CM-P00061691

Ref.TH.2307-CERN

Archives

A SOLUBLE FIELD THEORETICAL MODEL FOR STRONG INTERACTIONS
INSPIRED BY STATISTICAL BOOTSTRAP

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ABSTRACT

We construct a fully relativistic, four dimensional field theoretical model for strong interactions in which the interaction is described by a bootstrap integral equation instead of a Lagrangian. One of the most prominent features of strong interaction, recursive clustering, has been built in by means of a new algebraic operation, called "cluster". We develop the new, non-associative, cluster algebra and use it to formulate the bootstrap equation, which can be solved explicitly yielding the exponential hadron mass spectrum known from statistical bootstrap and dual models. From the solution of the bootstrap equation for "fireball operators" a non-trivial, unitary S matrix can be calculated in closed form, in which fireballs and clusters do no longer appear. While having much of the flavour of strong interactions, the model suffers from yielding (in its present form) only isotropic final momentum distributions with, however, the typical exponential momentum cut-off known from statistical bootstrap.

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1. INTRODUCTION

In this paper we develop a field theoretical model for strong interactions, of which at least one variant, inspired by the statistical bootstrap model¹⁻⁸⁾ (SBM), has a closed solution in terms of recursion relations. While we consider here only this particular version, the framework allows for generalizations.

The version we present here is unrealistic in several respects and cannot be applied in its present form to real collision processes; it contains, however, a good part of the properties of strong interactions to be non-trivial. The fate of being unrealistic is shared with all soluble models in field theory. Such models are unrealistic in different directions and/or at different degrees. We believe that our model is again sufficiently different from all others to be worth studying: we work in four-dimensional space time, fully relativistic and with a (non-Lagrangian) interaction which generates the full average (exponential) hadronic mass spectrum; it is unrealistic insofar as starting with a single real scalar field with physical mass m_0 (we call it pion), all collision and production processes lead to isotropic final distributions. We hope that the generalizations which the framework permits can be narrowed down to some realistic model. The freedom to generalize is similar to that in Lagrangian field theory, where the problem is to find the right Lagrangian (supposing solutions exist), while here it might be to find a suitable integral equation.

The central ideas of this paper come from SBM: the first is that strong interactions are strong enough to "cluster" particles together into resonances and fireballs and the second is, that this takes place in a recursive way: "fireballs consist of fireballs". This requires that clusters or resonances or fireballs (roughly all the same thing bearing different names) should be treated as genuine new particles, each having its own field, while at the same time also being explicitly composed of others. This requires a new algebraic operation for combining field operators, called "cluster", which supplements the usual addition and multiplication. The "cluster algebra" will turn out to be non-associative with respect to the cluster operation.

The plan of the paper is:

- Section 2 : Description of SBM as far as needed here;
- Section 3 : Cluster algebra;
- Section 4 : Bootstrap equation for fireball operators; solution;
- Section 5 : Physical interpretation; S-matrix;
- Section 6 : Concluding remarks.

Our units are $\hbar = c = k$ (Boltzmann constant) = 1; metric: $p^2 = p_0^2 - \vec{p}^2$; we use curly brackets to indicate sets: $\{a\}$ is the set of all elements denoted by a (with further specifications). The notation $A = B$ means: A is defined by this equation.

2. CONTENT AND MOTIVATION OF SBM

Like all statistical models, SBM starts from the observation that for a complete statistical description of a system enclosed in a volume V one only needs the knowledge of its density of states, $\sigma(p^2, V)$ or, equivalently, the Laplace transform of σ , called partition function $Z(\beta^2, V)$; β is a time-like four vector and $\sqrt{\beta^2}$ is the inverse of the temperature T .

The density of states of a strongly interacting gas enclosed in V can be written⁷⁾

$$B\sigma(p^2, B) = B\delta_0(p^2 - m_0^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n B\rho(p_i^2) d^4p_i \quad (2.1)$$

$$\delta_0(p^2 - m_0^2) := \theta(p_0) \delta(p^2 - m_0^2)$$

Comments: We work in invariant momentum space rather than invariant phase space⁹⁾; this is only a question of convenience; B has dimension (length²) and replaces the volume.

- : The basic particle is a "pion" of mass m_0 .
- : $\rho(p_i^2)$ is the hadronic mass spectrum. Each of the momenta p_i can run through the whole mass spectrum. Since the sum over the momentum space integrals goes from 2 to ∞ , this sum represents the density of states of an interacting gas, where all that part of interaction, which can be represented by creation and annihilation of particles (resonances) of arbitrary mass, is included, provided $\rho(p_i^2)$ is the full spectrum of pionic resonances plus background.
- : The term $B\delta_0(p^2 - m_0^2)$ deals with the case that the system considered contains just one single pion.
- : The unknown mass spectrum $\rho(p^2)$ is now found by arguing that if we make B small enough to ensure that each particle in the box is in the range of interaction of every other one, that is: if we give B the

magnitude B_0 corresponding to the size of a hadron, then the density $\sigma(p^2, B_0)$ is the density of states of a hadron and therefore $\sigma(p^2, B_0) = \rho(p^2)$.

For a more detailed argumentation, see Refs. 10), 11), 8). By putting $\sigma(p^2, B_0) = \rho(p^2)$ we obtain the statistical bootstrap equation (SBE), which says "fireballs consist of fireballs"; namely:

$$B_0 \rho(p^2) = B_0 \delta_0(p^2 - m_0^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n B_0 \rho(p_i^2) d^4 p_i$$

$$B_0 = (4\pi m_0^2)^{-1} \quad (2.2)$$

The sum is finite for finite p^2 :

$$n_{\max} = \text{int}(\sqrt{p^2}/m_0) \quad (2.3)$$

where $\text{int}(x) = \text{integral part of } x$.

SBE can be solved in two ways, one suitable for small p^2 , the other for asymptotic values $p^2 \rightarrow \infty$ [Refs. 3), 5), 6), 7)]. For finite p^2 the closed solution first given by Yellin⁵⁾, is obtained by the ansatz

$$\rho(p^2) = \sum_{n=1}^{\infty} g_n \Omega_n(p^2, B_0)$$

$$\Omega_n(p^2, B_0) = \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) d^4 p_i \quad (2.4)$$

We call $\Omega_n(p^2, B_0)$ the n pion momentum space. Inserting (2.4) into (2.2) and using the recursion relation

$$\Omega_n(p^2, B_0) = \int \delta^4(p - \sum_{k=1}^K p_k) \prod_{k=1}^K \Omega_{n_k}(p_k^2, B_0) d^4 p_k$$

valid for any set $\{n_k : \sum_{k=1}^K n_k = n\}$ (2.5)

we find a recursion relation for the "Yellin coefficients"

$$g_n = \sum_{k=2}^n \frac{1}{k!} \sum_{n_i=1}^{n-1} \delta(n - \sum_{i=1}^k n_i) \prod_{i=1}^k g_{n_i} \quad (2.6)$$

$$g_1 = 1$$

Note the structural similarity between (2.6) and (2.2). As the g_n can be computed from (2.6), Eq. (2.4) represents a closed, exact solution [equal to the result of the $n_{\max}^{\text{th}}$ iteration of Eq. (2.2)].

For asymptotic $p^2 \rightarrow \infty$ it is more convenient to Laplace transform Eq. (2.2)^{1),3),6),7),12)}

$$Z(\beta^2) = \int e^{-\beta p} \rho(p^2) B_0 d^4 p$$

$$t(\beta^2) = \int e^{-\beta p} \delta_0(p^2 - m_0^2) B_0 d^4 p = 2\pi B_0 m_0 T K_1\left(\frac{m_0}{T}\right) \quad (2.7)$$

$$\beta = (\beta_0, \vec{\beta}) \quad \text{time-like "inverse temperature four vector"}$$

$$\beta^2 = \frac{1}{T^2} ; \quad \beta_0 > 0$$

to obtain the Laplace transformed SBE

$$t = 2Z - \exp(Z) + 1 \quad (2.8)$$

As t is a known (monotonously increasing) function of the temperature $T = (\beta^2)^{-\frac{1}{2}}$, this equation may be solved: $Z(T) = Z(t(T))$ and then $\rho(p^2)$ can be found by inverse Laplace transforming Z . By drawing the graph of Eq. (2.8) one sees immediately that real Z exist only for $T \leq T_0$, the limiting temperature, defined by the equation

$$2\pi B_0 m_0 T_0 K_1\left(\frac{m_0}{T_0}\right) = \ln 4 - 1 \quad (2.9)$$

$$T_0 \approx m_0$$

The mass spectrum grows asymptotically like^{3)-8),12)}

$$\rho(m^2) = \frac{C_0}{m^3} \exp\left(\frac{m}{T_0}\right) \cdot \left(1 + C_1 \frac{T_0}{m} + \dots\right) \quad (2.10)$$

where the constants $C_0, C_1 \dots$ can all be calculated from (2.8) and (2.9). This behaviour agrees reasonably well with the actual hadron mass spectrum (up to ~ 2 GeV) and the numerical value of T_0 yields the observed mean transverse momenta in high-energy collisions.

