Archives

ASYMPTOTIC FREEDOM BEYOND THE LEADING ORDER

A.J. Buras ^{*)}, E.G. Floratos, D.A. Ross and C.T. Sachrajda
CERN - Geneva

A B S T R A C T

We make a quantitative analysis of the full $\bar{g}^2$ interaction corrections to the leading Q^2 dependence of νW_2 at $x \geq 0.4$, as given by an asymptotically free gauge theory. It turns out that due to partial cancellations between various contributions the $\bar{g}^2$ corrections are small. The best fit with the SLAC ep data after including the $\bar{g}^2$ corrections is almost identical to that without these corrections, the only effect being a change in Λ , the one free parameter, which sets the scale of the theory. On the other hand the effect of including target mass corrections is to improve the agreement of the prediction for νW_2^{ep} with data for large values of x . We study the implications of $\bar{g}^2$ and target mass corrections for $R = \sigma_L / \sigma_T$ and conclude, in agreement with other authors, that although both types of corrections are vital they do not account for values of R for $x \geq 0.5$ as measured at SLAC.

*) Address after 30 September 1977 :
Fermi National Accelerator Laboratory, P.O.Box 500,
Batavia, IL 60510, USA.

1. INTRODUCTION

During the past few years it has become popular to interpret the experimentally observed phenomenon of Bjorken scaling, in terms of "asymptotic freedom", a property of certain field theories ^{1),2)}. In particular quantum chromodynamics (QCD), an asymptotically free theory has emerged as the leading candidate for the theory of strong interactions. Asymptotic freedom implies that in certain kinematic situations, such as in the Bjorken limit of deep inelastic scattering, one can define a perturbation theory in terms of an effective coupling constant ($\bar{g}$), which decreases as the process gets deeper, i.e., as Q^2 increases. The hope is that this perturbation expansion in $\bar{g}^2$ converges rapidly even for moderate values of Q^2 and also that the first few terms of this expansion (perhaps just the first term) explain the present data, the higher order terms being negligible. Indeed, keeping only the first term in this perturbation expansion, one can obtain very satisfactory comparisons with the data. However, the value of the effective strong coupling constant at presently accessible values of Q^2 ($2 \text{ GeV}^2 \leq Q^2 \leq 50 \text{ GeV}^2$) is usually taken to lie in the range $0.2 \leq \bar{g}^2/4\pi \leq 0.5$, which is not very small. It is therefore interesting to ask how big the higher order terms in the expansion in $\bar{g}^2$ are, and whether they render the comparison of just the lowest order term with the data meaningless. In this paper we systematically study all the contributions to the second term in the perturbation expansion in $\bar{g}^2$ (for valence quarks).

In QCD, the dominant behaviour of a given moment of the deep inelastic structure functions [i.e., $\int_0^1 dx x^N F_i(x, Q^2)$ where x is the Bjorken scaling variable] is given by three local operators (the three operators of twist 2). Two of these three operators are singlets in the flavour group, whereas the third is a non-singlet and hence controls the contribution of the valence quarks. Thus this set of operators (one for each moment) controls the asymptotic behaviour of purely non-singlet quantities (such as the difference between the structure functions of the proton and the neutron), and also of the individual structure functions in the kinematic region where the sea is negligible (e.g., $x \geq 0.4$). In this paper we restrict our attention to those quantities which are given by the non-singlet operators. These operators are given by

$$O_n^a = \frac{1}{2} i^{n-1} S \bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \tau^a \psi - \text{trace terms} \quad (1.1)$$

$(2 \leq n \leq \infty)$

where S is the symmetrization of the Lorentz indices, τ^a are the flavour group generators and $D_{\mu} = \partial_{\mu} + ig\lambda_{\mu}^S A_{\mu}^S$ is the covariant derivative with λ^S the colour group generators and A_{μ}^S the gluon fields.

If indeed QCD is the true theory of the strong interactions, and if one could compare the theory to experiment at asymptotic values of Q^2 , then all the violations of Bjorken scaling would be the logarithmic ones due to the interaction effects. The available data, however, are at relatively low values of Q^2 and therefore mass corrections are expected to be relevant. Under the assumption that the operators defined in (1.1) are the dominant ones, the target mass corrections can be calculated ³⁾⁻⁵⁾, and we include these in our discussion. The corrections due to the effective quark masses are model dependent and we do not consider these.

We would like to remark that some of the interaction corrections to the leading order as well as target mass corrections have already been discussed in the literature ^{5),6)}. We differ from these analyses in several ways.

First of all we give a quantitative estimation of all $\bar{g}^2$ corrections to the moments of the non-singlet parts of the structure functions, whereas in the previous analysis only a small part of these corrections has been included. In particular we take into account those corrections which come from the two-loop contributions ^{*}) to the anomalous dimensions of the non-singlet operators.

Secondly we investigate which of the various $\bar{g}^2$ corrections coming from the various sources (anomalous dimensions, two-loop contributions to the β function, corrections to the parton model relations) are most important, how this importance varies with x and Q^2 , and how their interaction corrections compare with target mass effects. We study the latter in the manner suggested by other authors ^{5),6)}.

