FINITENESS AND DUAL VARIABLES FOR LORENTZIAN SPIN FOAM MODELS

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ABSTRACT. We describe here some new results concerning the Lorentzian Barrett-Crane model, a well-known spin foam formulation of quantum gravity. Generalizing an existing finiteness result, we provide a concise proof of finiteness of the partition function associated to all non-degenerate triangulations of 4-manifolds and for a class of degenerate triangulations not previously shown. This is accomplished by a suitable re-factoring and re-ordering of integration, through which a large set of variables can be eliminated. The resulting formulation can be interpreted as a “dual variables” model that uses hyperboloid variables associated to spin foam edges in place of representation variables associated to faces. We outline how this method may also be useful for numerical computations, which have so far proven to be very challenging for Lorentzian spin foam models.

1. INTRODUCTION

Spin foam models offer a promising new quantum mechanical picture of space-time geometry. The spin foam program has been developed as a path integral formulation of loop quantum gravity\(^1\). Of the spin foam models proposed in the literature, perhaps the most studied have been the Barrett-Crane models. Originally constructed using representations of the group Spin(4) associated to manifolds of Riemannian signature \(3\), a subsequent model given by Barrett and Crane in \(3\) based upon representations of the Lorentz group SO(3,1) is considered to be more physically realistic.

While important strides have been made in numerical computations within the Riemannian framework \(2,3,13\), numerical computations with the Lorentzian model have proven much more difficult. An important difference between the models lies in the space of representations, which in the Lorentzian case is continuous rather than discrete. In addition, the homogeneous space used to define vertex amplitudes is non-compact in the Lorentzian case.

The main result of the present work is that the Lorentzian partition function can, by a process we shall refer to as “face factoring”, be rearranged so that integration over the representation variables is performed exactly. After applying this transformation, computational or analytic effort can focus entirely on the homogeneous space integrals and the one-dimensional integrals associated to each edge of the spin foam.

We note that the model that results from face factoring is strongly in the spirit of the dual variables picture proposed by Pfeiffer in \(20\). While \(20\) dealt specifically with a Riemannian version of the Barrett-Crane model, our face-factored model can be understood as the realization of a dual variables picture in the Lorentzian case.

The organization of the paper is as follows. In the next section, we recall the form of the Lorentzian Barrett-Crane amplitude and describe the face factoring method. Included in this section is an illustration of the method applied to the partition function of a simple 2-complex. In Section 3, we prove that the partition function associated to triangulations of a certain type is absolutely convergent in the face factored formulation, which justifies the interchange of integration required by the method. It follows that the original partition function is finite for triangulations of this type, which includes both the non-degenerate triangulations proven finite in \(11\) as well as certain degenerate triangulations that have not previously been proven finite. It should be noted here that in the present work and in \(11\), finiteness is shown for a specific choice of edge and face amplitudes; throughout this work our main interest is in the choice\(^2\) due to Perez and Rovelli \(18\). We also show how the method can be applied to an explicitly causal version of the Barrett-Crane model due to Livine and Oriti \(16\) for certain types of triangulations. In Section 4, we briefly discuss the

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\(^1\)The reader is referred to \(2\) for a recent review of loop quantum gravity, and \(17\) for a review of the spin foam program specifically.

\(^2\)The physical content of these amplitude choices remains controversial; for example, see \(2\) for a critical discussion of this issue. In Section 3, we examine how certain changes in the face amplitude affect the type of degenerate triangulations for which we can show finiteness.
potential for numerical applications of the method and end with some concluding remarks. A bound required in Section 3 is proven in Appendix A, and in Appendix B we show that the finiteness results of Section 3 (in which a closed 4-manifold is assumed for clarity) also hold in the case of a 4-manifold with boundary.

2. Description of the Face Factoring Method

Review of the Lorentzian Barrett-Crane Model. To begin our discussion, we recall the form of a Lorentzian spin foam model. Spin foam models are defined on simplicial 2-complexes constructed from a set of vertices \( V \), a set of edges \( E \), and a set of polygonal faces \( F \). Because our present work is with the Barrett-Crane model, we restrict ourselves to those 2-complexes which are the dual 2-skeleton of a 4-manifold triangulation\(^3\). The dual 2-skeleton is formed as follows: for every 4-simplex there is an associated dual vertex, and any two dual vertices whose associated 4-simplices meet at a tetrahedron are connected by a dual edge. For every face of the triangulation, the closed loop of 4-simplices that share that face gives rise to a two-dimensional polygon in the dual 2-skeleton, the vertices of which are dual to the 4-simplices and the edges of which are dual to the tetrahedra that are shared between neighboring 4-simplices. An important property of this construction is that the dual polygonal faces are in one-to-one correspondence with the faces of the original 4-manifold triangulation. Further background, including illustrations of dual skeleta, can be found in \[4\].

We now introduce the Lorentzian Barrett-Crane model, which assigns to each dual 2-complex \( \Delta \) a partition function \( Z_\Delta \) as follows:

\[
Z_\Delta = \prod_{f \in \mathcal{F}} A_f \prod_{e \in \mathcal{E}} A_e \prod_{v \in \mathcal{V}} A_v \prod_{f \in \mathcal{F}} p_f^2 \, dp_f,
\]

where factors of the form \( p_f^2 \) arise from the measure on the principal series of \( SO(3,1) \) representations; the amplitudes \( A_f, A_e, \) and \( A_v \) will be defined shortly. For a given 2-complex, the state space associated with this partition function is the product space of representation variables \( p_f \) (one for each face) and the multiple integration over this space is thought of as a path integration over all spin configurations. To find the transition amplitude between two spin networks (the basis states of 3-geometry in loop quantum gravity), a sum of \( Z_\Delta \) over all 2-complexes interpolating between the spin networks is required. In the sum over 2-complexes, the amplitude \( Z_\Delta \) for each 2-complex will be weighted by an appropriate measure to provide a normalization and possibly to regulate which 2-complexes can contribute. The choice of measure and questions of convergence have yet to be resolved — progress is likely to depend on what results can be found for the partition function of individual 2-complexes, which is the focus of our present work.

In the expressions to follow, we write \( \prod \) to indicate a formal product of symbols such as integral signs or measures. When a relation such as \( f \ni v \) appears below a product symbol (multiplicative or formal), the product is taken over all of the objects on the left hand side of relation that satisfy the relation; for example, in the \( f \ni v \) case the product would be over all faces \( f \) such that \( v \) is a member of \( f \). The symbols \( v, e, \) and \( f \) will always denote members of the sets \( \mathcal{V}, \mathcal{E}, \) and \( \mathcal{F} \), respectively.