The Yellin solution (2.4) and the "Laplace solution" are related; $Z(t)$ is the generating function for the Yellin coefficients g_n [one can derive Eq. (2.8) from the recursion relation (2.6) and vice versa; see Appendix]:

$$Z(t) = \sum_{n=1}^{\infty} g_n t^n \quad (2.11)$$

with $t_0 = \ln 4 - 1$ as radius of convergence [Eq. (2.9)].

Recently, G.C. Rossi^{*)} discovered that the Yellin coefficients g_n , the generating equation (2.8) and its solution (2.11) as well as an asymptotic representation ($n \rightarrow \infty$) for the g_n [Eq. (C.10) of Ref. 7)] are already more than a hundred years old. They turned up in "Vier kombinatorische Probleme" solved in 1870 by E. Schröder^{13,14)}. In his "problem IV" he asked, in how many ways one could place brackets in an algebraic expression consisting of n elements, if it is permitted to exchange the sequence of elements in order to combine elements in complete arbitrariness. It is highly amusing to read his visualization of the bracketing process by a spatial model: (free translation). "Given n elements, e.g., material points freely movable in space, one divides the set of n elements

*) private communication.

arbitrarily into subsets; some possibly containing only one element, some with two, three ... elements; each set of more than one element is to be surrounded by a closed surface, which, together with its content, shall be considered as a new element on an equal footing with all the others. This new set of (obviously less than n) elements is submitted to the same procedure as long as one likes or until it ends because the next step would result in one single element. We wish to find the number of different complexions which can be generated by this procedure".

This is a picturesque description of the "fireballs-consist-of-fireballs" structure of SBM. Schröder finds that the number of possible complexions is $N(n) = n!g_n$, where g_n are our above Yellin coefficients. In the course of his investigation he derives our equations (2.8) and (2.11) as well as Eq. (C.10) of Ref. 7). This is Schröder's "problem IV"; his "problem III" is what today is called "two body truncated SBM"^{15),7)}. In deriving his results, he makes use of the "reciprocity principle" which states in our present language that "the decay structure of a fireball (i.e., the ensemble of all processes leading from a fireball to its ultimate pion distribution) is the exact counterpart of its composition"⁷⁾ and terminates his paper with the prophetic words: "Considerably more difficult would be the solution of the above problems in the case where among the original elements several are identical"; indeed, SBM with Bose and Fermi statistics turned out to be more complicated to solve^{16),17),18)} (except in the grand canonical language¹⁾).

The important observation is that Schröder got essential results of SBM without knowing anything about particle physics and relativistic invariant momentum space; therefore his results depend only on the way clustering is defined, not on the kinematics imposed on his n elements. This becomes very clear if we rewrite the Yellin solution of SBM, Eq. (2.4) as follows:

$$\varrho(p^2) = \sum_{n=1}^{\infty} (n!g_n) \cdot \frac{1}{n!} \Omega_n(p^2, B_0) \quad (2.12)$$

Now the first factor $n!g_n$ is Schröder's number of possible ways to cluster n elements, while $\Omega_n/n!$ is the number of quantum states of n (identical^{*)}) particles enclosed in B_0 . Thus the Yellin solution and Schröder's result teach us that SBM is the combination of two independent problems: cluster counting and momentum space counting. The ability of strong interactions to form clusters, and clusters of clusters, is represented by the first problem: cluster counting

*) but with Boltzmann statistics.

alone leads already to the main results [Eqs. (2.6), (2.8), (2.11)]; using phase space as kinematical constraint is merely a supplementary condition chosen for convenience [but with good reason^{10),11)}], which allows for a simple connection between the maximum temperature T_0 and the singularity of the series (2.11).

We conclude that the most important task is to provide a mathematical representation of the cluster property, while the exact form of the kinematical constraints (expressed by the structure of the bootstrap equation) is of secondary importance, giving us the freedom to try several models.

3. CLUSTER FIELD ALGEBRA

The main idea is suggested by SBE (2.2), where the density of states at p^2 is a convolution of densities of states at $p_i^2 < p^2$. What we would like is a similar equation for the creation operator of a field associated with a cluster: the creation operator at p should be composed of creation operators of clusters at other p_i ($\sum p_i = p$). As the product of several creation operators is not a creation operator (the commutator is an operator), an equation of the form (2.2) cannot hold for creation operators, unless we introduce a new algebraic operation similar to, but different from multiplication, called "cluster", which guarantees that a cluster of several creation operators is again a creation operator. In that case an equation of the type of (2.2), with "product" replaced by "cluster", would make sense and would represent the ability of strong interactions to build up clusters recursively.

3.1 The input field

We chose as input a neutral scalar ("pion") field (obeying the Klein-Gordon equation)

$$\varphi(x) = \int d^4p \delta_0(p^2 - m_0^2) [a(p)e^{-ipx} + a^\dagger(p)e^{ipx}] \quad (3.1)$$

with the usual commutation relations

$$[a(p), a^\dagger(q)] = \frac{2p_0}{(2\pi)^3} \delta^3(\vec{p} - \vec{q}) \quad , \quad \text{etc.} \quad (3.2)$$

In order to make SBE dimensionally consistent, the objects clustered together must have dimension $[B_0 \rho(p^2)] = (\text{mass}^{-4})$, while $[a(p)] = (\text{mass}^{-1})$. We therefore introduce operators with the required dimension by putting

$$a_o(p) : = \frac{\pi \sqrt{2}}{m_o} \delta_o(p^2 - m_o^2) a(p) \quad (3.3)$$

$$\varphi_o(x) : = \frac{\pi \sqrt{2}}{m_o} \varphi(x)$$

so that

$$\begin{aligned} \varphi_o(x) &= \int d^4p \left[\bar{a}_o(p) e^{-ipx} + a_o^+(p) e^{ipx} \right] \\ &= \varphi_o^{(+)}(x) + \varphi_o^{(-)}(x) \end{aligned} \quad (3.4)$$

where $\varphi_o^{(\pm)}$ denote positive and negative frequency parts respectively.

$$[a_o(p), a_o^+(q)] = B_o \delta_o(p^2 - m_o^2) \delta^4(p - q) \quad (3.5)$$

$$B_o := (4\pi m_o^2)^{-1} \quad [\text{same as in SBM, (2.2)}]$$

As usual, the vacuum is defined by

$$a_o(p) |0\rangle = \langle 0| a_o^+(p) = 0 \quad (3.6)$$

Nothing would prevent us from introducing fermions (quarks) or charged bosons as input fields.

3.2 The cluster algebra

Let be

$$\begin{aligned} \{a_0\} &:= \text{set of all } a_0(p) \\ \mathcal{A}_0 &:= \mathcal{A}_{(a_0)} = \text{algebra generated from } \{a_0\} \\ \mathcal{A}_0^+ &:= \mathcal{A}_{(a_0^+)} = \text{algebra generated from } \{a_0^+\} \end{aligned} \quad (3.7)$$

where the algebraic operations are as usual.

We shall distinguish between

$$\begin{aligned} \text{i) elementary operators [like } a_0(p)] \\ \text{ii) aggregates [like } a(p)a(q)] \\ \text{iii) mixed aggregates [like } a(p)a^+(q)] . \end{aligned} \quad (3.8)$$

In the following we shall introduce the cluster operation and thereby enlarge the algebras $\mathcal{A}_0$ and $\mathcal{A}_0^+$; the new algebras will contain new elementary operators and aggregates, not contained in $\mathcal{A}_0$ and $\mathcal{A}_0^+$ respectively. For these more general operators we use the notation

$$\begin{aligned} A_\ell; B; C; Q_k; \dots & \text{ for elementary operators (one letter)} \\ A_i A_k; BC; Q_k Q_\ell Q_m; \dots & \text{ for aggregates (two or more letters).} \end{aligned} \quad (3.9)$$

The cluster operation is written:

$$A = \overline{A_i A_k} \neq A_i A_k \quad (3.10)$$

For the cluster operation we postulate:

- I) Cluster is defined only for aggregates
- II) The result of cluster is an elementary operator

$$\text{III) } \overline{A_i A_k}^+ = \overline{A_k^+ A_i^+}$$

$$\text{IV) } \overline{A_i A_k} |0\rangle = \langle 0 | \overline{A_i A_k}^+ = 0$$

$$\text{V) } [\overline{A_i A_k}, \overline{A_j A_l}^+] = \langle 0 | [A_i A_k, A_l^+ A_j^+] | 0 \rangle \quad (3.11)$$

$$[\overline{A_i A_k}, a_o^+] = 0$$

$$[\overline{A_i A_k}, a_o] = 0$$

Comments: (I) says that elementary operators and mixed aggregates cannot be clustered;
 : (II) says that cluster cannot be applied twice to the same aggregate;
 : (V) is the defining property: the commutator of two clusters is a C number (in contradistinction to the commutator of two aggregates, which is an operator); therefore clusters are creation or annihilation operators associated with new fields.