Our principal findings are :

- i) for νW_2 the effects of $\bar{g}^2$ corrections can be almost completely absorbed by a redefinition of a certain parameter (which is equivalent to choosing the value of the effective coupling constant at a fixed Q^2), resulting in almost no change in the fit to the data relative to the leading order ⁷⁾ (which was already quite good). On the other hand, the addition of target mass corrections, together with a suitable change in this parameter, improves the agreement of the theory with the data for $x \geq 0.6$.

^{*}) Although these are of order $\bar{g}^4$, they contribute to the order $\bar{g}^2$ corrections to the leading behaviour of the structure functions.

- ii) for $R = \sigma_L/\sigma_T$ both interaction corrections and target mass corrections are vital but in this case the poorness of the data does not allow for a definite conclusion.

The plan of the paper is as follows. In Section 2 we present our formulae, which include all the first order corrections to the Born term of the expansion in $\bar{g}^2$. In Section 3 we compare the formulae to the experimental values of νW_2 and discuss the relevance of various interaction and target mass corrections. Section 4 contains our predictions for $R = \sigma_L/\sigma_T$ and the corresponding comparison with the data. R is zero in lowest order and it is therefore necessary to calculate the first order corrections in order to obtain non-zero values. Section 5 consists of our conclusions and comments.

2. BASIC FORMULAE

In this Section we present the formulae which we have used in the phenomenological applications (see Sections 3 and 4). Since the data for νW_1 and R are much poorer than for νW_2 , we find it convenient to discuss first the latter structure function.

2.1 Interaction (asymptotic freedom) effects

We start by discussing the Q^2 dependence of νW_2 arising from interaction effects as given by an asymptotically free gauge theory. The target mass corrections will be included in the next subsection.

In an asymptotically free gauge theory the Q^2 dependence of $^*)$

$$F_2(x, Q^2) \equiv \nu W_2(x, Q^2) \quad (2.1)$$

is given in terms of the following moment sum rules

$$\begin{aligned} M_n(Q^2) &\equiv \int_0^1 dx \, x^{n-2} F_2(x, Q^2) \quad , \quad n \geq 2 \\ &= A_n C_n(1, \bar{g}(Q^2)) \exp \left[- \int_{\bar{g}(Q^2)}^{\bar{g}(Q^2)} d\tau \frac{\gamma_n(\tau)}{\beta(\tau)} \right] . \end{aligned} \quad (2.2)$$

*) In this paper we restrict ourselves to $x \geq 0.4$ where the Wilson expansion for the structure functions is dominated by operators whose mixing with gluon operators is forbidden by charge conjugation and whose anomalous dimensions are equal to those of the non-singlet operators.

Here

$$x = \frac{Q^2}{2m\nu}$$

is the Bjorken scaling variable, Q^2 and ν are the standard invariants and m is the target mass. Furthermore $\bar{g}(Q^2)$, $\gamma_n(\bar{g})$ and $\beta(\bar{g})$ are the effective coupling constant, anomalous dimension of the spin n operator in the Wilson expansion and the standard β function, respectively. All these three quantities, as well as $C_n(1, \bar{g}(Q^2))$, can be calculated in perturbation theory. On the other hand, the constants A_n are to be found from experiment at some arbitrary value of Q^2 equal to Q_0^2 .

In order to make practical use of Eq. (2.2), we shall expand it in powers of

$$\bar{g}_0^2(Q^2) = \frac{48\pi^2}{(33-2f)\ln\frac{Q^2}{\Lambda^2}} \quad (2.3)$$

and keep the first two terms of this expansion. Here $\bar{g}_0^2(Q^2)$ is the leading behaviour of $\bar{g}^2(Q^2)$ for large Q^2 and f is the number of flavours which we shall take to be 4^{*)}. The parameter Λ , which is related to the subtraction point at which our theory is defined, is the only scale in the theory in question and will be found from the data.

We first expand $\gamma_n(\bar{g})$, $\beta(\bar{g})$ and $C_n(1, \bar{g})$ in powers of $\bar{g}$

$$\gamma_n(\bar{g}) = \gamma_0^n \frac{\bar{g}^2}{16\pi^2} + \gamma_1^n \left(\frac{\bar{g}^2}{16\pi^2} \right)^2 + \dots \quad (2.4)$$

$$\beta(\bar{g}) = -\beta_0 \frac{\bar{g}^3}{16\pi^2} - \beta_1 \frac{\bar{g}^5}{(16\pi^2)^2} + \dots \quad (2.5)$$

$$C_n(1, \bar{g}) = 1 + H_n \frac{\bar{g}^2}{12\pi^2} + \dots \quad (2.6)$$

*) Although the values of Q^2 for which we have data cross the charm threshold, the effect of considering the transition from $f=3$ to $f=4$ is negligible.