We now turn to the definition of the amplitudes \( A_f, A_e, \) and \( A_v \) that appear in (2.1). Up to a regularization that will be defined below, the Lorentzian Barrett-Crane vertex amplitude of \[3\] is defined as

\[
A_v(p_f) = \prod_{x \in \mathcal{V}} \int_{H^2_+} dx_e \prod_{f \ni v} K_{p_f}(x_{e_+(f,v)}, x_{e_-(f,v)}),
\]

where the kernel function \( K_{p_f} \) is

\[
K_{p_f}(x, y) = \frac{\sin(p_f \phi(x, y))}{p_f \sinh(\phi(x, y))}.
\]

The kernels \( K_{p_f} \) are functions of the variables \( x_{e_+(f,v)}, x_{e_-(f,v)} \in H^2_+ \) associated to the two dual edges that both contain \( v \) and are contained in \( f \); the + and − serve to distinguish the two tetrahedra within a 4-simplex that share a face, but otherwise have no significance — any fixed convention can be chosen. As there are ten faces in the 4-simplex to which a vertex is dual, the integrand is a product of ten such kernels. The

\(^3\)In Section 3, we extend the strict definition of triangulation to include degenerate triangulations which can be associated to topological 4-manifolds.
kernel function $K_{p_f}$ depends on the hyperboloid variables through their hyperbolic distance $\phi$ on $H_+^3$, which is defined as
\begin{equation}
\phi(x, y) = \cosh^{-1}(x \cdot y)
\end{equation}
for $x, y \in H_+^3$, where $H_+^3 = \{ x \in \mathbb{R}^4 \mid x \cdot x = 1, x_0 > 0 \}$; here $x \cdot y$ is the Minkowski inner product $x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$.

Although the expression for the vertex amplitude given above is generally infinite, we shall regularize it by fixing the value of one $H_+^3$ variable and dropping the corresponding integral over $H_+^3$ — the answer is independent of the choices $\mathbf{r}$.

It should be observed that the future 3-hyperboloid $H_+^3$ is a homogeneous space of $SL(2, \mathbb{C})$ and, unlike the homogeneous space $S^3$ that arises in the Riemannian model, it is non-compact. The other basic difference with the Riemannian model is that the relevant representations of $SL(2, \mathbb{C})$ are indexed by a continuous parameter $p_f$; in place of a summation for each face the Lorentzian model requires an integral.

While the original model of [2] specifies the vertex amplitude $A_v$ as given in (2.2), it leaves unspecified the edge and face amplitudes $A_e$ and $A_f$. A proposal introduced by Perez and Rovelli in [13] and considered in further work [14, 19] chooses $A_f = 1$ and $A_e = \Theta_4(p_1, p_2, p_3, p_4)$ for edges in the interior of the 2-complex. The function $\Theta_4(p_1, p_2, p_3, p_4)$ is known as the edge diagram and can be defined as follows:
\begin{equation}
\Theta_4(p_1, p_2, p_3, p_4) = \frac{2}{\pi p_1 p_2 p_3 p_4} \int_0^\infty \frac{\sin(p_1 r_e) \sin(p_2 r_e) \sin(p_3 r_e) \sin(p_4 r_e)}{\sinh^2(r_e)} dr_e,
\end{equation}
where $p_1, \ldots, p_4$ are the spin variables labelling the four faces containing $e$.

The results we have obtained with the face factoring technique have primarily been for the amplitudes $A_f = 1$ and $A_e = \Theta_4(p_1, p_2, p_3, p_4)$; we shall henceforth refer to this choice as the Perez-Rovelli Model.

The Face Factoring Method. The major difficulty in computing $Z_\Delta$ as presented in (2.1) relates to the computation of the vertex amplitude (2.2). For each choice of the ten $p_f$ variables, calculating a single vertex amplitude requires extensive computational effort in the form of Monte Carlo or quasi Monte Carlo integration. In the absence of an exact expression or some other significantly more efficient means of calculation, it is prohibitively expensive computationally to evaluate the partition function for even the simplest 2-complexes. This situation has provided the main motivation for finding an alternative approach to calculating $Z_\Delta$, in which no vertex amplitudes are explicitly computed.

The idea of the present work is to perform the integration over the spin variables $p_f$ first, leaving a formulation in which the states to be integrated over are the configurations of $H_+^3$ variables originating with the vertex amplitude integrand and the $r_e$ variables originating with the edge amplitude integrand. This reverses the conventional integration order implicit in (2.1) that has been adopted in most computational approaches to date. For example, in computations for the Riemannian case [3] the vertex amplitudes (equal to integrals over the homogeneous space) are computed for each spin configuration and the results summed to give the partition function.

Given the difficulty in evaluating a Lorentzian vertex amplitude for a single choice of spins, it is a rather fortuitous result that all the spin dependence can be integrated out — indeed, as we shall see shortly, one obtains exact expressions in terms of the remaining $H_+^3$ and $r_e$ variables.

We define the method explicitly as follows. Assuming that the original partition function is not affected by interchanging the order of integration, we can write:
\begin{equation}
Z_\Delta = \left( \prod_{f \in F} \int_0^\infty dp_f \right) \left( \prod_{f \in F} p_f^2 A_f(p_f) \right) \left( \prod_{e \in E} A_e(p_e) \right)
\end{equation}
\begin{equation}
\left( \prod_{e \in V} \left( \prod_{e \in E, e \not\in e} \int_{H_+^3} dx_e(v) \right) \prod_{f \in F} K_{p_f}(x_{e+}(f,v), x_{e-}(f,v)) \right)
\end{equation}

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\textsuperscript{4}In this we follow the original Lorentzian model of Barrett and Crane and use exclusively the $(0, p)$ representations from the principal series. While some proposals [12] have been made which also include half integer labelled representations $(k, 0)$, most work has been with the original proposal.

\textsuperscript{5}Some authors absorb the $p_f$ factors from the measure into the definition of $A_f(p_f)$; in the present work we keep the measure and face amplitude distinct.
\[
= \left( \prod_{v \in V} \left( \prod_{e \ni v, e \neq e^v} \int_{H^3_+} dx(e_v) \right) \right) \left( \prod_{f \in F} \int_0^\infty dp_f \right) \left( \prod_{f \neq H^3_+} p_f^2 A_f(p_f) \right) \left( \prod_{e \in E} A_e(p_f) \right) \]
\]

On the RHS of (2.6), all dependence on the \( p_f \) variables is integrated out before performing the integrals over \( H^3_+ \). As called for by the regularization, one integration with respect to \( H^3_+ \) has been dropped from each vertex (the edges at \( v \) for which integration is dropped is denoted by \( e^v \)). To perform integration over the \( p_f \) variables first, we consider the product of all the edge and vertex amplitude integrands, and organize this product into factors that depend only upon a single \( p_f \) variable.

If we consider for the moment only the factors contributed by the vertex amplitudes \( A_e \) given in (2.2), we note the following. Due to the form of the integrand of \( A_e \) in terms of the functions (2.3), we see that for any given face \( f \) of the 2-complex all of the factors involving the spin \( p_f \) can be grouped together into a vertex face factor \( F^V_f \) of the form

\[
F^V_f(p_f, \phi^f_i) = \frac{\sin(p_f \phi^f_1) \sin(p_f \phi^f_2) \cdots \sin(p_f \phi^f_{\text{deg}(f)})}{p_f^{\text{deg}(f)}},
\]

where the \( \phi^f_i \) denote distances on \( H^3_+ \) and \( \text{deg}(f) \) denotes the number of vertices contained in the dual face \( f \). With regard to the subscript \( i \) indexing the \( \phi^f_i \) variables, we assume that a numbering of vertices in every dual face \( f \) has been chosen; the \( i \) subscript runs over the values of the indexing map \( N_f(v) \) and the corresponding vertex \( v \) is found by inverting \( N_f(v) \). Explicitly, we choose for each \( f \) a bijective map of the form \( N_f : V_f \rightarrow \{1, \ldots, \text{deg}(f)\} \), where \( V_f \) is the set of vertices in \( f \). Although we have suppressed the arguments of each \( \phi^f_i \) in the formula above, for a given \( \phi^f_i \) letting \( v = N_f^{-1}(i) \) we see that \( \phi^f_i = \phi^f(x_{e_-(f,v)}, x_{e_+(f,v)}) \) is a function of the \( H^3_+ \) variables \( x_{e_-(f,v)} \) and \( x_{e_+(f,v)} \).