In the case of more general input fields (bosons and/or fermions) the above commutators would have to be replaced by anticommutators for clusters with odd fermion content. We now call "Cluster Algebra" the algebras

$$\mathcal{O}_c := \mathcal{O}_c(a_o) \quad \begin{array}{l} \text{= algebra generated from } \{a_o\} \\ \text{by the usual operations and cluster} \end{array} \quad (3.12)$$

$$\mathcal{O}_c^+ := \mathcal{O}_c(a_o^+) \quad \text{similarly}$$

and "cluster field algebra" the algebra

$$\mathcal{A}_{CF} := \mathcal{A}(\mathcal{A}_C, \mathcal{A}_C^+) \quad (3.13)$$

that is, the algebra generated from $\{\mathcal{A}_C\} \cup \{\mathcal{A}_C^+\}$ by the usual operations (no cluster).

3.3 Evaluation of commutators

Let us evaluate $[A, B]$ where $A, B \in \{\mathcal{A}_{CF}\}$.

a) if A and B are clusters: $A = \overline{A_1 A_2}$, $B = \overline{B_1 B_2}$

$$[A, B] = \langle 0 | [A_1 A_2, B_1 B_2] | 0 \rangle \quad (3.14)$$

b) if A and/or B are aggregates or mixed aggregates, use the ordinary rules for expanding commutators until each remaining commutator either fulfils condition (a) then proceed, or has one of the following forms:

$$\begin{aligned} [\overline{A_\ell A_k}, a_c^+] &= 0 \\ [a_c, \overline{B_\ell B_m}^+] &= 0 \end{aligned} \quad (3.15)$$

$$[a_c(p), a_c^+(q)] = B_c \delta_c(p^2 - m_c^2) \delta^4(p - q)$$

whereby the end is reached.

3.4 Tree-graph representation

Let us use the shorthand notation $a_1 = a_0(p_1)$, etc. Clusters and/or aggregates may conveniently be represented by tree graphs (giving an intuitive decay picture):

while sequential decay would appear as

For n elements $N(n) = n!g_n$ different tree topologies are available;
these may be labelled by some number

$$\alpha(n) = 1, 2, 3, \dots, N(n) \quad (3.18)$$

3.5 Cluster states

Let Q stay for any cluster. One shows, using the postulates, that

$$[Q_1, Q_2^+] = 0 \quad \text{for clusters of different topology} \quad (3.19)$$

$$\overline{\lambda Q_1 (Q_2 + \mu Q_3)} = \lambda \overline{Q_1 Q_2} + \lambda \mu \overline{Q_1 Q_3} \quad (3.20)$$

$$\overline{\overline{Q_1 Q_2 Q_3}} \neq \overline{Q_1 \overline{Q_2 Q_3}} \quad (\text{non associativity}) . \quad (3.21)$$

Furthermore, as $[Q, Q^+] = C$ number, Q^+ is a creation operator and Q an annihilation operator. Thus cluster states can be defined by applying cluster creation operators to the vacuum; Eq. (3.19) ensures orthogonality for topologically different cluster states. As all clusters can be written in terms of a_0 operators, clusters may be labelled uniquely by their momenta and their tree topology $\alpha(n) = 1, 2, 3 \dots N(n)$

$$Q(\alpha(n); p_1 \dots p_n) := \overline{\overline{\overline{\overline{\overline{a_0(p_1)} a_0(p_2)} \dots a_0(p_n)}}} \} \alpha(n) \quad (3.22)$$

Correspondingly we have states $|\{\alpha, p\}_n\rangle = |\alpha(n), p_1 \dots p_n\rangle$ and by repeated application $|\{\alpha, p\}_{n_1}, \{\alpha, p\}_{n_2} \dots\rangle$, whereby the whole cluster Fock space H_c is spanned. For finite normalization volume this space has a countable basis. If $A \in \{Q_{CF}\}$, then its matrix elements are (apart from normalization factors)

$$\begin{aligned} & \langle \{\alpha p\}_{m_1} \dots \{\alpha p\}_{m_e} | A | \{\alpha' p'\}_{m'_1} \dots \{\alpha' p'\}_{m'_j} \rangle \\ &= \langle 0 | [Q(\alpha p)_{m_1} \dots Q(\alpha p)_{m_e}, A Q^+(\alpha' p')_{m'_1} \dots Q^+(\alpha' p')_{m'_j}] | 0 \rangle \quad (3.23) \end{aligned}$$

which can be evaluated according to the rules 3.3. Thus all operators $\in \{\mathcal{O}_{CF}\}$ have an explicit matrix representation, and no such operator leads out of the cluster Fock space H_c , which, of course, contains the Fock space H_0 generated by a_0 as a subspace.

3.6 Clusters with wave functions

Consider the example

$$A_f := \frac{1}{2!} \int f(p_1 p_2) a_0(p_1) a_0(p_2) d^4 p_1 d^4 p_2$$

$$C_f := \frac{1}{2!} \int f(p_1 p_2) \overline{a_0(p_1) a_0(p_2)} d^4 p_1 d^4 p_2$$
(3.24)

where C_f is a cluster while A_f is not. We have

$$[C_f, C_g^+] = \langle 0 | [A_f, A_g^+] | 0 \rangle =$$

$$= B_0^2 \int f(p_1 p_2) g^*(p_1 p_2) \delta_0(p_1^2 - m_0^2) \delta_0(p_2^2 - m_0^2) d^4 p_1 d^4 p_2$$
(3.25)

This is zero if the two functions f, g are orthogonal with respect to the two-particle momentum space measure. Thus clusters may have an internal structure independently of their topological structure:

$$\text{cluster structure} = (\text{topological structure}) \otimes (\text{wave function structure})$$
(3.26)

and each may cause orthogonality independently of the other.

3.7 Space-time clusters

The cluster operator C_f has a space-time representation

$$C_f^{(+)} = \frac{1}{2!} \int \tilde{f}(x, y) \overline{\varphi_0^{(+)}(x) \varphi_0^{(+)}(y)} d^4 x d^4 y$$
(3.27)

where $\tilde{f}(x,y)$ is the Fourier transform of $f(p,q)$ and $\varphi_0^{(+)}$ the positive frequency part of φ_0 . Local clusters are generated from momentum space wave functions containing a $\delta^4(p - \sum p_i)$, e.g.,

$$C_f(p) = \frac{1}{2!} \int \delta^4(p-p_1-p_2) \overline{f(p_1, p_2)} a_0(p_1) a_0(p_2) d^4 p_1 d^4 p_2 \quad (3.28)$$

$$C_f^{(+)}(x) = \frac{1}{2!} \int \tilde{f}(x_1, x_2) \overline{\varphi_0^{(+)}(x_1-x)} \varphi_0^{(+)}(x_2-x) d^4 x_1 d^4 x_2$$

The generalization to n pion clusters is obvious.

Now consider the expression

$$\varphi_0(x) \varphi_0(y) = \varphi_0^{(+)}(x) \varphi_0^{(+)}(y) + \varphi_0^{(-)}(x) \varphi_0^{(-)}(y) + \varphi_0^{(+)}(x) \varphi_0^{(-)}(y) + \varphi_0^{(-)}(x) \varphi_0^{(+)}(y) \quad (3.29)$$

If we try to cluster this equation, the last two terms would represent clusters of mixed (creation and annihilation) aggregates, which are forbidden by the first postulate of (3.11); had we not forbidden it, the cluster of $\varphi_0(x) \varphi_0(y)$ would contain, apart from a pure creation part and a pure annihilation part, also a mixed part. As the cluster operation should define a new independent and quasi-elementary field, we must require its unique decomposability into a pure creation and a pure annihilation part. The obvious general definition is then, in accordance with postulate (I):

$$\overline{\prod_{i=1}^n \varphi_0(x_i)} = \overline{\prod_{i=1}^n \varphi_0^{(+)}(x_i)} + \overline{\prod_{i=1}^n \varphi_0^{(-)}(x_i)} \quad (3.30)$$

$$\overline{f}(\varphi_0) = \overline{f}(\varphi_0^{(+)}) + \overline{f}(\varphi_0^{(-)})$$

where $\bar{\Pi}$ and $\bar{f}$ indicate clustering; f is supposed to be expressed as a power series.