From Eq. (2.5) we have the following expansion for the effective coupling constant

$$\bar{g}^2(Q^2) = \bar{g}_0^2(Q^2) - \frac{\beta_1}{\beta_0} \frac{\bar{g}_0^4(Q^2)}{16\pi^2} \ln \ln \frac{Q^2}{\Lambda^2} + O(\bar{g}_0^6) \quad (2.7)$$

where $\bar{g}_0^2(Q^2)$ is given in Eq. (2.3). The arbitrary parameter Λ (which is the one constant of integration) has been chosen so that there are no further terms of order $\bar{g}_0^4$ in Eq. (2.7).

Inserting (2.4)-(2.7) into (2.2) and expanding in $\bar{g}_0^2$, we obtain

$$M_n(Q^2) = A_n \left[1 + \frac{\bar{g}_0^2(Q^2)}{12\pi^2} (H_n + P_n + L_n(Q^2)) \right] \exp\left(-\frac{\gamma_0^n}{2\beta_0} s\right) \quad (2.8)$$

where

$$s = \ln \left[\frac{\ln \frac{Q^2}{\Lambda^2}}{\ln \frac{Q_0^2}{\Lambda^2}} \right] , \quad (2.9)$$

$$H_n = 4 \sum_{k=1}^n \frac{1}{k} - 2 - 8 \sum_{k=1}^n \frac{1}{k^2} + \frac{10}{n+1} + \frac{4}{n^2} - \frac{4}{(n+1)^2} \quad (2.10)$$

$$P_n = \frac{3}{8} \left[\frac{\gamma_1^n}{\beta_0} - \frac{\gamma_0^n \beta_1}{\beta_0^2} \right] \quad (2.11)$$

and

$$L_n(Q^2) = -\frac{3}{8} \frac{\beta_1}{\beta_0^2} \gamma_0^n \ln \ln \left(\frac{Q^2}{\Lambda^2} \right). \quad (2.12)$$

The term H_n comes from expanding $C_n(\bar{g}^2)$ in $\bar{g}^2$ as indicated in Eq. (2.6). It has been calculated in Ref. 6) and Ref. 8). The term P_n arises from the expansion of the integrand in Eq. (2.2). Finally the term L_n arises from the second term in the r.h.s. of Eq. (2.7). The terms H_n and P_n are not individually physical quantities and indeed depend on the renormalization prescription. Using a particular example of a more general theorem of Stueckelberg and Peterman ⁹⁾ it has been shown in Ref. 10) that the renormalization prescription dependence of H_n and P_n cancels.

The various parameters which enter into Eqs. (2.11) and (2.12) have been already calculated in the literature and are given for an $SU(3)$ gauge theory with f flavours as follows

$$\gamma_0^n = \frac{8}{3} \left[1 - \frac{2}{n(n+1)} + 4 \sum_{j=2}^n \frac{1}{j} \right] \quad (2.13)$$

$$\beta_0 = \frac{1}{3} [33 - 2f] \quad (2.14)$$

and

$$\beta_1 = 102 - \frac{38}{3} f \quad (2.15)$$

γ_0^n and β_0 have been calculated in Ref. 11). β_1 has been obtained in Ref. 12) and Ref. 13). The analytic expressions for parameters γ_1^n are very complicated and we do not present them here. They can be found in Ref. 10) where also their numerical values for the first twelve moments are given.

For phenomenological applications it is useful to express A_n in Eq. (2.8) through the moments $M_n(Q_0^2)$, which can be determined from the data at a particular value of $Q^2 = Q_0^2$. After trivial algebra we obtain

$$M_n(Q^2) = M_n(Q_0^2) \frac{\left[1 + \frac{\bar{g}_0^2(Q^2)}{12\pi^2} (H_n + P_n + L_n(Q^2)) \right]}{\left[1 + \frac{\bar{g}_0^2(Q_0^2)}{12\pi^2} (H_n + P_n + L_n(Q_0^2)) \right]} \exp\left(-\frac{\gamma_0^n}{2\beta_0} s\right) \quad (2.16)$$

2.2 $vW_2(x, Q^2)$ with interaction and target mass corrections

In what follows it will be useful to work with Q^2 dependent quark distributions. Such distributions have been already discussed in the literature ^{7),14)-16)} in the leading approximation in which terms H_n , P_n and L_n appearing in Eq. (2.16) have been neglected. It is of interest to see how the terms H_n , L_n and P_n modify the previously found Q^2 dependence

of quark distributions and consequently of νW_2 . At present we are only in a position to investigate this question for valence quark distributions since the corresponding parameters γ_1^n have not been calculated for sea and gluon distributions. Therefore our analysis will be restricted to the values $x \geq 0.4$, where as is expected and has been indicated in Refs. 6) and 7), the sea contributions to νW_2 are negligible.

We introduce up and down valence quark distributions

$$x u_v(x, Q^2) \quad \text{and} \quad x d_v(x, Q^2) .$$

The Q^2 dependence of the moments of each of these distributions is given by Eq. (2.16) [just replace $F_2(x, Q^2)$ by $x u_v(x, Q^2)$ or $x d_v(x, Q^2)$]. The Q^2 dependence of the distributions itself is obtained by inverting the formula (2.16). How it can be done will be shown in Subsection 2.3.