Although the face amplitude for the Perez-Rovelli model has no spin dependence \((A_f = 1)\), the edge amplitude (2.5) is a non-trivial function of the four spins variables associated to an edge of the 2-complex. While the eye diagram can be integrated to an exact expression in terms of the product of spin variables and the hyperbolic cotangent function \( \text{ctanh} \), the result does not have a form in which the spins \( p_f \) appear in separate factors. However, we observe from (2.5) that before integration with respect to \( r_e \), the \( p_f \) dependent part of the integrand is a product of factors of the form \( \sin(p_f r_e) \). Hence, if we collect factors from the different edge amplitude integrands that depend on the same \( p_f \), we have a product for each face which has the same form as the vertex face factor (2.7), but with the \( r_e \) variables playing the role of the hyperbolic distance variables \( \phi^f_i \). Therefore, if the \( r_e \) variables are integrated only after integrating with respect to the \( p_f \), we can define an overall face factor that takes into account contributions from both edge and vertex amplitude kernels. Our final form for the face factors are then

\[
F_f(p_f, \phi^f_i, r_e) = \frac{\sin(p_f \phi^f_1) \cdots \sin(p_f \phi^f_{\text{deg}(f)}) \sin(p_f r_e(f,1)) \cdots \sin(p_f r_e(f,\text{deg}(f)))}{p_f^{\text{deg}(f)+\text{deg}(f)-2}},
\]

where \( \text{deg}(f) \) denotes the number of edges contained in the face \( f \) and the \( e(f,i) \) selects the \( i \)th edge contained in the face \( f \). It should be noted that we have absorbed the \( p_f^2 \) terms from the measure into our definition of \( F_f \) by lowering the power of \( p_f \) in the denominator by two.

Having defined our face factors, we can now rewrite the partition function in terms of the integrals of the face factors \( F_f(p_f, \phi^f_i, r_e) \) and products of sinh functions that depend only upon the \( r_e \) and \( \phi^f_i \):

\[
\mathcal{Z}_\Delta = \left( \prod_e \int_0^\infty dr_e \right) \left( \prod_v \left( \prod_{e \ni v, e \neq e^v} \int_{H^3_+} dx(e_v) \right) \right) \left( \prod_e \frac{2}{\pi \sinh^2(r_e)} \right)
\]

\text{As with the face-vertex numbering \( N_f(v) \), we assume that for every face a numbering of its edges \( M(f,e) \) has been chosen. We use an edge valued subscript for the \( r_e \) as there is only one such variable for every edge.}
\[
\left( \prod_{v} \left( \prod_{f \ni v} \frac{1}{\sinh(\phi_{N_{f}}(v))} \right) \right) \left( \prod_{f} \int_{0}^{\infty} F_{f}(p_{f}, \phi_{f}^{i}, r_{c}) \, dp_{f} \right).
\]

Formula (2.9) is our explicit \textit{face factored formulation} for the partition function of the Perez-Rovelli model; in Section 3 we shall prove it is equal to the original model for all non-degenerate triangulations and certain degenerate triangulations.

From (2.9), we see that face factoring requires interchanging the order of integration among the \( p_{f} \), the \( r_{c} \), and the hyperboloid integrals — all of which are improper. In order to justify the interchange, it is sufficient that at least one of the orderings of integration is absolutely integrable. This follows from the Tonelli-Hobson test, a corollary of the well-known Fubini theorem in the case of integrals on an unbounded domain; see for example [11]. In Section 3 below, we justify this interchange for a general class of triangulated 4-manifolds by establishing absolute integrability for the face factored formulation of the partition function.

We consider next an illustration of the face-factoring method for the 2-complex dual to a particular triangulation of the 4-sphere \( S^{4} \).

\textbf{Example face factoring of a Barrett-Crane model with trivial edge amplitude.} For simplicity, we consider in this section a modified Perez-Rovelli model with (2.5) replaced by a trivial edge amplitude so that

\[ A_{f} = 1, \quad A_{e} = 1. \]

The standard Barrett-Crane vertex amplitude

\[ A_{v} = \left( \prod_{e \ni v, e \neq e_{i}^{0}} \int_{H_{e}} \frac{dx_{e}}{2} \right) \left( \prod_{f \ni v} K_{p_{f}}(x_{e+}(f, v), x_{e-}(f, v)) \right) \]

is used; recall from (2.3) that the kernel functions are

\[ K_{p_{f}}(x, y) = \frac{\sin(p_{f} \phi(x, y))}{p_{f} \sinh(\phi(x, y))}. \]

Consider the triangulation of \( S^{4} \) resulting from taking the boundary of the 3-simplex. In this triangulation, there are 6 four-simplices, 15 tetrahedra, and 20 triangles. We shall work with the dual 2-skeleton of this triangulation, in which there are 6 vertices, 15 edges, and 20 polygonal faces; the relevant face factors \( F_{f}(p_{f}, \phi_{f}^{i}) \) will be deduced from counting and symmetry arguments. Because each vertex is contained in ten faces, there will be ten \( K_{p_{f}}(x, y) \) factors per vertex for a total of sixty \( K_{p_{f}}(x, y) \) factors. In our present case, all faces have an associated face factor of the same form by symmetry. Since by definition there is a face factor for each face \( f \) and there are twenty faces for the entire 2-complex, we see that each face factor has the form

\[ F_{f}(p_{f}, \phi_{f}^{i}) = \frac{\sin(p_{f} \phi_{1}^{i}) \sin(p_{f} \phi_{2}^{i}) \sin(p_{f} \phi_{3}^{i})}{p_{f}}, \]

where the three \( \phi_{f}^{i} \) are associated to the three vertices in \( f \). Having found the face factor, we consider the integral that appears in the face-factored form of the partition function definition given in (2.9):

\[ \int_{0}^{\infty} F_{f}(p_{f}, \phi_{f}^{i}) \, dp_{f} = \int_{0}^{\infty} \frac{\sin(p_{f} \phi_{1}^{i}) \sin(p_{f} \phi_{2}^{i}) \sin(p_{f} \phi_{3}^{i})}{p_{f}} \, dp_{f} = \frac{1}{8\pi} (\text{sign}(\phi_{1}^{i} + \phi_{2}^{i} + \phi_{3}^{i}) - \text{sign}(\phi_{1}^{i} + \phi_{2}^{i} - \phi_{3}^{i}) - \text{sign}(\phi_{1}^{i} + \phi_{2}^{i} + \phi_{3}^{i}) + \text{sign}(\phi_{1}^{i} + \phi_{2}^{i} - \phi_{3}^{i})). \]

As suggested above, this integral is not difficult to evaluate to an exact closed form in terms of the distance variables \( \phi_{f}^{i} \). We note here that given the difficulty of evaluating a single Lorentzian vertex amplitude, it may be surprising that integrating over all the spin variables can be done in such a way that the result is a product of simple functions of the \( \mathbb{H}_{3}^{+} \) variables.