3.8 Decay

In real physics we will have to do with unstable clusters; thus we shall need a decay-operation written $D(\dots)$ and having the following properties ($A, B \in \mathcal{Q}_{CF}$; λ any complex number):

$$D(\lambda A + B) = \lambda D(A) + D(B)$$

$$D(AB) = D(A)D(B)$$

$$D(\overline{AB}) = AB \tag{3.31}$$

$$D(a_0) = a_0$$

$$D^n; = \quad n \text{ fold applied } D$$

Each D applied to a cluster or aggregate of clusters will remove the uppermost bar (or bars). Repeated application will eventually lead to an aggregate of a_0 's. Consider the aggregate $a_0(p_1)a_0(p_2) \dots a_0(p_n) \equiv a_1 a_2 \dots a_n$. From it $N(n) = n!g_n$ different clusters can be constructed; for each of them the decay operation D , repeated sufficiently often, gives back $a_1 a_2 \dots a_n$ and each application of D will correspond to one decay step along the tree graph, which therefore may be called decay tree. Complete decay is denoted by D^∞ .

Note that D is not the inverse of cluster, but it is always well defined. Equation (3.31) is a formalization of the "principle of reciprocity"⁷⁾.

4. BOOTSTRAP

The cluster algebra developed in Section 3 is so far a set of rules which formalizes the combinatorial problem of counting states in SBM, but does not constitute a dynamical description. As a simple non-trivial realization of dynamics we shall consider an integral equation analogous to SBE (Eq. 2.2) and take it as representing our "equation of motion". It describes the internal structure of (a particular kind of) clusters, but does not say how incoming pions become clusters and how clusters become finally pions again; thus clustering and decay are put in by hand, while between these two operations clusters are formally stable particles. SBM describes only isotropically decaying clusters and so will this simple model.

4.1 Operator bootstrap equation

As discussed at the end of Section 2, SBM combines the most general topological structure (sum over all tree topologies) with the most primitive wave function structure, namely just invariant momentum space; this will then be also the case for the model represented by the following "equation of motion" for the "fireball operator" $C(p)$:

$$C(p) = a_0(p) + \sum_{n=2}^{\infty} \frac{\gamma_n}{\sqrt{n!}} \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n C(p_i) d^4 p_i \quad (4.1)$$

$$\prod_{i=1}^n C_i = \overline{C_1 C_2 \dots C_n}$$

If $\gamma_n = 1/\sqrt{n!}$, the analogy to SBE is perfect; here we allow the γ_n to take arbitrary complex values (including 0). Further generalization might be achieved by introducing "wave functions" $F_n(p_1 \dots p_n, p)$ under the integral.

Equation (4.1) defines a particular cluster field operator; the cluster is composed of other clusters of the same structure: it is a fireball. As the input field has finite mass m_0 , it is obvious that (4.1) can be solved exactly by a finite number $n_{\max}(p^2)$ of iterations [see Eq. (2.3)]; with increasing $n_{\max}$, these solutions become rapidly unmanageable.

4.2 Commutation relations for $C(p)$

We can compute $[C(p), C^\dagger(q)]$ without solving Eq. (4.1). As we know that $C(p)$ is a cluster, the commutator must be a C number; it must contain a $\delta^4(p-q)$ and it is an invariant function. Hence it must have the form

$$[C(p), C^\dagger(q)] = \langle 0 | [C(p), C^\dagger(q)] | 0 \rangle = f(p^2) \delta^4(p-q) \quad (4.2)$$

In order to find $f(p^2)$, we replace $C(p)$ and $C^\dagger(q)$ by the right-hand sides of their bootstrap equations and find

$$[C(p), C^+(q)] = B_0 \delta_0(p^2 - m_0^2) \delta^4(p - q) +$$

(4.3)

$$+ \sum_{n=2}^{\infty} \frac{\gamma_n \gamma_k^*}{\sqrt{n! k!}} \int \delta^4(p - \sum_{i=1}^n p_i) \delta^4(q - \sum_{j=1}^k q_j) \langle 0 | \left[\prod_{i=1}^n C(p_i), \prod_{j=1}^k C^+(q_j) \right] | 0 \rangle \times d^4 p_i d^4 q_j$$

Using the ordinary rules for evaluating commutators of products we arrive at

$$\begin{aligned} \langle 0 | \left[\prod_{i=1}^n C(p_i), \prod_{j=1}^k C^+(q_j) \right] | 0 \rangle &= \\ &= \delta_{nk} \sum_{P(i')} \prod_{i=1}^k \langle 0 | [C(p_i), C^+(q_{i'})] | 0 \rangle \\ &= \delta_{nk} \sum_{P(i')} \prod_{i=1}^k f(p_i^2) \delta^4(p_i - q_{i'}) \end{aligned} \quad (4.4)$$

where $P(i')$ means: sum over all permutations of $i' = \{1 \dots k\}$, and where Eq. (4.2) was used in the last step. Inserting (4.4) into (4.3), carrying out the integrals and comparing the result to Eq. (4.2), yield an integral equation for $f(p^2)$:

$$\begin{aligned} f(p^2) &= B_0 \delta_0(p^2 - m_0^2) + \\ &+ \sum_{n=2}^{\infty} |\gamma_n|^2 \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{i=1}^n f(p_i^2) d^4 p_i \end{aligned} \quad (4.5)$$

which is the SBE (2.2) except for $|\gamma_n|^2$ replacing $1/n!$. As SBE has a unique solution, it follows that

$$f(p^2) = B_0 \rho(p^2) \quad (4.6)$$

where $\rho(p^2)$ is the mass spectrum of SBM (with $|\gamma_n|^2$ instead of $1/n!$). Hence

$$[C(p), C^+(q)] = B_0 \rho(p^2) \delta^4(p-q) \quad (4.7)$$

Thus the mass spectrum belonging to $C(p)$ is the hadron mass spectrum of SBM; it grows exponentially, making the model non-local.

We have an explicit proof that it is impossible to achieve that the mass spectrum of Eq. (4.7) obeys SBM without introducing cluster in Eq. (4.1). Replacing $\bar{\Pi}$ by Π will not only make $[C(p), C^+(q)]$ an operator, but even its vacuum expectation value would no longer have a mass spectrum obeying SBM; instead it would correspond to the density of states of non-interacting pions. This, by the way, was how we realized that a cluster operation was needed.

4.3 Yellin solution

As the solution of Eq. (4.1) is a sum over topologies of clusters of clusters, the Yellin solution will be more complicated than in SBM. We make the ansatz

$$C(p) = \sum_{n=1}^{\infty} \int d^4(p - \sum_{i=1}^n p_i) g_n(p_1 \dots p_n) \prod_{i=1}^n d^4 p_i \quad (4.8)$$

Inserting this into Eq. (4.1) yields a recurrence relation very similar to that of the Yellin coefficients [Eq. (2.6)]

$$g_1(p) = a_0(p) \quad (4.9)$$

$$g_n(p_1 \dots p_n) = \sum_{k=2}^n \frac{\gamma_k}{\sqrt{k!}} \sum_{n_i=1}^{n-1} \delta(m - \sum_{i=1}^k n_i) \prod_{i=1}^k g_{n_i}(p_1^{(i)} \dots p_{n_i}^{(i)})$$

$$\{p_1 \dots p_n\} \equiv \{p_1^{(1)} \dots p_{n_1}^{(1)}, p_1^{(2)} \dots p_{n_2}^{(2)}, \dots, p_{n_k}^{(k)}\}$$

in this order.

Note that the g_n are cluster operators.

From this all $g_n(p_1 \dots p_n)$ can be calculated. Actually, this is possible only for low n , because the recursion becomes very complicated. Fortunately, we shall see later that for the calculation of physical transition amplitudes we do not need the explicit g_n . Therefore we content ourselves here with the statement that Eq. (4.9) represents (in principle) the complete solution of the bootstrap equation (4.1).

Just as an example we give

$$\begin{aligned}
 g_1(p) &= a_0(p) \\
 g_2(p_1 p_2) &= \frac{\gamma_2}{\sqrt{2!}} \overline{a_0(p_1) a_0(p_2)} \\
 g_3(p_1 p_2 p_3) &= \frac{\gamma_2^2}{2!} \left[\overline{a_0(p_1) a_0(p_2) a_0(p_3)} + \overline{a_0(p_1) a_0(p_3) a_0(p_2)} + \right. \\
 &\quad \left. + \overline{a_0(p_2) a_0(p_1) a_0(p_3)} \right] + \\
 &\quad + \frac{\gamma_3}{\sqrt{3!}} \overline{a_0(p_1) a_0(p_2) a_0(p_3)}
 \end{aligned} \tag{4.10}$$

It is recommended to the reader to evaluate $[g_3(p_1 p_2 p_3), g_3^+(q_1 q_2 q_3)]$ and to calculate g_4, g_5, g_6 .

