AND-gated (( \chi^A_{\alpha, \beta, \gamma, \delta} ))</td>
<td>32</td>
<td>( y_1 = \alpha, y_2 = \beta, b' = (b_1 \oplus \gamma)(b_2 \oplus \delta) \oplus \epsilon )</td>
</tr>
<tr>
<td>Sequential (( \chi^S_{\alpha, \beta, \gamma, \delta} ))</td>
<td>32</td>
<td>( y_2 = \beta, y_{(1 \oplus \alpha)} = b_0 \oplus \gamma, b' = (b_1 \oplus \alpha) \oplus b_1 \oplus \delta \oplus \epsilon )</td>
</tr>
</tbody>
</table>

TABLE I: Table giving \( \chi(b', b_1 b_2 y_1 y_2) \) for a representation of all 82 extremal points of \( \chi \), corresponding to potential couplers (elements satisfying the listed constraints are one, while all other elements are zero). The potential couplers are divided into 5 classes, with the members in each class parameterized by \( \alpha, \beta, \gamma, \delta, \epsilon \in \{0, 1\} \). The second column shows the number of different potential couplers in each class.
discussed in section IV, this means that it is impossible to implement an analogue of entanglement swapping for PR-states.

VI. CONCLUSIONS

Considering correlation experiments in terms of abstract black-boxes allows us to separate the information-theoretic content of non-locality from the underlying physical detail.

For correlations between a set of time-independent measurements, no-signalling box-states represent every possibility which is consistent with relativity. However, quantum theory also allows the possibility of entanglement-swapping, in which non-local correlations can be introduced between two subsystems which are initially uncorrelated, by performing a joint measurement on two subsystems with which they are entangled and announcing the result.

We have shown that it is impossible to achieve an analogue of entanglement-swapping between two bipartite non-local box-states using a sequence of individual (yet conditional) box inputs. This led us to introduce the concept of couplers, which are an analogue of measurements with entangled eigenstates in quantum theory.

Under very general assumptions of universality (a coupler can be applied to any box with the appropriate inputs and outputs), completeness (a coupler acts identically on states with the same probability distribution), and no-signalling, we found that any allowed coupler must be linear. We then proceeded to investigate the allowed couplers which act on two binary-input/binary-output boxes. Perhaps surprisingly, we found that no couplers of this type exist. In particular, this means that when Alice and Bob share a PR-state, and Bob and Charlie share a PR-state, there is no way that Bob can generate any non-local correlations between Alice and Charlie.

As we have so far only explicitly considered couplers acting on binary-input/binary-output boxes, it would be interesting to see if couplers exist in more general cases. Of particular interest would be the class of couplers which act on two ternary-input/binary-output boxes and generate a two-bit output. As qubit states can be characterised by measurements of the three Pauli matrices, and Bell measurements have four outputs, this would enable a closer analogy between the quantum case and that of general correlated box states.

Couplers also correspond to a first step into the dynamics of correlated boxes, transforming \( n \) boxes in the initial state into one effective box (with output \( b' \) and no input) in the final state. It is straightforward to generalise the constraints introduced for couplers to apply to more general dynamical processes (taking \( n \) boxes in the initial state to \( m \) boxes in the final state). This opens the possibility for deeper studies of the dynamics of correlated boxes (as desired in it).

The results obtained so far for couplers suggest that by allowing stronger non-local correlations than are attainable in quantum theory, the dynamics of the model actually become weaker (to the extent that no couplers exist for two binary-input/binary-output boxes). This offers a possible insight into why such super-strong correlations are not attainable in nature: In order to allow richer dynamics.

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References

[17] Note that this setup differs from that of some other authors (eg. [13]), who consider that all parties share a single extended black box. Treating the subsystems as separate (yet correlated) boxes provides us with a closer analogue to the quantum case, and allows us to utilize the concept of measurement-induced ‘collapse’ in later sections. The mathematical formalism is the same in both cases.
[18] Note that when \( P(O|I) \) satisfies the no-signalling condition, \( P(O|R|I) \) will also satisfy it. \( P(O|R|I) \) is therefore an allowed correlated box state.
[19] Note that we do not give the coupler an input, as we intend it to correspond to a specific measurement (analogous to the Bell measurement in quantum mechanics), whereas the standard box inputs correspond to a selection of possible measurements.

[20] We obtained the polytope $\mathcal{X}$ using the program LRS, written by D.Avis.