We again emphasize that to equate the face factored form of (2.9) with the original formulation of the partition function, interchanging the order of the improper integrals needs to be justified. We won’t provide such a justification for this example case — however, in Section 3 we prove that the face factoring method is
valid for the actual case of interest, the Lorentzian Barrett-Crane model with the Perez-Rovelli face and edge amplitudes. Before doing so, we shall give an exact closed form expression for the face factor integrations that arise in the Perez-Rovelli model.

**Closed form for the Perez-Rovelli face factor integrations.** In the Perez-Rovelli model, the most general face factor is given by \((2.8)\), and the integrals required to eliminate dependence on \(p_f\) for a given \(f\) are of the form

\[
\int_0^\infty \left( \prod_{v \in f} \frac{\sin(p_f \phi^f_v)}{p_f} \right) \left( \prod_{e \in f} \frac{\sin(p_f r_e)}{p_f} \right) p_f^2 \, dp_f.
\]

Clearly, this integral can be expressed using a product of sinc functions as

\[
\left( \prod_{v \in f} \phi^f_v \right) \left( \prod_{e \in f} r_e \right) \int_0^\infty \left( \prod_{v \in f} \frac{\sin(p_f \phi^f_v)}{p_f} \right) \left( \prod_{e \in f} \frac{\sin(p_f r_e)}{p_f} \right) p_f^2 \, dp_f.
\]

Were it not for the factor of \(p_f^2\) due to the measure, one would be integrating a product of sinc functions with arguments scaled by the \(\phi^f_v\) and \(r_e\) variables. If this were the case, a rather remarkable closed form given by Borwein et. al. in [10] could be directly applied. This formula, which will be useful for our actual case as well, we give here as follows.

Without loss of generality, let an arbitrary face \(f\) in the 2-complex be chosen. We define a new set of variables \(d_i\) so that \(d_i = \phi^f_{i+1} - \phi^f_i\) for \(0 \leq i < \deg_V(f)\) and \(d_i = r_{e(f,i+1-\deg_V(f))}\) for \(\deg_V(f) \leq i < \deg_V(f) + \deg_E(f)\). As the \(d_i\) are all non-negative, we can apply from [10] the result:

\[
\int_0^\infty \left( \prod_{i=0}^{\deg_V(f)} \frac{\sin(p_f d_i)}{p_f} \right) \left( \prod_{i=0}^{\deg_E(f)} \frac{\sin(p_f r_i)}{p_f} \right) p_f^2 \, dp_f = \frac{\pi}{2} \frac{1}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma b_\gamma \text{sign}(b_\gamma)
\]

where \(n = \deg_V(f) + \deg_E(f) - 1\) and \(\gamma = (\gamma_1, \cdots, \gamma_n) \in \{-1,1\}^n\). The \(b_\gamma\) and \(\epsilon_\gamma\) are defined by

\[
\epsilon_\gamma = \prod_{k=1}^n \gamma_k, \quad b_\gamma = d_0 + \sum_{k=1}^n \gamma_k d_k.
\]

Remarkably, it turns out that this formula enables us to handle the Perez-Rovelli case as well — we simply differentiate both sides of (2.12) twice with respect to (any) one of the \(d_i\) parameters. On the LHS the differentiation is passed under the integral sign; choosing \(d_0\) as our parameter to differentiate we have:

\[
-\frac{\partial^2}{\partial d_0^2} \int_0^\infty \left( \prod_{i=0}^{\deg_V(f)} \frac{\sin(p_f d_i)}{p_f} \right) \left( \prod_{i=0}^{\deg_E(f)} \frac{\sin(p_f r_i)}{p_f} \right) p_f^2 \, dp_f = \int_0^\infty -\frac{\partial^2}{\partial d_0^2} \left( \prod_{i=0}^{\deg_V(f)} \frac{\sin(p_f d_i)}{p_f} \right) \, dp_f
\]

\[
= \int_0^\infty \left( \prod_{i=0}^{\deg_V(f)} \frac{\sin(p_f d_i)}{p_f} \right) p_f^2 \, dp_f = -\frac{\partial^2}{\partial d_0^2} \left( \frac{\pi}{2} \frac{1}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma b_\gamma \text{sign}(b_\gamma) \right)
\]

Note that differentiating \(\sin(p_f d_0)\) twice with respect to \(d_0\) gives the desired factor of \(p_f^2\) multiplying the negative of \(\sin(p_f d_0)\). Thus (2.13) provides a closed form expression for all the face factor integrals that may appear in the Perez-Rovelli model. In practice, one can also evaluate these types of integrals using symbolic integration with software such as Maple or Mathematica.

3. **Finiteness Results**

The finiteness of the Lorentzian Barrett-Crane partition function with the Perez-Rovelli choice for face and edge amplitudes was an important finding established\(^7\) in [14] for arbitrary non-degenerate 2-complexes. In this section, we prove that the partition function is absolutely integrable for a more general class of 2-complexes. This proof both justifies the interchange of improper integrals required by face factoring and

\(^7\)See also [12] for further detail.
at the same time proves finiteness of the model for all non-degenerate triangulations and a certain class of degenerate triangulations. We characterize this class of degenerate triangulations and show how it is sensitive to certain changes in the face amplitude. Note that the proof we give in this section applies to closed two-complexes. The result also holds for 2-complexes with boundary; we describe in Appendix B how the proof of the closed case is changed to account for the presence of boundaries.

We start by introducing an absolute bound on the Perez-Rovelli partition function in its face-factored form (2.9) above.

\[
|\mathcal{Z}_\Delta| \leq \left( \prod_e \int_0^\infty dr_e \right) \left( \prod_v \left( \prod_{e \in \{e \neq e_0\} \in H^3} \int dx_{(e,v)} \right) \right) \left( \prod_e \frac{2}{\pi \sinh^2(r_e)} \right) \left( \prod_v \left( \prod_{f \in \{f \in N_f(v)\}} \frac{1}{\sinh(\phi_{N_f(v)})} \right) \right) \left( \prod_f \int_0^\infty \left| F(p_f, \phi^f_{i,j}, r_e) \right| dp_f \right).
\]

Introducing the inequality (A.5) proven in Appendix A:

\[
\prod_f \left( \int_0^\infty \left| F(p_f, \phi^f_{i,j}, r_e) \right| dp_f \right) \leq \prod_f \left( \frac{4}{3} \left( \prod_{i=1}^{\deg_V(f)} \phi^f_i \prod_{j=1}^{\deg_E(f)} \phi^f_j \prod_{e \in \{e \neq e_0\} \in H^3} \right)^{1 - \frac{3}{4 \deg_V(f) + \deg_E(f)}} \right).
\]

Hence, if the RHS of the bound

\[
|\mathcal{Z}_\Delta| \leq \left( \prod_e \int_0^\infty dr_e \right) \left( \prod_v \left( \prod_{e \in \{e \neq e_0\} \in H^3} \int dx_{(e,v)} \right) \right) \left( \prod_e \frac{2}{\pi \sinh^2(r_e)} \right) \left( \prod_v \left( \prod_{f \in \{f \in N_f(v)\}} \frac{1}{\sinh(\phi_{N_f(v)})} \right) \right) \left( \prod_f \left( \frac{4}{3} \left( \prod_{i=1}^{\deg_V(f)} \phi^f_i \prod_{j=1}^{\deg_E(f)} \phi^f_j \prod_{e \in \{e \neq e_0\} \in H^3} \right) \right)^{1 - \frac{3}{4 \deg_V(f) + \deg_E(f)}} \right)
\]

is finite then our face factored form of the partition function (and hence the original form) is finite.