4.4 Commutators in the Yellin solution

We calculate once more the commutator $[C(p), C^+(q)]$, this time using the Yellin solution (4.8). We are led to calculate the commutator $[g_n(p_1 \dots p_n), g_\ell^+(q_1 \dots q_\ell)]$. In spite of the complicated structure of the g 's, this is not so difficult, because we know that the g commutator is a c number and that it reduces to a sum over products of a_0 commutators. It must therefore necessarily be of the form

$$\begin{aligned}
 &[g_n(p_1 \dots p_n), g_\ell^+(q_1 \dots q_\ell)] = \\
 &= \delta_{n\ell} \sum_{P(i')} w_n(P) \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) \delta^4(p_i - q_{i'})
 \end{aligned} \tag{4.11}$$

where $w_n(P)$ is the weight of the contribution of the permutation P of the indices $\{i'\}$. With this result we obtain

$$\begin{aligned} [C(p), C^+(q)] = & \\ = \sum_{n=1}^{\infty} \sum_{P(i')} w_n(P) \int & \delta^4(p - \sum_{i=1}^n p_i) \delta^4(q - \sum_{i=1}^n q_i) \times \\ & \times \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) \delta^4(p_i - q_{i'}) d^4 p_i d^4 q_{i'} \end{aligned} \quad (4.12)$$

The integral is invariant under the permutations and equals

$$\delta^4(p - q) \Omega_n(p^2, B_0)$$

Hence

$$[C(p), C^+(q)] = \delta^4(p - q) \sum_{n=1}^{\infty} w_n \Omega_n(p^2, B_0) \quad (4.13)$$

$$w_n : = \sum_{P_n} w_n(P)$$

We compare this result with Eq. (4.7) and see that

$$B_0 \rho(p^2) = \sum_{n=1}^{\infty} w_n \Omega_n(p^2, B_0) \quad (4.14)$$

Therefore the w_n , which are the sums of the weights $w_n(P)$, are the Yellin coefficients of SBM and obey the recurrence relation

$$w_1 = 1$$

$$w_n = \sum_{k=2}^{\infty} |g_k|^2 \sum_{n_i=1}^{n-1} \delta(n - \sum_{i=1}^k n_i) \prod_{i=1}^k w_{n_i} \quad (4.15)$$

The commutator (4.11) of g_n consists of $n!$ contributing terms and therefore $w_n/n!$ represents the average weight per term.

4.5 Normalized cluster states

For the present model we shall only need pion and fireball states, obtained by applying $a_0^+(p)$ and/or $C^+(p)$ any number of times to the vacuum. From the commutation relations (3.5) and (4.7) it follows that properly normalized states are obtained by putting

$$|p_1 p_2 \dots p_n\rangle = \frac{1}{\sqrt{n!}} \prod_{i=1}^n \frac{C^+(p_i)}{\sqrt{B_0 \rho(p_i^2)}} |0\rangle \quad (4.16)$$

We remember that $\rho(p^2)$ has a single line $B_0 \delta_0(p^2 - m_0^2)$ at m_0 and is continuous for $p^2 \geq 4m_0^2$; if $p_1^2 = m_0^2$, the momentum is carried by a pion; if $p_1^2 \geq 4m_0^2$, it is carried by a fireball and if any p_i^2 meets neither of these two conditions, the state does not exist. Hence (4.16) is the most general state consisting of pions and/or fireballs. It follows

$$\begin{aligned} & \langle p_1 p_2 \dots p_n | q_1 q_2 \dots q_\ell \rangle \\ &= \delta_{ne} \frac{1}{n!} \sum_{P(i')} \prod_{i=1}^n \delta^4(p_i - q_{i'}) \end{aligned} \quad (4.17)$$

where $P(i')$ means sum over all permutations of $\{i'\}$. Hence

$$\int \langle p_1 \dots p_n | q_1 \dots q_\ell \rangle \prod_{i=1}^{\ell} d^4 q_i = \delta_{ne} \quad (4.18)$$

We may define fireball states with fixed mass M by introducing a wave function $F(p)$:

$$|F_M\rangle := \int F(p) |p\rangle d^4p \quad (4.19)$$

From (4.17) we have

$$\langle F_M | F_{M'} \rangle = \int F_M^*(p) F_{M'}(p) d^4p \quad (4.20)$$

and with the normalization

$$\int F_M^*(p) F_{M'}(p) d^4p = \delta_0(M^2 - M'^2) \quad (4.21)$$

we have

$$\langle F_M | F_{M'} \rangle = \delta_0(M^2 - M'^2) \quad (4.22)$$

5. PHYSICAL INTERPRETATION

We have now constructed a world consisting of pions and (stable) fireballs -- but so far these objects do not interact with each other. Our fireballs are a kind of Leibniz monads: their highly complicated internal structure is a picture of the world of strong interactions, but each of them possesses it independently of all others: "they have no windows". This is certainly an incomplete picture; what we actually observe in scattering processes long before and long after the mutual interaction, are free pions and not fireballs. Hence we need some operations which carry us from pions to clusters and then back to pions. This is not

provided by our "equation of motion", the bootstrap equation, which only deals with the structure of our monadic fireballs. Thus the two operations: clustering incoming pions and decay of fireballs, must be put in by hand.

In other words: clustering the incoming pions is equivalent to switching on their mutual (not self-) interaction; letting clusters decay is equivalent to switching it off again. Between these two operations the bootstrap equation cares for the rest. As, however, the bootstrap equation is supposed to describe strong interactions and since clustering and decay must be considered also as part of the interaction, these two operations which will be "put in by hand" must be consistent with the structure imposed by the bootstrap equation and cannot be defined arbitrarily. These questions will be discussed in Sections 5.2 and 5.3.

5.1 Normalized decay states

The situation just described reminds one somehow of the measuring process in quantum mechanics: in between two measurements the system follows its equation of motion, while the event of making a measurement falls outside this frame. As the measuring process leaves us (at least in textbooks) with an eigenstate of the operator corresponding to it, and with a certain amplitude, the new state has to be renormalized to one, if we wish to use it to calculate the possible results of further measurements. Similarly here: applying D^∞ to some cluster state will destroy its normalization, and we shall see why and how. The normalization of cluster states followed from the commutator $[C(p), C^+(q)]$. In order to find the normalization of decay states we consider the commutator of two C operators after complete decay (see Section 3.8).

From the Yellin solution (4.8) we have

$$D^\infty C(p) = \sum_{n=1}^{\infty} \int \delta^4(p - \sum_{i=1}^n p_i) D^\infty g_n(p_1 \dots p_n) \prod_{i=1}^n d^4 p_i \quad (5.1)$$

$g_n(p_1 \dots p_n)$ is a "weighed" sum (with complex "weights") over all possible tree topologies which can be built up from $a_0(p_1) \dots a_0(p_n)$; under D^∞ each of these topologies will end up with the same product of a_0 's. Hence $D^\infty g_n(p_1 \dots p_n)$ is a sum of identical products $\prod a_0$ with different coefficients [see Eq. (4.10)]

$$D^\infty g_n(p_1 \dots p_n) = \beta_n \prod_{i=1}^n a_0(p_i) \quad (5.2)$$

where β_n , the sum of all coefficients, is a complex number which, due to the recursion relation for the g 's (4.9) obeys itself the recursion relation for Yellin coefficients

$$\begin{aligned} \beta_1 &= 1 \\ \beta_n &= \sum_{k=2}^n \frac{\delta_k}{\sqrt{k!}} \sum_{n_i=1}^{n-1} \delta(n - \sum_{i=1}^k n_i) \prod_{i=1}^k \beta_{n_i} \end{aligned} \quad (5.3)$$

Herewith the β_n are known; we defer to the Appendix some remarks about more efficient recurrence relations which may in certain cases replace (5.3). Note that we have now two sets of numbers, w_n [Eqs. (4.11) to (4.15)] and β_n , both being Yellin coefficients; the w_n with respect to $|\gamma_k|^2$, hence real, and the β_n with respect to $\gamma_k/\sqrt{k!}$ hence in general complex. As the w_n generate the mass spectrum, the β_n cannot:

$$B_0 \rho(p^2) = \sum_{n=1}^{\infty} w_n \Omega_n(p^2, B_0) \neq \sum_{n=1}^{\infty} \beta_n \Omega_n(p^2, B_0) \quad (5.4)$$

except if $\gamma_k = 1/\sqrt{k!}$ when both are equal.

Given the β_n from (5.3) we find with (5.1) after a short calculation

$$\langle 0 | [D^\infty C(p), D^\infty C^\dagger(q)] | 0 \rangle = \delta^4(p-q) B_0 R(p^2) \quad (5.5)$$

$$B_0 R(p^2) = \sum_{n=1}^{\infty} n! |\beta_n|^2 \Omega_n(p^2, B_0) \neq B_0 \rho(p^2)$$

In order to give to the decay states the same norm as to the corresponding cluster states, we define normalized decay states

$$\begin{aligned} |D(p)\rangle &= \sqrt{N(p^2)} D^\infty |p\rangle \\ &= \sqrt{N(p^2)} D^\infty \frac{C^+(p)}{\sqrt{B_0 \rho(p^2)}} |0\rangle \end{aligned}$$

$$N(p^2) := \rho(p^2) / R(p^2) < 1 \quad (5.6)$$

$$\langle D(p) | D(q) \rangle = \delta^4(p-q)$$

With the help of Eqs. (5.1) and (5.2) we have

$$\begin{aligned} |D(p)\rangle &= \\ &= \frac{1}{\sqrt{B_0 R(p^2)}} \sum_{n=1}^{\infty} \beta_n^* \int d^4(p - \sum p_i) \prod_{i=1}^n a_0^+(p_i) |0\rangle d^4 p_i \end{aligned} \quad (5.7)$$

which gives the last line of Eq. (5.6).