It turns out that the comparison of the theory with the data is easier if we work with the pair of distributions

$$x V(x, Q^2) \equiv x u_v(x, Q^2) + x d_v(x, Q^2)$$

and $x d_v(x, Q^2)$ rather than with $x u_v(x, Q^2)$ and $x d_v(x, Q^2)$. We are now in a position to state our final formulae. Let us define a function relevant for scattering on proton targets

$$F_2(x, Q^2) \equiv \frac{4}{9} x V(x, Q^2) - \frac{1}{3} x d_v(x, Q^2) . \quad (2.17)$$

Then in the case of interaction effects alone

$$\nu W_2(x, Q^2) = F_2(x, Q^2) \quad (2.18)$$

whereas if also target mass effects are present, we have ⁴⁾⁻⁶⁾

$$\begin{aligned} \nu W_2(x, Q^2) = & \frac{1}{(1 + 4x^2 \frac{m^2}{Q^2})^{3/2}} \frac{x^2}{\xi^2} F_2(\xi, Q^2) \\ & + 6 \frac{m^2}{Q^2} \frac{x^3}{(1 + 4x^2 \frac{m^2}{Q^2})^2} \int_{\xi}^1 d\xi' \frac{F_2(\xi', Q^2)}{\xi'^2} \\ & + 12 \frac{m^4}{Q^4} \frac{x^4}{(1 + 4x^2 \frac{m^2}{Q^2})^{5/2}} \int_{\xi}^1 d\xi' \int_{\xi'}^1 d\xi'' \frac{F_2(\xi'', Q^2)}{\xi''^2} \end{aligned} \quad (2.19)$$

with the new "scaling" variable ξ ³⁾ given by

$$\xi = \frac{2x}{\left[1 + \left(1 + 4x^2 \frac{m^2}{Q^2}\right)^{1/2}\right]} \quad (2.20)$$

Obviously for $Q^2 \rightarrow \infty$ Eq. (2.19) approaches Eq. (2.18). In other words for very large values of Q^2 only interaction effects matter. At small Q^2 values, however, target mass corrections are also of some importance.

In order to complete our analysis we have to show how to find the Q^2 dependence of $V(x, Q^2)$ and $d_V(x, Q^2)$ as given by Eq. (2.16). In other words we have to tackle the inversion problem. There exist many techniques to solve this problem, but since the n dependence in Eq. (2.16) is very complicated, the inversion has to be done numerically. However, as has been pointed out in Ref. 16) and studied more carefully in Ref. 7), it is possible to find simple analytic expressions for the Q^2 dependence of quark distributions which, to a very good accuracy, represent the Q^2 dependence predicted by ASF. Although in Ref. 7) only the leading behaviour of the moments has been discussed, the procedure for finding analytic expressions presented there can also be applied to the more general case of (2.16). We now briefly present the procedure by applying it to $xV(x, Q^2)$ and $xd_V(x, Q^2)$. Further details can be found in Refs. 7) and 16).

2.3 Analytic expressions for $xV(x, Q^2)$ and $xd_V(x, Q^2)$

The procedure developed in Ref. 7) consists of three steps :

i) write

$$x V(x, Q^2) = \frac{3}{B(\eta_1(s), 1 + \eta_2(s))} x^{\eta_1(s)} (1-x)^{\eta_2(s)} \quad (2.21)$$

and

$$x d_V(x, Q^2) = \frac{1}{B(\eta_3(s), 1 + \eta_4(s))} x^{\eta_3(s)} (1-x)^{\eta_4(s)} \quad (2.22)$$

where

$$\eta_i(s) = \eta_i(0) + \eta'_i G s.$$

Here $G = 4/25$ and s is given by Eq. (2.9). The appearance of Euler's beta function is necessary if we want to satisfy the known sum rules ¹⁷⁾

$$\int dx \, V(x, Q^2) = 3 \quad \text{and} \quad \int dx \, d_V(x, Q^2) = 1$$

- ii) find $\eta_i(0)$'s from the data at $Q^2 = Q_0^2$ and calculate the moments $M_n(Q_0^2)$ which appear on the r.h.s. of Eq. (2.16). In this way the r.h.s. of Eq. (2.16) is known for any Q^2 up to the single parameter Λ which is to be determined by comparing the scaling violations predicted by the theory with those observed in experiment.
- iii) finally determine $\eta_i^!$ by fitting the moments obtained from (2.21) and (2.22) to those predicted by the theory, i.e., to Eq. (2.16).

It turns out that the expressions (2.21) and (2.22) reproduce up to the accuracy of 1-2% the Q^2 dependence of the first 12 moments in (2.16) for $0 \leq s \leq 1.6$. This is sufficient to study the range $x \leq 0.8$ and $Q^2 \leq 10'000 \text{ GeV}^2$. For $x > 0.8$ we cannot trust our formulae anyhow since for such high values of x the contributions from higher twist operators, which we have neglected, are believed ⁶⁾ to be of importance.