To integrate out the \( r_e \) variables in our bound, for each \( e \) we collect together all the \( r_e \) dependence into a single term of the form

\[
\int_0^\infty \frac{r_e^{\alpha_e}}{\sinh^2(r_e)} dr_e,
\]

where \( \alpha_e > 0 \) is the overall power of \( r_e \) appearing in (3.3) due to contributions from the four faces that contain \( e \).

We claim that for \( \alpha_e > 1 \), quantities of this form are always finite. To see this we divide the integration as follows:

\[
\int_0^\infty \frac{r_e^{\alpha_e}}{\sinh^2(r_e)} dr_e = \int_0^{\frac{\ln 2}{2}} \frac{r_e^{\alpha_e}}{\sinh^2(r_e)} dr_e + \int_{\frac{\ln 2}{2}}^\infty \frac{r_e^{\alpha_e}}{\sinh^2(r_e)} dr_e.
\]

Consider the first term. Although the integrand is unbounded at \( r_e = 0 \), \( \sinh(r_e) \to r_e \) as \( r_e \to 0 \), so near zero the integrand behaves as \( r_e^{-(2 - \alpha_e)} \), which is integrable for \( \alpha_e > 1 \), but will otherwise diverge. Because \( \frac{\ln 2}{16} < \sinh^2(x) \) for \( x > \frac{\ln 2}{2} \), we can bound the second term as

\[
\int_{\frac{\ln 2}{2}}^\infty \frac{r_e^{\alpha_e}}{\sinh^2(r_e)} dr_e < 16 \int_{\frac{\ln 2}{2}}^\infty r_e^{\alpha_e} e^{-2r} dr_e,
\]

which is finite, as the polynomial \( r_e^{\alpha_e} \) is exponentially damped as \( r_e \to \infty \). Thus all factors involving \( r_e \) can be integrated to give finite factors if for each edge \( e \) the inequality \( \alpha_e > 1 \) holds. We will return below to a discussion of this \( \alpha_e > 1 \) condition and the constraints it places on which degenerate triangulations can be shown finite.
Next we need to check that integration over the hyperboloid variables always yields finite factors. Upon integrating out the \( r_e \) as described, we have reduced the bound (3.3) to:

\[
|Z_\Delta| \leq C \left( \prod_v \left( \prod_{e \in \Delta, e \neq 0} \int_{H^+_e} dx(e,v) \right) \prod_{f \ni v} \left( \prod_{e \ni v, e \neq 0} \frac{1}{\sinh(\phi_{N_f}(v))} \right) \prod_{f \ni v} \left( \prod_{e \ni v} \frac{\deg_V(f)}{\deg_E(f)} \right)^{1-\deg_Y(f) + \deg_E(f)} \right) \prod_{f \ni v} \left( \prod_{e \ni v} \frac{\deg_Y(f)}{\deg_E(f)} \right)^{1-\deg_Y(f) + \deg_E(f)},
\]

where the \( r_e \) integrations and all other constants have been absorbed into an overall constant \( C \).

From this expression, we see that to complete our proof we require

\[
\left( \prod_{e \ni v, e \neq 0} \int_{H^+_e} dx(e,v) \right) \prod_{f \ni v} \left( \prod_{e \ni v} \frac{\deg_Y(f)}{\deg_E(f)} \right)^{1-\deg_Y(f) + \deg_E(f)} \prod_{f \ni v} \left( \prod_{e \ni v} \frac{\deg_Y(f)}{\deg_E(f)} \right)^{1-\deg_Y(f) + \deg_E(f)}
\]

for each \( v \) to be finite. Note that these integrals are of the same form of the vertex amplitude (2), but with modified kernel functions. In recent work by Dan Christensen [12], a proof is given in which a class of integrals that includes those of the form (3.6) are shown to be finite. From this we can conclude that all hyperboloid integrations produce finite factors.

Therefore, absolute integrability for our bound on the face factored formulation depends only upon finiteness of the \( r_e \) variable integrations, for which we found above that \( \alpha_e > 1 \) is a necessary and sufficient condition.

We analyze the \( \alpha_e > 1 \) condition as follows. For all non-degenerate triangulations, \( \deg_V(f) + \deg_E(f) \geq 6 \), hence the power of \( r_e \) appearing in each face factor is at least \( 1 - \frac{3}{4} = \frac{1}{2} \). Since each \( r_e \) is associated to a tetrahedron, it will appear in 4 face factors. Hence \( \alpha_e \geq (4 \cdot \frac{1}{2}) = 2 > 1 \), and so we conclude that all non-degenerate triangulations are absolutely integrable. This justifies the face factoring method for these triangulations and gives an alternative finiteness proof to that of [14].

More generally, the condition on \( \alpha_e \) in terms of the vertex and edge degrees \( \deg_V(f) \) and \( \deg_E(f) \) can be given as \( \alpha_e = \sum_{f \ni v} \left( 1 - \frac{3}{\deg_V(f) + \deg_E(f)} \right) > 1 \). If we consider degenerate triangulations where \( \deg_V(f) \geq 2 \) and \( \deg_E(f) \geq 2 \), this condition is equivalent to the following statement:

**Theorem 3.1.** The Perez-Rovelli partition function \( Z_\Delta \) is absolutely integrable if for every \( e \in \Delta \), \( \deg_V(f) + \deg_E(f) > 4 \) for at least one \( f \) containing \( e \).

This is a generalization of the finiteness proof of [14], which was limited to non-degenerate triangulations. We shall see next that some degenerate triangulations satisfy this condition and some do not.

**Finiteness for Degenerate Triangulations.** In the case where a dual 2-complex contains a face \( f \) where \( \deg_V(f) \) or \( \deg_E(f) \) is less than 3, it is dual to a degenerate triangulation. Such a "triangulation" is not strictly speaking a triangulation in the usual sense of a simplicial complex, as it contains 4-simplices whose intersection contains more than one tetrahedra. While visualizing 4-dimensional geometry is challenging, we can draw lower-dimensional analogs of degenerate triangulations; two examples are shown in Figure 3.1.

Using Theorem 3.1, the 4-dimensional analog8 of Figure 3.1(a) can be shown finite. Observe that although the interior degenerate edges (indicated with thickened lines) have a face where \( \deg_E(f) = 2 \) and \( \deg_V(f) = 2 \), they also have 3 faces where \( \deg_E(f) > 2 \) and \( \deg_V(f) > 2 \), and so applying our criterion above we see that \( Z_\Delta \) is finite.