That $N(p^2) < 1$ follows from

$$N(p^2) = \frac{\int \langle 0 | [C(p), C^+(q)] | 0 \rangle d^4 q}{\int \langle 0 | [D^\infty C(p), D^\infty C^+(q)] | 0 \rangle d^4 q} \quad (5.8)$$

where we imagine the Yellin solution inserted; in the denominator all decay trees of $C(p)$ combine with all decay trees of $C^+(q)$, while in the numerator contributions come only from combinations of pairwise equal decay trees. This is why and how the normalization is destroyed by D^∞ ; Eq. (5.6) repairs this.

Decay states of superpositions of fireball states with a wave function (Section 4.5) are defined as

$$|D(F_M) \rangle = \int F_M(p) |D(p) \rangle d^4 p \quad (5.9)$$

so that with (4.21)

$$\langle D(F_M) | D(F_{M'}) \rangle = \delta_0(M^2 - M'^2) \quad (5.10)$$

5.2 Decay amplitudes

Given a fireball with four momentum q , what is the decay amplitude to a final specified n pion state?

In order to make decay consistent with the bootstrap equation, we invoke the "principle of reciprocity"⁷⁾ [which has been criticized as not rigorously true¹⁹⁾, but in our formalism is hardly avoidable]; it says that the tree graph of a given cluster is its decay tree; the decay operation (3.31) is the formal expression of this reciprocity. Therefore, in the present model, the decay amplitude to a given final state is just the scalar product of that state with $|D(q)\rangle$. As all our states are now properly normalized, we have

$$\langle p_1 p_2 \dots p_n | D(q) \rangle = \beta_n^* \sqrt{\frac{n!}{B_0 R(q^2)}} \delta^4(q - \sum_{i=1}^n p_i) \prod_{i=1}^n \sqrt{B_0 \delta_0(p_i^2 - m_0^2)} \quad (5.11)$$

where (4.16), (5.7) and the commutation relations for the a_0 's have been used.

From this we find for the transition probability

$$\begin{aligned} & |\langle p_1 p_2 \dots p_n | D(q) \rangle|^2 = \\ &= \delta^4(q - \sum_{i=1}^n p_i) \frac{n! |\beta_n|^2}{B_0 R(q^2)} \delta^4(q - \sum_{i=1}^n p_i) \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) \end{aligned} \quad (5.12)$$

As usual we interpret one of the two $\delta^4(q - \sum p_i)$ as the total infinite space-time volume τ , so that the remaining part of the right-hand side of Eq. (5.12) is the transition probability per space-time unit.

The transition probability (per space-time unit) into just n pions is then

$$\frac{1}{\tau} \int |\langle p_1 \dots p_n | D(q) \rangle|^2 \prod_{i=1}^n d^4 p_i = \frac{n! |\beta_n|^2}{B_0 R(q^2)} \Omega_n(q^2, B_0) \quad (5.13)$$

which corresponds to the SBM result [Ref. 7), Eq. (2.20)] without being equal to it; the difference is due to the interference terms missing in SBM. The sum of (5.13) over n gives 1 on account of Eq. (5.5).

The infinite factor $\delta^4(q - \sum p_i) = \delta^4(0) = \tau$ does not appear, if we use n -pion wave packets $F_n(p_1 \dots p_n)$ instead of plane waves:

$$|F_n\rangle = \frac{1}{\sqrt{n!}} \int F_n(p_1 \dots p_n) \prod_{i=1}^n a_0^\dagger(p_i) |0\rangle d^4 p_i \quad (5.14)$$

where the wave function is normalized as

$$\int |F_n(p_1 \dots p_n)|^2 \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) d^4 p_i = 1 \quad (5.15)$$

$$\langle F_m | F_n \rangle = \delta_{mn}$$

so that

$$\langle F_n | D(q) \rangle = \beta_n^* \sqrt{\frac{n!}{B_0 R(q^2)}} \Omega_n(q^2, B_0) \cdot \langle F_n^* \rangle$$

$$\langle F_n^* \rangle = \frac{1}{\Omega_n(q^2, B_0)} \int F_n^*(p_1 \dots p_n) \delta^4(q - \sum_{i=1}^n p_i) \prod_{i=1}^n B_0 \delta_0(p_i^2 - m_0^2) d^4 p_i \quad (5.16)$$

that is: $\langle F_n \rangle$ is the n pion wave function averaged over n pion invariant momentum space. Now

$$\sum_{n=1}^{\infty} |\langle F_n | D(q) \rangle|^2 = \quad (5.17)$$

$$= \sum_{n=1}^{\infty} n! |\beta_n|^2 \Omega_n(q^2, B_0) |\langle F_n \rangle|^2 \Omega_n(q^2, B_0) \cdot \frac{1}{B_0 R(q^2)}$$

is finite for any finite q^2 and arbitrary (normalized) F_n . In what follows, we use again plane waves, for convenience.

5.3 The S operator

How do incoming pions go over into fireballs? Insofar as no explicit interaction between clusters and pions has been introduced (it is implicitly present in the internal structure of the fireballs), the question is open (cf., the beginning of Section 5).

The simplest prescription, consistent with the bootstrap equation and not needing an additional assumption, is that the amplitude for the transition of the initial pion configuration into a fireball shall be equal to the time-reversed amplitude for the decay of the same fireball (with its three momentum reversed) into the same pion configuration (all three momenta reversed), namely

$$A_{init. \rightarrow fireball} = \langle D(q) | init. \rangle \quad (5.18)$$

Interpretation:

$|D(q)\rangle$, the final decay configuration of the fireball state $|q\rangle$, reflects still the internal structure of the latter: it remembers that the bootstrap equation was at work. Therefore the above amplitude expresses, how well the initial pion configuration would match the fireball state $|q\rangle$, if the pions had been subject to the interaction described by the bootstrap equation, i.e., if all possible ways of clustering had been allowed: the amplitude is equal to one, if the initial pion configuration could have been itself the result of a fireball decay.

With the amplitude (5.18) we can define an operator Ω which transforms, with the correct amplitude, any given initial pion configuration into the corresponding intermediate fireball state:

$$\begin{aligned}\Omega &: = \int d^4q \, |q\rangle \langle D(q)| \\ \Omega^+ &: = \int d^4q \, |D(q)\rangle \langle q|\end{aligned}\tag{5.19}$$

the Hermitean conjugate, Ω^+ , transforms from any fireball state into its decay state.

Equation (5.19) is the formal expression for "putting in cluster and decay by hand in a manner consistent with the bootstrap equation", as we required at the beginning of this section.

With these assumptions the transition amplitude from an initial to a final pion configuration is given by

$$\begin{aligned}A_{init. \rightarrow fin.} &= \\ &= \int d^4q d^4q' \langle fin. | D(q) \rangle \langle q | q' \rangle \langle D(q') | init. \rangle \\ &= \int d^4q \langle fin. | D(q) \rangle \langle D(q) | init. \rangle \\ &= \langle fin. | \Omega^+ \Omega | init. \rangle\end{aligned}\tag{5.20}$$

From Eq. (5.11) the transition amplitude from an ℓ pion state into an n pion state is then

$$\int \langle p_1 \dots p_n | D(q) \rangle \langle D(q) | q_1 \dots q_e \rangle d^4 q =$$

$$= \delta^4 \left(\sum_{i=1}^n p_i - \sum_{j=1}^e q_j \right) \frac{\beta_m^* \beta_e \sqrt{m!e!}}{B_0 R(p^2)} \prod_{i=1}^m \sqrt{B_0 \delta_0(p_i^2 - m_0^2)} \prod_{j=1}^e \sqrt{B_0 \delta_0(q_j^2 - m_0^2)}$$

(5.21)

$$p: = \sum_{i=1}^n p_i = \sum_{j=1}^e q_j$$

The operator

$$F: = \int d^4 q |D(q)\rangle \langle D(q)| = \Omega^+ \Omega$$

(5.22)

corresponds to switching on strong interactions between the l incoming pions, leading through an interaction regime described by the bootstrap equation and switching off the interaction again:

$$\text{transition amplitude} = \langle p_1 \dots p_n | F | q_1 \dots q_e \rangle$$

(5.23)

This amplitude contains explicitly the factor $\delta^4(\sum p_i - \sum q_j)$. We can write also an explicit expression for F by using Eq. (5.7)

$$F = \sum_{m,e=1}^{\infty} \beta_m \beta_e^* \times$$

$$\times \int \frac{d^4 q}{B_0 R(q^2)} \delta^4 \left(q - \sum_{i=1}^m q_i \right) \delta^4 \left(\sum_{j=1}^e p_j - \sum_{i=1}^m q_i \right) \left[\prod_{j=1}^e a_0^+(p_j) |0\rangle \langle 0| \prod_{i=1}^m a_0(q_i) \right] d^4 p_j d^4 q_i$$

(5.24)

Equation (5.21) is the matrix representation of this operator in the basis of the pion Fock space H_0 . Nothing more is needed here; the cluster states as well as the manual operations cluster and decay have disappeared.