We end this Section by summarizing our method of comparing the predictions of asymptotic freedom to the data. It consists of two steps.

In the first step we fit expressions (2.21) and (2.22) to the equation (2.16) for a given input $\eta_i(0)$ keeping Λ free. In this way we find $\eta_i^!$. It should be remarked that for each input $\eta_i(0)$ there is a different set of $\eta_i^!$. In addition only in the case of the leading approximation are the $\eta_i^!$ independent of Λ *).

In the second step we fit expressions (2.18) or (2.19) to the experimental data and find the best value of Λ .

3. NUMERICAL RESULTS FOR $\nu W_2(x, Q^2)$

As we already stated in the Introduction, our study of $\bar{g}^2$ and target mass corrections to the leading approximation is twofold.

*) In this particular case the dependence on Λ enters only through the variable s .

First we want to investigate the importance of various $\bar{g}^2$ corrections H_n, L_n, P_n as well as of target mass corrections.

Second we want to see whether all these new corrections, when included in the analysis can improve the already good agreement of the leading order analysis ⁷⁾ with the data.

We begin with the investigation of $\bar{g}^2$ corrections neglecting target mass corrections for the moment. We take the values $\eta_1(0)$ and Λ found in Ref. 6) on the basis of leading order analysis and calculate $\nu W_2(x, Q^2)$ using Eqs. (2.16), (2.17), (2.21) and (2.22) for the following cases

- a) H_n included
- b) H_n and P_n included, and
- c) H_n, P_n and L_n included.

The result is presented in Fig. 1 where also the curve corresponding to $H_n = P_n = L_n = 0$, obtained in Ref. 7) is shown. We notice first that all corrections are much smaller in absolute magnitude than the scaling violations caused by the leading term on the r.h.s. of Eq. (2.16). Typically these corrections are of order of 5-20% of the scaling violations caused by the leading term, in the range of Q^2 between 2 GeV² and 50 GeV². This means that the predictions of the leading term will be effected only slightly and also assures that the perturbation expansion makes sense. We notice next the unique role of the first term in P_n , which contains the two-loop contribution to the anomalous dimension γ_n . It is the only positive term of order $\bar{g}_0^2$ in the basic formula (2.16). This term is, however, large enough to cancel substantially the remaining terms so that the total correction of order $\bar{g}_0^2$ is relatively small. The importance of various terms L_n, P_n and H_n as well as cancellations between them are clearly seen in Fig. 1. We only remark that at $x=0.4$ the P_n contribution cancels almost exactly that of L_n , so that the $\bar{g}_0^2$ correction is given by H_n . On the other hand at $x=0.75$ P_n substantially cancels H_n so that the $\bar{g}_0^2$ correction is well represented by L_n . From this we also conclude that the term H_n , the only term taken into account in Ref. 6) does not represent the $\bar{g}_0^2$ corrections well in the whole region of x . All the above observations are unchanged if we vary Λ from 0.2 to 0.5 GeV, the usually accepted range of this parameter.

We continue our discussion by comparing the interaction effects to target mass effects. We again take the values $\eta_1(0)$ and Λ found in Ref. 7) and calculate $\nu W_2(x, Q^2)$ using Eqs. (2.16)-(2.22) for the following cases :

- i) ASF I - case discussed in Ref. 7);
- ii) ASF II - all corrections L_n , H_n and P_n added to i);
- iii) MASF I - target mass corrections added to i);
- iv) MASF II - target mass corrections added to ii).

The result is presented in Fig. 2 where we also plot the SLAC data ¹⁸⁾ in order to show how the differences between various cases compare with the experimental errors. We would like to stress that one should not conclude from Fig. 2 which of the cases i) to iv) fits the data best, since the $\eta_1(0)$ and Λ used here are the best parameters only for the case i). We shall make the best fits for the remaining cases at the end of this Section. We think that by keeping $\eta_1(0)$ and Λ fixed for i) to iv) is the best way to illustrate the relative importance of the various corrections.

We notice first a very different behaviour of the target mass corrections compared with the $\bar{g}_0^2$ corrections. The effect of the full $\bar{g}_0^2$ correction is to flatten the Q^2 dependence of νW_2 at any x as we have already seen in Fig. 1. On the other hand, the effect of target mass corrections is to flatten the curves in Fig. 2 slightly for $x < 0.6$ and to make them steeper for $x > 0.6$. The flattening due to target mass corrections at small x where $\xi = x$ is mainly due to the increased range of the integrals in Eq. (2.19). At large x , on the other hand, the observed effect of target mass corrections is due to ξ which is there very different from x . Finally we notice that, whereas for intermediate x values the effects of $\bar{g}_0^2$ corrections and target mass corrections are of equal importance, at large values of x target mass effects are more important in the Q^2 range considered.