The four dimensional analog of Figure 3.1(b), in which two 4-simplices intersect along all five of their boundary tetrahedra, results in \( \deg_E(f) = 2 \) and \( \deg_V(f) = 2 \) for all faces inside the tetrahedron of the intersection; for edges dual to these tetrahedra \( \alpha_e = 4 \cdot \frac{1}{2} = 1 \) so integration of our face factor bound in the \( r_e \) variable diverges. Hence our proof fails to show that such 2-complexes are finite. We remark here that our Theorem 3.1 is a sufficient condition but we have not shown it to be necessary — our work does not rule out finiteness of the face factored form for this type of triangulation. However, even if the face factored

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8In the analog, two four-simplices are glued along two of their tetrahedra.
form can be shown not to be absolutely integrable, the original form of the partition function may still be finite but only conditionally convergent. Currently, numerical computations are underway by the author to explore some of these possibilities for 2-complexes arising from these degenerate triangulations.

A final observation relates to how possible changes in the face amplitude affect the integrability of degenerate triangulations. While the edge and vertex amplitudes we have used here are fairly well accepted and can be given some physical motivation, it may be of interest to consider alternatives to the $A_f(p_f) = 1$ choice. If we allow ourselves to consider a face amplitude of the form

$$A_f(p_f) = \frac{1}{p_f}$$

for $\gamma \geq 0$, we can construct a new bound (following the steps of Appendix A) in which the convergence condition becomes $\alpha_e = \sum f \geq e \left(1 - \frac{3-\gamma}{\deg_V(f) + \deg_E(f)}\right) > 1$. The analog of Theorem 3.1 is than the requirement that

$$\deg_V(f) + \deg_E(f) + \frac{4}{3} \gamma > 4$$

for some $f$ containing $e$.

It is interesting to observe that for $\gamma > 0$, this criterion is still met even if $\deg_V(f) = \deg_E(f) = 2$ for all the faces of a tetrahedron, the case which fails for the original amplitude $A_f(p_f) = 1$ corresponding to $\gamma = 0$. Hence, with the original model such 2-complexes give a bound that is divergent but at the "margin" of integrability — any non-zero $\gamma$ results in a model where all degenerate triangulations satisfying $\deg_E(f) \geq 2$ and $\deg_E(f) \geq 2$ are finite.

4. APPLICATIONS

Generalization to the Livine-Oriti Causal Barrett-Crane Model. In the Livine-Oriti causal model developed in [10], the kernel functions for the vertex amplitude $A_v$ are modified to take into account face orientations $\varepsilon^f_v \in \{-1, 1\}$ relative to a vertex $v$ that arise when imposing causality. The causal kernel replacing the original kernel (2.3) is given as

$$K_{p_f}(x, y) = \frac{e^{i \varepsilon^f_v \phi^f_{i_{v,j}}(x, y)}}{p_f \sinh(\phi^f_{i_{v,j}}(x, y))}.$$ (4.1)

From the form of this kernel, we can write the vertex face factors as

$$F^V_f(p_f, \phi^f_{i}) = \frac{1}{p_f^{\deg_V(f)}} \prod_{v \in f} e^{i \varepsilon^f_v \phi^f_{i_{v,j}}(v)}. $$ (4.2)

The integral $\int_0^\infty F^V_f(p_f, \phi^f_{i}) dp_f$ diverges in the general case, because the numerator approaches a constant as $p_f \to 0$ while the denominator approaches zero at least linearly; this gives a divergent integral for $\deg_V(f) \geq 1$, which applies to any triangulation. Hence, contributions to the overall face factor from face and edge amplitudes must be relied upon to give a finite face factored partition function.
Convergence criterion for the face amplitude: In [10], the edge amplitude remains the eye diagram function when passing to the causal case. Taking the causal vertex and edge amplitudes as fixed, we show in this section how the face amplitude has to be carefully tuned to avoid divergence of the face factored formulation. Let us consider the full causal face factor with the edge amplitude kernels and a face amplitude of the form \( A_f(p_f) = p_f^\beta \) for a real number \( \beta \):

\[
F_f(p_f, \phi_f^i, r_e) = \frac{1}{p_f^{\deg_V(f) + \deg_E(f)-\beta}} \left( \prod_{v \in f} e^{i p_f \phi_f^i(v)} \right) \left( \prod_{e \in f} \sin(p_f r_e) \right).
\]

Observing that the product of \( \deg_E(f) \) sine functions can cancel against the \( p_f^{\deg_E(f)} \) at \( p_f = 0 \), to avoid divergence we arrive at the condition

\[ \deg_V(f) - 3 < \beta. \]

In addition, if \( \beta \geq \deg_V(f) + \deg_E(f) - 2 \) then \( F_f(p_f, \phi_f^i, r_e) \) for general \( \phi \) fails to have a limit as \( p_f \to \infty \), and so its integral is not defined. Hence a necessary finiteness condition for a given \( \beta \) is

\[ \deg_V(f) - 1 < \beta + 2 < \deg_V(f) + \deg_E(f). \]  

For \( \beta = 0 \) as has been proposed, (4.4) becomes \( \deg_V(f) \leq 2 \) and \( \deg_V(f) + \deg_E(f) > 2 \); thus we see that the former condition restricts us to degenerate triangulations. For \( \beta \geq 1 \) however, degenerate triangulations are completely ruled out. Note here that in (4.4) the parameter \( \beta \) is bounded from both above and below.

We stress here that failure of convergence for the face factored partition function does not imply divergence of the original form; it may still be that the original partition function is conditionally integrable, precluding the identification of the two forms.

To summarize, we find that for the Oriti-Livine model there are some rather strong constraints on \( A_f(p_f) \) necessary for the existence of a finite face factored formulation. An interesting goal for future work is to determine whether the model with original integration ordering escapes these constraints by being conditionally convergent, or if it in fact is divergent in the same cases as are divergent in the face factored formulation.

**Numerical Applications.** Within the context of Monte Carlo methods applied directly to the partition function, the face factored formulation is advantageous in several respects. The integrations eliminated involved a highly oscillatory product of sine functions; the integration of these functions leads to much smoother functions of the remaining variables. In fact, it can be shown that the remaining functions are always non-negative [11]. The features of smoothing and positivity are both desirable in Monte Carlo methods for improving the accuracy achievable for a given number of samples. Currently, computations are underway to evaluate the partition function for sufficiently small 2-complexes where the number of integration dimensions is still not prohibitive.

Apart from direct Monte Carlo methods which have severe scaling limitations, the face factored form of the partition function may be amenable to statistical mechanical methods similar to those used in lattice quantum field theory computations. In such computations, one avoids calculating the partition function itself, but rather generates a sequence of samples in configuration space which has the same distribution as that determined by the partition function. In this way, expectation values of observables can be found to good approximation without calculating the entire partition function. The Metropolis algorithm is one such method that was successfully applied [13] in calculating expectation values for the Riemannian Barrett-Crane model.

Many statistical mechanical methods require that the partition function amplitude \(^9\) be non-negative for all configurations (i.e. the partition function amplitude is the exponential of some real-valued action). Clearly, before face factoring the Perez-Rovelli amplitude as a function of all the spin variables fails to have such positivity due to its definition in terms of sine functions. However, after the \( p_f \) variables have been integrated, it is rather remarkable that for closed spin foams the amplitude as a function of tetrahedral variables is always non-negative [11]. The potential for applying lattice field theory techniques to spin foam models using “dual” tetrahedral variables has been emphasized by Pfeiffer in [20], and may open the door to computing with much larger 2-complexes.