From Eq. (5.22) we see that, because of (5.6),

$$F = F^+ = F^2 \quad (5.25)$$

So that F is a projection; it is the projection onto the subspace consisting of states which can result from the decay of one single fireball (while Ω^+ is the projector onto the single fireball subspace). As our fireball is isotropic by construction [Eq. (4.1)], the final particle distribution is isotropic. There are now various ways to construct a unitary S operator out of F . For instance, we may consider

$$S = \frac{1 + iaF}{1 - iaF} = 1 + \frac{2ia}{1 - ia} F ; a \text{ real} \neq 0 \quad (5.26)$$

where the denominator has been formally expanded and $F^n = F$ been used. Another definition might be

$$S = e^{i\alpha F} = 1 + F \cdot (e^{i\alpha} - 1) ; \alpha \text{ real} \neq 0, 2\pi \quad (5.27)$$

Both definitions, (5.26) and (5.27), are, however, equivalent; given a in (5.26) we can find α (and vice versa) such that both S operators are the same.

We do not have a unique definition; in view of the large freedom which the model offers anyway, this might not be serious.

As to be expected, the total cross-section (two pions in)

$$\sigma_{\text{tot}}(q^2) \sim B_0 \frac{|\beta_2|^2}{R(q^2)} \quad (5.28)$$

vanishes for $q^2 = s \Rightarrow \infty$, because the amplitude for forming a single isotropic intermediate fireball goes exponentially to zero. While the model is unrealistic, the S matrix is non-trivial and the result (5.28) is consistent with the physics of the model.

5.4 Space-time fields

The present version of the model has a complete, non-perturbative solution with a calculable unitary S matrix. In this framework we would in principle not need a space-time description of fireball fields, nor would we need propagators and Feynman graphs.

Unfortunately this version is not realistic. It does not allow for non-isotropic fireballs and/or off-mass shell momenta as we would need in order to produce multi-fireball states out of two incoming pions. A possible solution to this might be to introduce in addition to the bootstrap equation some further interaction (e.g., Yukawa) between fireballs and/or pions. Such an interaction would not contradict the bootstrap equation but supplement it; and it might have a very small coupling constant, because the main part of strong interactions is already taken care of by the bootstrap equation (4.1). If, however, we would introduce such an additional interaction, space-time fields and propagators would be needed. We discuss them in the following.

We define $C_n(p)$ as the n^{th} term of the Yellin solution (4.8):

$$C(p) = \sum_{n=1}^{\infty} C_n(p) \quad (5.29)$$

$$C_n(p) : = \int \delta^4(p - \sum_{i=1}^n p_i) g_n(p_1 \dots p_n) \prod_{i=1}^n d^4 p_i$$

We introduce the Fourier transform of $C_n(p)$

$$\Gamma_n^{(+)}(x) : = \int C_n(p) e^{-ipx} d^4 p$$

$$\Gamma_n^{(-)}(x) : = \int C_n^+(p) e^{ipx} d^4 p$$

$$\Gamma_n(x) : = \Gamma_n^{(+)}(x) + \Gamma_n^{(-)}(x) \quad (5.30)$$

where $\Gamma_n(x)$ is defined in accordance with Section 3.7. Due to the momentum δ^4 function, $\Gamma_n(x)$ is local. For any finite p the sum (5.29) has a finite number of terms: $n_{\max} = \text{Int}(\sqrt{p^2}/m_0)$; but in the Fourier transform $p \rightarrow \infty$, so that also $n \rightarrow \infty$. We define

$$\Gamma(x) := \lim_{N \rightarrow \infty} \sum_{n=1}^N \Gamma_n(x) \quad (5.31)$$

This operator is no longer local; it obeys the formal equation

$$\begin{aligned} \Gamma(x) &= \int C(p) e^{-ipx} d^4p + \int C^+(p) e^{ipx} d^4p \\ \Gamma(x) &= \Gamma^{(+)}(x) + \Gamma^{(-)}(x) \end{aligned} \quad (5.32)$$

We carry out the formal Fourier transform on the bootstrap equation (4.1) with the result

$$\begin{aligned} \Gamma(x) &= \varphi_0(x) + \overline{G}(\Gamma(x)) - \gamma_1 \Gamma(x) - \gamma_0 \mathbb{1} \\ G(z) &:= \sum_{n=0}^{\infty} \frac{\gamma_n}{\sqrt{n!}} z^n \end{aligned} \quad (5.33)$$

This equation holds in the sense of Eqs. (3.30).

Consider the choice $\gamma_n = 1\sqrt{n!}$, then $G(z) = e^z$ and the above equation takes the form of Eq. (2.8):

$$\varphi_0(x) = 2\Gamma(x) - \overline{\exp}(\Gamma(x)) + \mathbb{1} \quad (5.34)$$

For this choice the Feynman propagator of a fireball field is

$$\langle 0 | T(\Gamma(x) \Gamma(0)) | 0 \rangle = i B_0 (2\pi)^3 \int dm^2 \varphi(m^2) \Delta_F(x, m^2) \quad (5.35)$$

For spacelike distances $x^2 < 0$ this becomes

$$\begin{aligned} \langle 0 | T(\Gamma(x) \Gamma(0)) | 0 \rangle_{x^2 < 0} &= \frac{2\pi B_0}{\sqrt{-x^2}} \int dm^2 \varphi(m^2) K_1(m \sqrt{-x^2}) = \\ &= Z \left(\frac{1}{\sqrt{-x^2}} \right) \end{aligned} \quad (5.36)$$

where Z is the SBM partition function and where $1/\sqrt{-x^2}$ corresponds to the temperature T .

From SBM we know that $Z(T)$ has a square root singularity at $T = T_0$ [see Section 2 and Refs. 3), 5), 6), 7)]

$$Z(T) = Z_0 \pm \text{const} \sqrt{T_0 - T} + \dots \quad (5.37)$$

$$T_0 \approx m_0 \quad \text{defined by Eq. (2.9);} \quad Z_0 = \ln 2$$

A corresponding singularity appears in the propagator; we put

$$R : = \sqrt{-x^2} \quad (5.38)$$

$$t : = x_0$$

$$\mu : = \frac{1}{\sqrt{-x^2}}$$

Then (still for the special choice $\gamma_n = 1/\sqrt{n!}$)

$$\begin{aligned} \langle 0 | T(\Gamma(x) \Gamma(0)) | 0 \rangle_{x^2 < 0} &= Z(\mu) = \\ &= Z_0 \pm \text{const} \sqrt{\mu_0 - \mu} + \dots \end{aligned} \quad (5.39)$$

Thus the fireball Feynman propagator has a square root singularity on the hyperboloid

$$-x^2 = r^2 - t^2 = \frac{1}{\mu_0^2} \approx \frac{1}{m_0^2} \quad (5.40)$$

and is double-valued; for $r^2 - t^2 > 1/\mu_0^2$ it decreases exponentially. For $r^2 - t^2 < 1/\mu_0^2$ the integral representation (5.36) does not exist.

The singularity at $r^2 - t^2 = 1/\mu_0^2 \approx 1/m_0^2$, which does not appear in conventional field theories, is a most important feature of this model and a manifestation of its inherent non-locality: $1/\mu_0$ is a sort of smallest length equal to the inverse of the limiting temperature T_0 .

The singularities just discussed are not particular to the choice $\gamma_n = 1/\sqrt{n!}$; any (non-trivial) choice of γ_n will lead to a square root singularity, because Z is the Laplace transform of $\rho(m^2)$ and thus depends only on the $|\gamma_n|^2$ [Eqs. (4.14), (4.15)]; we know, however, from SBM that any choice of $|\gamma_n|^2$ with at least one of them $\neq 0$ will lead to a square root singularity at $\mu_0 \approx m_0$.

6. CONCLUDING REMARKS

We have constructed a soluble, fully relativistic four-dimensional field theoretical model for strong interactions with two main results: the full exponential (average) mass spectrum of strong interactions and a non-trivial S matrix. The model is non-local as a consequence of its exponential mass spectrum. It possesses a kind of smallest length $r_0 = 1/T_0$ where T_0 is the limiting temperature belonging to the Laplace transform (partition function) $Z(T)$ of the (exponential) mass spectrum $\rho(m^2)$ of the model.

