Morse Theory in Field Theory

Peter Koroteev†,*
koroteev@mccme.ru

Andrey V. Zayakin†,*,‡
zayakin@theor.jinr.ru

† Institute of Theoretical and Experimental Physics, Moscow 117259, Russia
*Moscow Institute of Physics and Technology, Moscow 141701, Russia
*Moscow State University, Moscow 119992, Russia
‡Joint Institute for Nuclear Research, Dubna 141980, Russia

Abstract

In recent years it has become apparent that topological field theories (TFTs)[1] are likely to be the best candidates for the truly fundamental physical theory. Supersymmetry, for instance, can be motivated and expressed in terms of TFTs[2]. Here we build a simple example of TFT using Morse theory and Massey product. Action (invariant under supersymmetric transformations) is constructed and relations for correlators in this theory are obtained. While constructing this theory a theorem about the connection between Euler characteristic of a manifold and the sum of indices of critical points is proved for arbitrary dimension of the target space.
1 Introduction

Let \((M, g)\) be a smooth compact \(n\)-dimensional Riemannian manifold and \(v^i(x)\) a vector field on it. Suppose it has critical points (CP), i.e., \(A_i \in M\) where \(v(A_i) = 0\). According to Whitney theorem, one can embed this manifold into \(\mathbb{R}^{n+1}\).

Let \(f(x), x \in M\) be a smooth function on the manifold so that

1. all critical points of \(f\) (points where all first derivatives vanish) are nondegenerate,
2. \(v^i(x) = g^{ij}(x) \partial_j f(x), x \in M\), i.e. \(v^i\) is a gradient vector field.

We will refer to such functions as Morse functions on \(M\). The first statement is that one can decompose \(f\) in the vicinity of each CP \(A\) as follows

\[
f(x) = f(A) - \frac{1}{2} \sum_{j=0}^{p} (x^j - A^j)^2 + \frac{1}{2} \sum_{j=p+1}^{n} (x^j - A^j)^2 + O((x - A)^3). \tag{1}
\]

One calls \(p\) here a (Morse) index of CP \(A\). Let \(\Omega^\bullet(M, \mathbb{R}^1)\) be the space of differential forms on \(M\).

2 The Simplest Case: Zero–Dimensional World

Let us consider the simplest example — zero-dimensional theory.\(^1\) We deal with a 1-dimensional worldsheet (line \(\mathbb{R}^1\)) and \(M\) is a 1-dimensional compact target space. Since there are only two 1-dimensional compact manifolds: circle and segment, it would be convenient for us to deal with a boundaryless \(M\), therefore, case of \(S^1\) is considered.

To construct quantum mechanics one needs to determine two spaces: space of observables \(\mathcal{O}\) and space of states \(\mathcal{S}\). Let’s start with the observables. We assume \(\mathcal{O} = \Omega^\bullet(M)\). After quantization they become operators in a Hilbert space \(\mathcal{H}\).

We construct a partition functional (statistical sum) as a delta-functional on vector field

\[
Z = \int \delta[v(x)] d[v(x)] \tag{2}
\]

which yields a sum of indices over all CPs \(A\) of \(v\). Indeed, as \(v(x) = 0\) in a CP,

\[
Z = \sum_{A \in M} \text{ind}_{v(A)}. \tag{3}
\]

Note that we have a finite number of critical points. Therefore the statistical sum is represented as a finite-dimensional integral, not a path integral

\[
Z = \int dx dp d\psi d\pi \exp(-S[x, p, \psi, \pi]),
\]

\[
S = pv(x) - \psi v'(x)\pi \equiv Q(\pi v),
\]

where

\[
Q = p \frac{\partial}{\partial \pi} + \psi \frac{\partial}{\partial x}.
\]

\(^1\)Terminology is borrowed from string theory.
Here the exponent has been introduced formally and the variables $p, \psi, \pi$ don’t have any physical meaning. They will acquire it later, $p, \pi$ becoming canonical momenta, $x, \psi$ becoming their respective coordinates.

Quantum mechanical observables can be represented by differential forms on $\mathcal{M}$ with delta-functional support. In our simple case forms under consideration “live” on points $C \in \mathbb{R}^1$. Therefore, expectation value is given by the following expression

$$\langle \omega \rangle = \int_\mathcal{M} \delta_C \wedge \omega = \int dx \, dp \, d\psi \, d\pi \ e^{-S[x,p,\psi,\pi]} \omega = \text{Tr}(\rho \omega),$$

where $\delta_C = \int dp \, d\psi \, d\pi \ e^{-S}$ is a “delta function on a point $C$”. One can see that this approach enables us to introduce density matrix $\rho = \delta_C$ but not physical states. To calculate expectation value of several observables we write by an analogy

$$\langle O_1 \ldots O_n \rangle = \int_\mathcal{M} \delta_C \wedge O_1 \wedge \cdots \wedge O_n = \int dx \, dp \, d\psi \, d\pi \ e^{-S[x,p,\psi,\pi]} O_1 \wedge \cdots \wedge O_n$$

$$= \text{Tr}(\rho O_1 \ldots O_n),$$

(5)