\(^9\)The integrand of the partition function \( Z_\Delta \) has a function of \( p_f \) variables in the original formulation and a function of \( H^+_3 \) and \( r_e \) variables associated to tetrahedra in the face factored formulation.
On a cautionary note, the physical interpretation of a model defined purely in terms of tetrahedral variables seems ambiguous at this time. Based upon connections to the Regge action in certain limits and general geometric considerations, it has been argued\footnote{For instance in \cite{ref10}, a dual variable framework is also developed in some detail for the Euclidean case in \cite{ref13}.} that the hyperboloidal variables can be interpreted as the time-like normals to the tetrahedra (which are purely spacelike by the Barrett-Crane construction). While this viewpoint is certainly promising, it is unclear to the author how these interpretations can be related back to the canonical approach.

5. Conclusions

In this paper we have shown how the Lorentzian Barrett-Crane spin foam model can be reformulated so that integration with respect to the spin variables $p_f$ is performed before all other integrations to yield exact expressions in terms of tetrahedral variables. Absolute integrability was proven for the face factored formulation, justifying the interchange of integration and establishing finiteness for a large class of triangulations. This class includes all non-degenerate triangulations and a certain class of degenerate triangulations, extending the finiteness proof of \cite{ref14}. In addition to the face-factoring transformation, an essential ingredient of this proof is a recent finiteness result due to Christensen \cite{ref12} for a large class of 10j-like integrals.

We have also described how the type of degenerate triangulations proven finite by our method depends upon the choice of face amplitude. Given this result, it may be worthwhile to investigate more thoroughly the physical significance of the degenerate triangulations. While the method can also be applied to the causal Livine-Oriti model, we found that given their choice of edge and vertex amplitude, there is a rather stringent constraint between the form of the face amplitude and the types of triangulations that allow a finite face factoring formulation.

An interesting feature of the Barrett-Crane model revealed by our work is that edge and vertex amplitudes contribute kernels to the face factors of the same form; this is no accident and can be traced back to the diagrams used to derive the model \cite{ref8}. Hence, an obvious question is whether or not generalizations or modifications of the Barrett-Crane model may still allow a face factoring formulation. Specifically, one could consider whether the Barrett-Crane type model proposed in \cite{ref19}, with mixed representations, may be susceptible to a face factoring treatment. Moreover, it may be interesting to investigate whether any analog of face factoring is possible for spin foam models not of the Barrett-Crane type.

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Appendix A. Product Bound for General Face Factor Integrations

In this appendix we find functions of $r_e$ and $\phi_i^f$ that bound from above the possible integrated face factors that arise in the Perez-Rovelli model. The bounding functions have the simple form of a product of $r_e$ and $\phi_i^f$ variables raised to a power that depends only upon the number of edges and vertices that the face contains. These bounding functions are used in Section 3 to show the absolute integrability of the face-factored formulation.

To begin, we recall that the most general face factor that can arise for the Perez-Rovelli model has the form

$$F_f(p_f, \phi_i^f, r_e) = \frac{\sin(p_f \phi_i^f) \cdots \sin(p_f \phi_{\deg(f)}) \sin(p_f r_e(f,1)) \cdots \sin(p_f r_e(f,\deg(f)))}{p_f^{\deg(f) + \deg_E(f) - 2}},$$

where the $r_e$ and $\phi_i^f$ are non-negative real numbers coming from edge and vertex amplitudes, respectively. To see this, note simply that in both the edge and vertex amplitude each sine function in the numerator is accompanied by a factor of $p_f$ in the denominator. The measure then contributes a factor of $p_f^2$, reducing the degree of the denominator by 2. Based on this form, we can immediately establish two types of bounds:

$$|F_f(p_f, \phi_i^f, r_e)| \leq \frac{\prod_{i=1}^{\deg(f)} \phi_i^f \prod_{j=1}^{\deg_E(f)} r_e(f,j)}{p_f^{\deg(f) + \deg_E(f) - 2}} = p_f^2.$$
and

\[ F_f(p_f, \phi_i^f, r_e) = \frac{\sin(p_f \phi_i^f) \cdots \sin(p_f \phi_{\text{deg}_V(f)}) \sin(p_f r_e(f, 1)) \cdots \sin(p_f r_e(f, \text{deg}_E(f)))}{p_f^{\text{deg}_V(f) + \text{deg}_E(f) - 2}} \]

\[ \leq \frac{1}{p_f^{\text{deg}_V(f) + \text{deg}_E(f) - 2}}. \]

The first inequality (A.2) is valid for \( p_f \geq 0 \), while the second bound (A.3) can only be used for \( p_f > 0 \). Integrating both sides of these inequalities we can form the following estimate for the integrated face factor:

\[ \int_0^\infty |F_f(p_f, \phi_i^f, r_e)| \, dp_f \leq \int_0^M \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right) p_f^2 \, dp_f + \int_M^\infty \frac{1}{p_f^{\text{deg}_V(f) + \text{deg}_E(f) - 2}} \]

\[ = \frac{1}{3} \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right) M^3 + \frac{1}{M^{\text{deg}_V(f) + \text{deg}_E(f) - 3}}, \]

for any \( M > 0 \). Since we are integrating \( p_f \) for a given set of the \( \phi_i^f \) and \( r_e \), we can make this \( M \) a function \( M(\phi_i^f, r_e) \) of these variables. We make the following choice:

\[ M(\phi_i^f, r_e) = \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right)^{-\frac{1}{\text{deg}_V(f) + \text{deg}_E(f)}} \]

which upon substituting into (A.4) gives:

\[ \int_0^\infty |F_f(p_f, \phi_i^f, r_e)| \, dp_f \leq \frac{1}{3} \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right) M^3 + \frac{1}{M^{\text{deg}_V(f) + \text{deg}_E(f) - 3}} \]

\[ = \frac{1}{3} \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right)^{1 - \frac{3}{\text{deg}_V(f) + \text{deg}_E(f)}} + \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right)^{1 - \frac{3}{\text{deg}_V(f) + \text{deg}_E(f)}} \]

\[ = \frac{4}{3} \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right)^{1 - \frac{3}{\text{deg}_V(f) + \text{deg}_E(f)}}. \]

Although we assume \( M > 0 \) for this derivation to be valid, if any of the \( \phi_i^f \) or \( r_e \) are zero then the original integrand (A.1) and hence the integral over \( p_f \) vanishes; hence the bound (A.5) is valid for any choice of \( \phi_i^f \) and \( r_e \). As it relates to the discussion at the end of Section 3, we mention here that for a face amplitude of the form \( A_f(p_f) = \frac{1}{p_f} \) the above proof can be generalized to arrive at a bound:

\[ \int_0^\infty |F_f(p_f, \phi_i^f, r_e)| \, dp_f \leq \left( 1 + \frac{1}{3 - \gamma} \right) \left( \prod_{i=1}^{\text{deg}_V(f)} \phi_i^f \prod_{j=1}^{\text{deg}_E(f)} r_e(f, j) \right)^{1 - \frac{3}{\text{deg}_V(f) + \text{deg}_E(f)}} \]

by following essentially the same steps.
APPENDIX B. FINITENESS FOR SPIN FOAMS WITH BOUNDARY

As a path integral theory of quantum gravity, a spin foam model defines a transition amplitude from one 3-geometry to another for a given “history” — a triangulated 4-manifold whose boundary is the union of the ingoing and outgoing 3-geometries.