All this has been achieved by translating the basic philosophy of SBM into field theory; the price to pay was introducing cluster algebra. Cluster states, however, as well as cluster operators, disappear completely from the final result for the S matrix, which in its operator form is expressed by pion operators a_0 only. Computing the S matrix or better: any definite matrix element, involves only recursive computation of a finite number of coefficients β_n and n pion invariant momentum space integrals $\Omega_n [n_{\max} = \text{Int}(\sqrt{s}/m_0), \text{ if } \sqrt{s} = E_{\text{c.m.}}]$.

In contrast to conventional field theories, the pion Fock space H_0 is complete only with respect to the incoming and outgoing pion states, while the description of the interaction needs the full cluster Fock space H_c . Symbolically

$$\begin{array}{c}
 \begin{array}{c} H_0 \\ \text{final} \end{array} \leftarrow \left\{ \begin{array}{c} \xleftarrow[\text{decay}]{\Omega^+} \quad \xleftarrow[\text{cluster}]{\Omega} \\ \text{bootstrap} \end{array} \right\} \leftarrow \begin{array}{c} H_0 \\ \text{initial} \end{array} \quad (6.1)
 \end{array}$$

where the curly bracket contains the strong interaction and corresponds to the S operator, which in spite of the complicated intermediate steps, maps H_0 on- to H_0 .

While unconventional, these features of the model have the typical flavour of strong interactions. There are, however, many unrealistic properties:

- the mass spectrum has a line at m_0 and a continuum for $m \geq 2m_0$; no discrete resonances emerge. The spectrum $\rho(m^2)$ is to be interpreted as an average. Nothing prevents us to use several input particles taking thus account of the lowest resonances (see below);
- we have a purely pionic world. This is not a serious deficiency and has been chosen here to keep things simple. One can extend the scheme to several coupled bootstrap equations, each for one species of particle or even each carrying a whole basic multiplet of some symmetry; this would amount to put in by hand several of the lowest resonances. Such coupled bootstrap equations have been solved in SBM [Refs. 20)] and may be translated into our formalism;
- despite the commutation relations for bosons, our final momentum distributions are as if our pions had Boltzmann statistics. This is a consequence of using

the simplest possible bootstrap equation corresponding to invariant momentum space with Boltzmann measure [this is equivalent to imposing an additional interaction between pions and/or clusters which just balances the effect of Bose statistics in momentum space; for details see Ref. 9)]. There is no principal difficulty in introducing Bose measure⁹⁾ and writing the bootstrap equation in a manifestly Bose statistical form; such equations have been studied in SBM^{8),16)-18)} and the whole procedure could be translated into our formalism;

- we do not have bound states. It might be very fruitful to introduce quarks as input particles in a set of coupled bootstrap equations. In that case bound states are needed; they could be represented by clusters which do not decay under D (for kinematical reasons) while their collision with sufficient energy would of course allow free quarks to appear. The probability for such an event would, however, drop exponentially with the quark mass for the same reasons as in SBM²¹⁾. We would then have statistical confinement. Some steps in the direction of solving coupled bootstrap equations with strong quark binding have been done in SBM²²⁾ but a convincing scheme is still lacking. This is a problem worth studying in SBM and solving it there completely before trying to translate it into our formalism.

While the diseases mentioned so far are all curable by cures known from SBM studies, the next and last one is not:

- the present version allows only for isotropic final distributions, while in real collisions we see jets.

We do not yet know how to overcome this deficiency. There is only one straightforward way suggested by the model itself: angular momentum conservation. One would decompose the initial state into partial waves and treat each partial wave separately by coupling it to a final state via clusters with definite angular momentum. Similar attempts have been made in the Fermi statistical model in the years 1960-1965 without success²³⁾; the reason for the failure was found to lie in the probabilistic ansatz, where all phase relations between the different partial waves were lost. It is, however, just the phase relations which build up the jets. In our present model phases are not lost and in principle it could be that via partial waves a correct description would emerge -- if only bootstrap with angular momentum conservation had been solved in full generality. Approximate solutions are known^{24),25),26)} and an attempt to translate them into our formalism might be worth while. Apart from this way, which of course is most faithful to the spirit of the model, two others would be either to introduce some conventional

Γ^3 or Γ^4 fireball interaction on top of the bootstrap and solve it perturbatively; or else to introduce ad hoc non-isotropic momentum weight functions in the bootstrap equation; this, however, would degrade the model to a curve-fitting procedure with great complication and no insight.

Acknowledgement

We are grateful to many colleagues at Bielefeld and at CERN for discussions and criticism. Our particular thanks go to W. Nahm (CERN) who was the first to suggest that the non-associative nature of clustering should manifest itself somewhere in the field theoretical model, and to G.C. Rossi (CERN) who discovered the Schröder paper¹³⁾ and in long discussions helped us to see our way.

APPENDIX

RECURSION RELATIONS AND GENERATING EQUATIONS
FOR YELLIN COEFFICIENTS

From the bootstrap equation (4.1) with coefficients $\gamma_n/\sqrt{n!}$ we have derived two sets of Yellin coefficients, w_n , obeying a recursion relation (4.15) involving only $|\gamma_n|^2$, and β_n , obeying the recursion relation (5.3) involving $\gamma_n/\sqrt{n!}$. The structure of both equations is exactly the same, namely

$$b_1 = 1$$

$$b_n = \sum_{k=2}^n a_k \sum_{n_i=1}^{n-1} \delta(n - \sum_{i=1}^k n_i) \prod_{i=1}^k b_{n_i} \quad (A.1)$$

where b may stand for β or w and a_k for $\gamma_k/\sqrt{k!}$ or $|\gamma_k|^2$. We derive from (A.1) a generating equation by first observing that (A.1) can be rewritten as

$$b_1 = 1$$

$$b_n = \sum_{k=2}^{\infty} a_k \sum_{n_e}^* k! \frac{b_1^{n_1} b_2^{n_2} \dots b_e^{n_e} \dots}{n_1! n_2! \dots n_e! \dots} \quad (A.2)$$

$$\sum_{n_e}^* := \text{sum over } n_e \in \{n_e : \sum n_e = k; \sum e n_e = n\}$$

According to well-known combinatorial formulae²⁷⁾,

$$C_n := \sum_{n_e}^* k! \frac{b_1^{n_1} \dots b_e^{n_e} \dots}{n_1! \dots n_e! \dots} \quad (A.3)$$

is the coefficient of t^n of the expression

$$\left[\sum_{l=1}^{\infty} b_l t^l \right]^k = : [B(t)]^k = \sum_{n=1}^{\infty} C_n t^n \quad (\text{A.4})$$

This coefficient is by (A.2) only defined for $k \geq 2$ and for $n \geq 2$; we make it valid for $n = 1$ and $k = 1$ by rewriting (A.2) once more (adding and subtracting terms):

$$(1+a_1)b_n - \delta_{n1}b_1 = \sum_{k=1}^{\infty} a_k \sum_{n_l}^* k! \frac{b_1^{n_1} \dots b_l^{n_l} \dots}{n_1! \dots n_l! \dots} \quad (\text{A.5})$$

Now we multiply both sides by t^n and sum over n with the result

$$(1+a_1)B(t) - t = \sum_{k=1}^{\infty} a_k [B(t)]^k \quad (\text{A.6})$$

Now define

$$A(x) := \sum_{k=0}^{\infty} a_k x^k \quad (\text{A.7})$$

then Eq. (A.6) becomes

$$t = (1+a_1)B(t) - A(B(t)) + a_0 \quad (\text{A.8})$$

which we call the generating equation for the Yellin coefficients b_n . In the special case $a_n = 1/n!$ we obtain Eq. (2.8) with $B \equiv Z$.

The generating equation can often lead to an explicit solution: e.g., if $a_1 = 2a_2 = 1$, all others = 0; this is the "two-body truncated" bootstrap equation with the generating equation

$$t = B - \frac{B^2}{2} \quad (\text{A.9})$$

and

$$B(t) = 1 \pm \sqrt{1-2t} = \sum_{n=1}^{\infty} b_n t^n \quad (\text{A.10})$$

which can be expanded in powers of t leading immediately to the b_n (the - sign corresponds to the physical solution).

In more complicated cases $A(x)$ may at least obey a simple differential equation. Then one differentiates the generating equation and tries to eliminate $A(x)$ and its derivatives by means of the differential equation; then compare powers of t . If $A(x) = e^x$ ($a_n = 1/n!$) as in Eq. (2.8), then this is particularly simple and leads to another recurrence relation²⁸⁾ for the b_n :

$$b_n = -\frac{n-1}{n} b_{n-1} + \frac{2}{n} \sum_{k=1}^n k b_k b_{n-k} \quad (\text{A.11})$$

which is much easier to compute as the original equation (A.1). Similar simplifications might be possible in other cases, too.

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