### 2.1 Generalization. Vector Field on the Space of Paths of Steepest Descent.

We are going to study action $S = \{Q, A\}$, where

$$A = \pi_a(t) \left( \frac{dx^a}{dt} - v^a(t) \right) + \epsilon p_a \pi_b \eta^{ab}$$

(6)

Here we consider paths with boundary conditions

$$x(t) : [0, 1] \mapsto \mathcal{M}, \quad x(0) = A, \ x(1) = B \quad (7)$$

$$x(t) : (-\infty, +\infty) \mapsto \mathcal{M}, \quad x(-\infty) = A, \ x(+\infty) = B. \quad (8)$$

between critical points of vector field $v$.

$$S = \int_A^B dt \left[ p_a (\dot{x}^a - v^a) + \epsilon \psi^i \psi^j F_{ij}^b \pi_b \pi_c \eta^{ac} + \epsilon p_a p_b \eta^{ab} - \psi^i (\nabla_i v^a(x)) \pi_a \right]$$

$$+ \psi^c A_{ci}^a \dot{x}^i \pi_a, \quad (9)$$

Thus partition functional in our theory is

$$Z = \frac{1}{\sqrt{\epsilon}} \exp \int_A^B dt \left[ -\frac{1}{4\epsilon} (\dot{x}^a - v^a)^2 + \epsilon \psi^i \psi^j F_{ij}^b \pi_b \pi_c \eta^{bc} - (\nabla_i v^a(x)) \pi_a \right]$$

$$+ \psi^c A_{ci}^a \dot{x}^i \pi_a \right], \quad (10)$$

represents delta — functional on the path of steepest descent, determined by equation

$$\dot{x}^i = v^i(x).$$

(11)
2.2 Gauss-Bonnet-Hopf Theorem

Let’s make a transition $\epsilon \to \infty$. Partitional functional will tend to

$$
Z \to \frac{1}{\epsilon} \int \exp \int_{t_A}^{t_B} dt \left[ - \epsilon \psi^i \psi^j F_{ij}^b \pi_b \pi_c \eta^{ac} \right], \epsilon \to \infty \quad (12)
$$

Integrating out odd fields one can show that $Z$ tends to integral of curvature $F$. On the other hand, by definition, $Z$ counts CP of $v$ with their indices. Thus Hopf theorem is proved using QFT methods

$$
\int_M d^n x F = \chi(M) = \sum_{A \in \mathcal{M}} \text{ind}_v(A), \quad (13)
$$

where $\chi(M)$ is Euler characteristic of $M$.

3 Morse Theory in Nil Geometry

We’re going to consider an interesting example — a three-dimensional manifold $M$ which is known as Nil geometry. It is a uniform anisotropic space with zero scalar curvature but nonzero Ricci tensor. $\mathcal{M}$ can be represented as a nontrivial bundle over base $S^1$ with typical fiber $T^2$. Or vice versa, as a nontrivial bundle over base $T^2$ with typical fiber $S^1$. This bundle is very similar to Hopf bundle — $S^3$ over $S^2$.

We can define $\mathcal{M}$ by taking its universal covering space, $R^3$ and factorizing it via the following relations:

$$
x \sim x + m \\
y \sim y + mz + k \\
z \sim z + n
$$

$m, n, k \in \mathbb{Z}$. The fundamental region is a “deformed parallelepiped”. Note that its $Oxy$ and $Oyz$ facets are planes, whereas the former $Oxz$ facets have become deformed.

The non-trivial factorization relations essentially limit the class of Morse functions on this manifold. If we had a torus, we would deal with simple shifts of coordinates instead

$$
x \sim x + m \\
y \sim y + k \\
z \sim z + n
$$

and therefore we would employ trigonometric functions as Morse functions. Dealing with modular transformations as (14) results in appearance of $\Theta$-functions instead, the latter ones being invariant thereof.

We employ the following Morse function

$$
f(x, y, z) = (2 + \cos(2\pi x)) (2 + \cos(2\pi z)) \cdot \left( \theta_3(\pi x - iAz, e^{-A}) e^{2i\pi(y-xz)} + \theta_3(\pi x + iAz, e^{-A}) e^{-2i\pi(y-xz)} \right) e^{-Az^2}
$$

One can explicitly check this function is a real invariant of (14). This Morse function has six different critical points.
### 3.1 Differential Forms on $\mathcal{M}$

From (14) one can see that forms $dx$ and $dy$ live on $\mathcal{M}$. To understand whether any other forms exist we consider generic forms. As far as 1-form case is concerned we have

$$\omega^{(1)} = f dx + g dy + hdz$$

All necessary conditions are satisfied when

$$f(x + n, y + nz + m, z + k) = f(x, y, z),$$
$$g(x + n, y + nz + m, z + k) = g(x, y, z),$$
$$h(x + n, y + nz + m, z + k) - h(x, y, z) = -ng(x, y, z),$$

i.e., $f, g \in C^\infty(\mathcal{M}, \mathbb{R})$, whereas $h$ satisfies (5). For example,