Next we present the results of fitting the cases i)-iv) to the data. The parameters $\eta_1(0)$ and Λ and corresponding values for $\eta_1^!$ are given in Table I. It turns out that the best fits for cases i) and ii) are almost indistinguishable from each other. The effect of $\bar{g}_0^2$ corrections is only seen in the change of Λ . This is also the case for iii) and iv). Therefore in Fig. 3 we only present the cases i) and iv). We notice that the inclusion of target mass corrections further improves the agreement of the theory with the data relative to that obtained in Ref. 7). This is particularly the case for $x \geq 0.6$.

In summary, the effect of $\bar{g}_0^2$ corrections can be almost entirely absorbed by a suitable change in the value of Λ . No improvement on the agreement of the theory with the data can be attained. On the other hand the inclusion of target mass corrections leads to a visible improvement

(particularly seen at large x) of the agreement of the theory with the data. In addition in the latter case both Λ and $\eta_1(0)$ are changed relative to the case i). The whole situation is summarized in Fig. 4.

Finally, in Fig. 5 we present our predictions for νW_2 for two sample values of x at very large values of Q^2 , values which may be experimentally attainable in future accelerators¹⁹⁾.

4. THE RATIO OF THE LONGITUDINAL TO TRANSVERSE CROSS-SECTIONS

We now turn our attention to W_1 and $R = \sigma_L / \sigma_T$. We define a structure function F_1 . In the absence of mass corrections W_1 and R are given in terms of F_1 by

$$2m W_1(x, Q^2) = F_1(x, Q^2) \quad (4.1)$$

$$R(x, Q^2) = \frac{F_2(x, Q^2) - x F_1(x, Q^2)}{x F_1(x, Q^2)} \quad (4.2)$$

We define further the quantities $\tilde{M}_n(Q^2)$ in terms of the moments of F_1 and $M_n(Q^2)$ of Eq. (2.16)

$$\int_0^1 dx x^{n-1} F_1(x, Q^2) = M_n(Q^2) - \tilde{M}_n(Q^2) \quad (4.3)$$

Asymptotic freedom gives a formula for the Q^2 dependence of these moments analogous to Eq. (2.16)

$$M_n(Q^2) - \tilde{M}_n(Q^2) = M_n(Q_0^2) \frac{\left[1 + \frac{\bar{g}_0^2(Q^2)}{12\pi^2} (H_n - \tilde{H}_n + P_n + L_n(Q^2))\right]}{\left[1 + \frac{\bar{g}_0^2(Q_0^2)}{12\pi^2} (H_n + P_n + L_n(Q_0^2))\right]} \exp\left(-\frac{\gamma_0^n}{2\beta_0} s\right) \quad (4.4)$$

Comparing Eq. (4.4) with Eq. (2.16), we see that $P_n, L_n, \gamma_0^n, \beta_0$, being properties of the β function and the anomalous dimensions of operators, are the same for F_1 and F_2 , whereas H_n , which is obtained by calculating the one-loop corrections to the structure functions, changes by an amount $-\tilde{H}_n$, which is given by^{6),8)}

$$\tilde{H}_n = \frac{4}{n+1} \quad (4.5)$$

From Eqs. (4.4), (4.5) and (2.16) we get

$$\tilde{M}_n(Q^2) = \frac{\bar{g}_0^2(Q^2)}{12\pi^2} \frac{4}{n+1} M_n(Q_0^2) \exp\left(-\frac{\gamma_0^n}{2\beta_0} s\right) \quad (4.6)$$

where we have kept the terms of order $\bar{g}_0^2(Q^2)/12\pi^2$ and dropped the corrections of order

$$\frac{\bar{g}_0^2(Q^2)}{12\pi^2} \cdot \frac{\bar{g}_0^2(Q_0^2)}{12\pi^2}$$

multiplied by H_n , P_n and $L_n(Q^2)$. These terms are of the same order as the two-loop corrections to the perturbation expansions for the C_n of Eq. (2.6) so it would be inconsistent to consider them separately. The second term in the expansion for the β function and anomalous dimensions do not affect the calculation of R to lowest order. It is worth emphasizing, however, that taking the lowest order [which involves replacing the denominator in Eq. (4.4) by 1] is only reasonable because of the fortuitous partial cancellation of H_n , P_n and $L_n(Q^2)$.

By inverting Eq. (4.6) and using the values of $M_n(Q_0^2)$ obtained from the fit to νW_2 , $F_1(x, Q^2)$ and hence W_1 and R can be calculated. It is important to note that no further fit to the data is necessary. We invert Eq. (4.6) using a similar method to that used to invert the moments of νW_2 .