In the case where the 2-complex is dual to a triangulated 4-manifold with boundary, a straightforward extension of our finiteness proof for the closed case can be given. The spin variables associated to each face in the boundary are taken as boundary data, so these are not integrated in defining the partition function. As well, for dual edges and faces coming from boundary tetrahedra and triangles, one needs to take the square root of the edge and face amplitudes so that the partition function multiplies when two histories are “glued” along a common boundary. We shall denote the sets of edges and faces dual to the boundary tetrahedra and triangles as $\partial E$ and $\partial F$, respectively. Rather than working directly with the square root of the edge amplitude $\Theta_4(p_1, \ldots, p_4)$, in this proof we use the following bound:

$$\sqrt{\Theta_4(p_1, \ldots, p_4)} < 1 + \Theta_4(p_1, \ldots, p_4).$$

Applying this bound to the partition function of a 2-complex $\Delta$ with boundary we have

$$Z_\Delta = \int_0^\infty \cdots \int_0^\infty \left( \prod_{e \in \partial E} \sqrt{\Theta_4(p_1, \ldots, p_4)} \right) \left( \prod_{e \notin \partial E} \Theta_4(p_1, \ldots, p_4) \right) \left( \prod_{v \in V} A_v(p_1, \ldots, p_{10}) \right) \prod_{f \notin \partial F} p_f^2 \, dp_f$$

$$< \int_0^\infty \cdots \int_0^\infty \left( \prod_{e \in \partial E} 1 + \Theta_4(p_1, \ldots, p_4) \right) \left( \prod_{e \notin \partial E} \Theta_4(p_1, \ldots, p_4) \right) \left( \prod_{v \in V} A_v(p_1, \ldots, p_{10}) \right) \prod_{f \notin \partial F} p_f^2 \, dp_f.$$

Considering the RHS of (B.2), we see that expanding the product of $1 + \Theta_4$ functions results in a sum of partition functions in which boundary tetrahedra are either trivial ($A_e = 1$) or have the eye diagram amplitude $A_e = \Theta_4(p_1, \ldots, p_4)$. We will show that any partition function of this form is finite.

First we handle the integrals with respect to the edge variables $r_e$. For the given 2-complex $\Delta$, let a partition function be chosen where the amplitude for any boundary edge is either $A_e = 1$ or $A_e = \Theta_4(p_1, \ldots, p_4)$, and all other amplitudes agree with the Perez-Rovelli model. We can extend the face factoring method to a partition functions of this form as follows. We begin by defining a boundary face factor $F^\partial_f$ associated to any face on the boundary as:

$$F^\partial_f(p_f, \phi_f^j, r_e) = \frac{\sin(p_f \phi_1^j) \sin(p_f \phi_2^j) \cdots \sin(p_f \phi_{\deg_v(f)}^j) \sin(p_f r_1) \sin(p_f r_2) \cdots \sin(p_f r_{\deg_v(f)})}{p_f^{\deg_v(f) + \deg_v^2(f)}}$$

where the function $\deg_v^2(f)$ counts the boundary edges of type $\Theta_4$ contained in $f$. Observe that since the $p_f$ are fixed on the boundary rather than integrated over, the measure factor of $p_f^2$ does not enter into the definition of $F^\partial_f(p_f, \phi_f^j, r_e)$; the functions $F^\partial_f(p_f, \phi_f^j, r_e)$ and not their integrals with respect to $p_f$ multiply into the partition function directly. The analog of the inequality (A.2) for faces in the boundary can be given as

$$\left| F^\partial_f(p_f, \phi_f^j, r_e) \right| \leq \frac{p_f \phi_1^j p_f \phi_2^j \cdots p_f \phi_{\deg_v(f)}^j p_f r_1 p_f r_2 \cdots p_f r_{\deg_v(f)}}{p_f^{\deg_v(f) + \deg_v^2(f)}} = \left( \prod_{i=1}^{\deg_v(f)} \phi_i^j \prod_{j=1}^{\deg_v^2(f)} r_{e(f,j)} \right).$$

In the open spin foam case, the inequality (3.2) is modified to include the boundary face factors:

$$\prod_{f \notin \partial \Delta} \int_0^\infty \left| F_f(p_f, \phi_f^j, r_e) \right| \, dp_f \prod_{f \in \partial \Delta} \left| F^\partial_f(p_f, \phi_f^j, r_e) \right|$$

$$\leq \prod_{f \notin \partial \Delta} \left( \frac{3}{2} \prod_{i=1}^{\deg_v(f)} \phi_i^j \prod_{j=1}^{\deg_v^2(f)} r_{e(f,j)} \right) \prod_{f \in \partial \Delta} \left( \frac{\deg_v(f) \deg_v^2(f)}{2^{\deg_v(f) + \deg_v^2(f)}} \right).$$
For tetrahedra in the interior, finiteness goes through as in the closed case. For tetrahedra on the boundary, there is an integration over $r_e$ for any edge $e$ of type $\Theta_4$; for such edges one collects all the $r_e$ dependence to give factors of the form:

$$\int_0^\infty \frac{r_e^4}{\sinh^2(\pi r_e)} \, dr_e.$$  

As this integral is clearly finite, integrating with respect to the $r_e$ for all $e$ in the interior and boundary always yields finite factors. For boundary tetrahedra where $\mathcal{A}_e = 1$, there is no $r_e$ variable to be integrated so finiteness of the bound is not affected.

The final step is to show that the result of integrating the partition function bound over the hyperboloid variables is always finite. For a 4-simplex which has all of its faces on the interior, we recall from the proof of the closed case that integration with respect to hyperboloid variables factors into products of the form

$$\left( \prod_{e \ni v, v \neq e_0} \frac{\int_{H_+^3} d\delta_r(e, v)}{\sinh(\delta^e)} \right) \prod_{f \ni v} \frac{\left( \frac{\delta^f}{\delta^e} \right)^{1 - \deg_f(f)/deg_E(f)}}{\sinh(\delta^e)},$$

which each of which can shown finite by invoking the result of [12]. For 4-simplices that have one or more faces in the boundary, one instead has

$$\left( \prod_{e \ni v, v \neq e_0} \int_{H_+^3} d\delta_r(e, v) \right) \prod_{f \ni v} \frac{\left( \frac{\delta^f}{\delta^e} \right)^{1 - \deg_f(f)/deg_E(f)}}{\sinh(\delta^e)} \prod_{f \ni v} \frac{\delta^f}{\sinh(\delta^f)},$$

which again are shown to be finite in [12] — thus we have checked all hyperboloid integrations lead to finite factors.

As our bound (B.2) is a finite sum of partition functions which are finite, we conclude that the Perez-Rovelli Lorentzian partition function is finite for all 2-complexes dual to triangulated 4-manifolds with boundary, on the condition that the tetrahedron faces meet the same criteria with respect to $\deg_v(f)$ and $\deg_E(f)$ as were found for the closed case.

REFERENCES


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