$$\omega^{(1)} = -dy + xdz.$$

Analogously for 2 and 3 forms we obtain

$$\omega^{(2)} = \alpha dx \wedge dy + \beta dy \wedge dz + \gamma dx \wedge dz$$
$$\omega^{(3)} = \Omega dx \wedge dy \wedge dz, \quad \Omega \in C^\infty(\mathcal{M}, \mathbb{R})$$

with the following constraints

$$\alpha(x + n, y + nz + m, z + k) = \alpha(z, y, z),$$
$$\beta(x + n, y + nz + m, z + k) = \beta(x, y, z),$$
$$\gamma(x + n, y + nz + m, z + k) - \gamma(z, y, z) = -n\alpha(x, y, z)$$

i.e., $\alpha, \beta \in C^\infty(\mathcal{M}, \mathbb{R})$, but $\gamma$ satisfies (6). For example,

$$\omega^{(2)} = dx \wedge dy - x dx \wedge dz$$

### 3.2 Cohomology Groups of Nil Geometry

De-Rham cohomologies are defined as follows [5]

$$H^k(\mathcal{M}, \mathbb{R}) = \frac{\text{Ker}(\Omega^k \rightarrow \Omega^{k+1})}{\text{Im}(\Omega^{k-1} \rightarrow \Omega^k)}$$
We recall that $T^n$ cohomologies are groups generated by $C^n_k$ monomials in corresponding dimensionalities. Let’s calculate cohomologies for our case. Closed forms

\[
\begin{align*}
\text{const} & \quad \text{in dimension 0} \\
dx, dz & \quad \text{in dimension 1} \\
dx \wedge dz, dy \wedge dz, & \\
dx \wedge dy - x dx \wedge dz & \quad \text{in dimension 2} \\
dx \wedge dy \wedge dz & \quad \text{in dimension 3}
\end{align*}
\]

Exact forms

\[
\begin{align*}
\text{no} & \quad \text{in dimension 0} \\
\text{no} & \quad \text{in dimension 1} \\
dx \wedge dz & \quad \text{in dimension 2} \\
\text{no} & \quad \text{in dimension 3}
\end{align*}
\]

Cohomologies

\[
\begin{align*}
H^0 &= \mathbb{R} \\
H^1 &= \{dx, dz\} \\
H^2 &= \{dx \wedge dy - x dx \wedge dz, dy \wedge dz\} \\
H^3 &= \{dx \wedge dy \wedge dz\}
\end{align*}
\]

4 Massey Product in Morse Theory

Let’s introduce Massey product on differential forms

\[
\text{MP}(\omega_1, \omega_2, \omega_3) = \int_{\mathcal{M}} \Omega_1 \wedge \omega_1 \wedge d^{-1}(\omega_2 \wedge \omega_3) + \Omega_2 \wedge d^{-1}(\omega_1 \wedge \omega_2) \wedge \omega_3,
\]

where $\omega_i \in H^*(\mathcal{M}, \mathcal{R}^1)$. For the whole integrand to be an n-form, $\Omega_i$ is introduced, which should be a monomial in the space complementary to the space where the other part of integrand “lives”. Symbolically Massey product can be depicted as a sum of diagrams shown above.

If the case of Nil geometry is concerned there are only two nontrivial Massey products

\[
\begin{align*}
\text{MP}(dx, dx, dz) &= \int_{\mathcal{M}} \Omega_1 \wedge (dx \wedge dy - x dx \wedge dz) \\
\text{MP}(dz, dx, dz) &= 2 \int_{\mathcal{M}} \Omega_2 \wedge dy \wedge dz
\end{align*}
\]

6
Here we are to put $\Omega_1 = dz$, $\Omega_2 = dx$ for the integrand to be a 3-form. Note that that whole integrand should be well defined on $\mathcal{M}$ so we have only one choice for $\Omega_1$, since multiplying by $dy$ yields the form $xdx \wedge dy \wedge dz$, non-existent on $\mathcal{M}$ and multiplying by $dx$ yields zero.

Introduction of the following normalization (volume of $\mathcal{M}$ is equal to unity)

\[ \int_{\mathcal{M}} dx \wedge dy \wedge dz = 1 \]  \hspace{1cm} (22)

gives the following results for MP

\[ \text{MP}(dx, dx, dz) = 1 \]  \hspace{1cm} (23)
\[ \text{MP}(dz, dx, dz) = 2 \]  \hspace{1cm} (24)

Thus one can make MP an integer.

5 Outlook

Here we have constructed the simplest example of TFT — supersymmetric quantum mechanics. Its correlators are integers — numbers of intersections of paths of steepest descent with cycles corresponding to each dynamical operator[1].

Massey product of forms on $\mathcal{M}$ can be made integer by an appropriate choice of normalization.

Thus an interesting question arises. Is there any connection between Massey product MP with necessary normalization and correlator in TFT? Another question is how one can express MP (19) in terms of quantum mechanics, i.e., as a correlator with an appropriate Hamiltonian.

All this questions are now under investigation.

6 Acknowledgements

The authors are grateful to Dr. Andrey S. Losev for fruitful discussions. This work is partly supported by grants RFBR 01-02-17227 (P. Koroteev) and 04-01-00637 (A. Zayakin).

References