We define $\tilde{V}(x, Q^2)$ and $\tilde{d}_V(x, Q^2)$ such that

$$F_2(x, Q^2) - x F_1(x, Q^2) = \frac{4}{9} x \tilde{V}(x, Q^2) - \frac{1}{3} x \tilde{d}_V(x, Q^2) \quad (4.7)$$

and we assume that $\tilde{V}(x, Q^2)$ and $\tilde{d}_V(x, Q^2)$ have the form

$$x \tilde{V}(x, Q^2) = \frac{\bar{g}_0^2(Q^2)}{12\pi^2} \frac{3 \epsilon^\nu}{\beta(\tilde{\eta}_1(s), 1+\tilde{\eta}_2(s))} x^{\tilde{\eta}_1(s)} (1-x)^{\tilde{\eta}_2(s)} \quad (4.8)$$

$$x \tilde{d}_V(x, Q^2) = \frac{g_o^2(Q^2)}{12\pi^2} \frac{\epsilon^d}{B(\tilde{\eta}_3(s), 1+\tilde{\eta}_4(s))} x^{\tilde{\eta}_3(s)} (1-x)^{\tilde{\eta}_4(s)} \quad (4.9)$$

where
$$\tilde{\eta}_i(s) = \tilde{\eta}_i(0) + \tilde{\eta}'_i G s$$

The appearance of ϵ^V and ϵ^d in Eqs. (4.8) and (4.9) arises because we have no sum rule which governs the integrals of $\tilde{V}$ and $\tilde{d}_V$. From Eqs. (4.7), (4.9) the moments, $\tilde{M}_n(Q^2)$, can be calculated and we have found values of the 10 parameters ϵ^V , ϵ^d , $\tilde{\eta}_i(0)$, $\tilde{\eta}'_i$ ($i=1, \dots, 4$), such that Eq. (4.6) is obeyed to better than 2% for the first twelve moments in the range $0 \leq s \leq 1.6$. The values of these parameters are given in Table II. We emphasize that these are not free parameters but are given by the theory.

When target mass corrections are included, Eqs. (4.1) and (4.2) are modified to

$$\begin{aligned} 2m W_1(x, Q^2) &= \frac{1}{(1 + 4x^2 \frac{m^2}{Q^2})^{1/2}} \frac{x}{\xi} F_1(\xi, Q^2) \\ &+ 2 \frac{m^2}{Q^2} \left(\frac{x^2}{(1 + 4x^2 \frac{m^2}{Q^2})} \right) \int_{\xi}^1 d\xi' \frac{F_2(\xi', Q^2)}{\xi'^2} \\ &+ 4 \frac{m^4}{Q^4} \frac{x^3}{(1 + 4x^2 \frac{m^2}{Q^2})^{3/2}} \int_{\xi}^1 d\xi' \int_{\xi'}^1 d\xi'' \frac{F_2(\xi'', Q^2)}{\xi''^2} \end{aligned} \quad (4.10)$$

$$R(x, Q^2) = \frac{\psi W_2(1 + 4x^2 \frac{m^2}{Q^2}) - 2m x W_1(x, Q^2)}{2m x W_1(x, Q^2)} \quad (4.11)$$

where $W_2(x, Q^2)$ and $W_1(x, Q^2)$ are given by Eqs. (2.19) and (4.10), respectively. However, the programme for finding F_1 remains unchanged. In the case of mass corrections, the fit to the data at Q_0^2 for F_2 is different so we get different values of $M_n(Q_0^2)$. Using these different values we also get different values of the ten parameters of $xF_1 - F_2$ required to satisfy Eq. (4.6). This set of parameters is also given in Table II.

Finally, we present our results for R in Fig. 6. In the lowest order in perturbation theory and without mass corrections R would be equal to zero. We see from the graphs that about half the contribution to R comes from the non-leading term in the perturbation expansion and the other half from mass corrections. Despite the large error bars on the data it can be clearly seen that the predicted values are consistently too low (especially at large values of x). This conclusion was also reached in Refs. 8) and 15) *). Figure 7 shows our predictions for R for $x=0.4$ and $x=0.75$ for very large values of Q^2 .

5. CONCLUSIONS

The main conclusion of this paper is that the interpretation of approximate Bjorken scaling of the electroproduction structure functions in terms of the first term in the expansion of $\bar{g}$, the effective coupling constant, is already valid at SLAC values of Q^2 . The effect of the next term in the expansion is small, however this is due to a partial cancellation between individual contributions. The predicted behaviour of $\sqrt{W_2}$ with Q^2 is in good agreement with the data, particularly when corrections due to the target mass are taken into account. Because the higher order terms are already small, it is possible to make predictions for very large values of Q^2 , ones which will be accessible in future experiments.

When the higher order corrections are taken into account, the Callan-Gross relation is violated, and one gets a non-zero prediction for the quantity $R = \sigma_L / \sigma_T$. Since the first order correction to the prediction for the structure functions is small, the prediction for R is reliable. The data for this quantity are very poor, nevertheless the prediction falls consistently below the data, particularly at larger values of x . We have no explanation of this discrepancy.

ACKNOWLEDGEMENTS

The authors are indebted to J. Ellis, K. Gaemers, M.K. Gaillard, and A. Peterman for useful discussions.

*) We disagree with the results of a recent paper by Elias²⁰⁾, whose results have been obtained by neglecting the m^2/Q^2 corrections to the Callan-Gross relation, while considering these corrections in the definition of R . By considering all target mass effects (but neglecting all interaction effects) we have obtained results for R , which are much lower than his.

Table I

Parameters for the best fits to νW_2 in various cases.
The value of Q_0^2 is taken to be 1.8 GeV² throughout.

Case	$\eta_1(0)$	$\eta_2(0)$	$\eta_3(0)$	$\eta_4(0)$	η'_1	η'_2	η'_3	η'_4	Λ
ASF I	0.70	2.60	0.85	3.35	-1.10	5.0	-1.5	5.1	0.3
ASF II	0.70	2.60	0.85	3.35	-0.92	4.73	-1.2	4.93	0.346
MASF I	0.79	2.74	0.94	3.49	-1.31	4.96	-1.64	5.1	0.428
MASF II	0.79	2.74	0.94	3.49	-0.98	5.42	-1.25	5.68	0.474

Table II

Parameters for $F_2 - xF_1$, with and without mass corrections.

Case	$\tilde{\eta}_1(0)$	$\tilde{\eta}_2(0)$	$\tilde{\eta}_3(0)$	$\tilde{\eta}_4(0)$	$\tilde{\eta}'_1$	$\tilde{\eta}'_2$	$\tilde{\eta}'_3$	$\tilde{\eta}'_4$	ϵ^v	$\epsilon^{\bar{d}}$
ASF II	0.39	3.60	0.47	4.37	-0.62	5.86	-0.78	6.15	0.96	0.92
MASF II	0.48	3.75	0.53	4.51	-0.72	5.86	-0.88	6.13	1.32	1.27

R E F E R E N C E S

- 1) H.D. Politzer, Phys.Reports 14, 129 (1974).
- 2) D.J. Gross, Proceedings of the 1975 Les Houches Summer School - "Methods in Field Theory", Eds. R. Balian and J. Zinn-Justin, p.141 (North-Holland Publ., 1976).
- 3) O. Nachtmann, Nuclear Phys. B63, 237 (1973).
- 4) H. Georgi and H.D. Politzer, Phys.Rev. D14, 1829 (1976).
- 5) R. Barbieri, J. Ellis, M.K. Gaillard and G.G. Ross, Phys.Letters 64B, 171 (1976); Nuclear Phys. B117, 50 (1976).
- 6) A. de Rujula, H. Georgi and H.D. Politzer, Annals of Phys. 103, 315 (1977).
- 7) A.J. Buras and K.J.F. Gaemers, CERN preprint TH.2322 (1977).
- 8) M. Calvo, Phys.Rev. D15, 730 (1977).
- 9) E.C.G. Stueckelberg and A. Peterman, Helv.Phys.Acta 26, 499 (1953).
- 10) E.G. Floratos, D.A. Ross and C.T. Sachrajda, CERN preprint TH.2326 (1977).
- 11) D.J. Gross and F. Wilczek, Phys.Rev. D8, 3633 (1973); Phys.Rev. D9, 980 (1974);
H. Georgi and H.D. Politzer, Phys.Rev. D9, 416 (1974).
- 12) D.R.T. Jones, Nuclear Phys. B75, 531 (1974).
- 13) W. Caswell, Phys.Rev.Letters 33, 224 (1974).
- 14) M. Glück and E. Reya, Phys.Letters 64B, 169 (1976); Phys.Rev. D14, 3034 (1976) and Mainz preprint MZ-TH-76-10 (1976).
- 15) I. Hinchliffe and C.H. Llewellyn Smith, Phys.Letters 66B, 281 (1977); Oxford University preprint OUTP36/77 (1977).
- 16) A.J. Buras, CERN preprint TH.2285 (1977).
- 17) S.L. Adler, Phys.Rev. 143, 1144 (1965).
- 18) E.M. Riordan et al., SLAC-PUB-1634 (1975).
- 19) C.H. Llewellyn Smith and B.H. Wiik, DESY preprint 77/38 (1977).
- 20) V. Elias, University of Maryland preprint 77-074 (1977).

FIGURE CAPTIONS

- Figure 1 : Illustration of the importance of various $\bar{g}^2$ contributions for $\Lambda=0.3$ and fixed values of $\eta_i(0)$.
- Figure 2 : νW_2 for various cases discussed in the text for $\Lambda=0.3$ and fixed values of $\eta_i(0)$. The data are from Ref. 18).
- Figure 3 : Best fits for νW_2 for ASF I and MASF II. The data are from Ref. 18).
- Figure 4 : Logical flow chart summarizing the results of Section 3.
- Figure 5 : Predictions for νW_2 at very high Q^2 .
- Figure 6 : Comparison of R to the data of Ref. 18) for MASF II [—] and ASF II [----].
- Figure 7 : Predictions for R at very high Q^2 . The meaning of the curves is as in Fig. 6.

FIG.1

FIG.1 cont.

FIG. 2

FIG. 2 cont.

FIG. 3

FIG. 3 cont.

FIG. 4

FIG. 5

FIG. 6

FIG. 6 cont.

FIG.6 cont.

FIG.7