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PERTURBATION THEORY AT LARGE ORDER
IN REGGEON FIELD THEORY

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A B S T R A C T

We estimate the behaviour of the perturbation expansion and the ϵ expansion at large orders in Reggeon field theory. The perturbation expansion is divergent but Borel summable for $\alpha_0 < 1$. At fixed total rapidity, the Borel summability extends into $\alpha_0 > 1$.

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1. - INTRODUCTION

The recently developed techniques ¹⁾⁻³⁾ for determining the behaviour of perturbation theory at large order in quantum field theory provide a useful tool for extracting reliable estimates from low order perturbation theory. Recently ⁴⁾, we have used these methods to calculate the critical exponents of Reggeon field theory ⁵⁾ [which determine the asymptotic behaviour of diffraction scattering when $\alpha(0)=1$] from the known low order expansions ⁶⁾⁻⁹⁾. In this paper we describe the derivation of the asymptotic formulae which were crucial in carrying out these calculations. We derive asymptotic expressions for the large order coefficients and the renormalization group functions. We also determine the large order behaviour of the ϵ expansion of the critical exponents.

We consider the Reggeon field theory defined by the action

$$A[\bar{\psi}, \psi] = \int dy d^D b \left\{ \frac{1}{2} \bar{\psi} \partial_y^2 \psi + \alpha'_0 \nabla \bar{\psi} \cdot \nabla \psi + \Delta_0 \bar{\psi} \psi + \frac{1}{2} i r_0 \bar{\psi} (\bar{\psi} + \psi) \psi \right\} \quad (1.1)$$

Here y is the rapidity and $\vec{b}$ is the impact parameter. The number D of dimensions of impact parameter space lies between zero and four. We include only a constant triple-Pomeron coupling since this is sufficient to discuss the critical behaviour. We also write

$$\Delta_0 = \Delta + \delta\Delta \quad (1.2)$$

where $\alpha(0)=1-\Delta$ is the physical Pomeron intercept ($\Delta \geq 0$) and $\delta\Delta$ is a counter term. For $D=4$ we should also include a vertex counter term in the Lagrangian to cancel the additional divergences which then arise.

All quantities of interest are expanded in terms of the dimensionless bare (renormalized) coupling $g_0(g)$, formed out of the appropriate bare (renormalized) quantities. The $(\bar{m} \rightarrow m)$ Green function $G^{(\bar{m}, m)}$ is, in the case $(\bar{m}+m)$ even, an alternating series in g_0^2 ; and in the case $(\bar{m}+m)$ odd, equal to g_0 times an alternating series in g_0^2 . The large order behaviour of such a series can be found by assuming a dispersion relation for the Green functions in g_0^2 . For the case $(\bar{m}+m)$ even, this will take the form

$$G^{(\bar{m},m)}(g_0^2) = \frac{1}{\pi} \int_{-\infty}^0 \frac{dg_0'^2 \operatorname{Im} G^{(\bar{m},m)}(g_0'^2)}{g_0'^2 - g_0^2} \quad (1.3)$$

Subtractions may be necessary, but they do not affect the argument. The coefficient of $(-g_0^2)^n$ in $G^{(\bar{m},m)}$ can then be written

$$G_n^{(\bar{m},m)} = \frac{1}{\pi} \int_{-\infty}^0 \frac{dg_0'^2}{(-g_0'^2)^{n+1}} \operatorname{Im} G^{(\bar{m},m)}(g_0'^2) \quad (1.4)$$

From (1.4) it is clear that the $n \rightarrow \infty$ limit of $G_n^{(\bar{m},m)}$ is controlled by the behaviour of $G^{(\bar{m},m)}(g_0^2)$ as $g_0^2 \rightarrow 0^-$, i.e., for a small unphysical coupling constant. This behaviour is found by applying the method of steepest descent to the path integral representation of the Green functions. Doing this, we obtain

$$G^{(\bar{m},m)}(g_0^2) \underset{g_0^2 \rightarrow 0^-}{\sim} i \frac{\bar{C}^{(\bar{m},m)}}{(-g_0^2)^b} e^{\bar{A}/g_0^2} \quad (1.5)$$

In (1.5), $\bar{A}$ is a number which depends only on the dimensionality D , b is an integer or half-integer which is simply related to D and the number $(\bar{m}+m)$ of external lines, and $\bar{C}^{(\bar{m},m)}$ is a function of the coordinates (or momenta), but independent of g_0^2 . Substituting in (1.4) we find

$$G_n^{(\bar{m},m)} \underset{n \rightarrow \infty}{\sim} C^{(\bar{m},m)} \Gamma(n+b) \bar{A}^{-n} (1 + O(1/n)) \quad (1.6)$$

This equation is also true for $(\bar{m}+m)$ odd if we take half-integer values for n .

Once the large order behaviour of the bare perturbation expansion of the Green functions is established, certain simple properties of series which diverge like $n!$ enable us to deduce the large order behaviour of the renormalized perturbation series, and the renormalization group functions. They have the same character as (1.6). Only the parameter b depends on which function is being calculated.

That the perturbation expansion for RFT should diverge roughly like $n!$ was first shown by Suranyi¹⁰⁾, who established bounds on the n^{th} order coefficient in the expansion of the two-point function. Such a behaviour also agrees with the exact calculations of Bronzan, Shapiro and Sugar¹¹⁾ for the case $D=0$. In fact, the values of $\bar{A}$ and b which we calculate agree with the exact results. This is the first analytic check of the method of Lipatov¹⁾ in a non-trivial theory. (Previous checks for the anharmonic oscillator have been numerical¹²⁾.)

The lay-out of this paper is as follows. In Section 2 we present a detailed calculation of the $D=0$ case. This is the simplest to deal with as all quantities can be evaluated exactly. Section 3 deals with the renormalized perturbation expansion in $0 < D < 4$ for $\Delta > 0$. This is relevant to the calculation of the critical exponents in the loop expansion. Section 4 covers the case $D=4, \Delta=0$, from which can be deduced the large order behaviour of the ϵ expansion. All our results are summarized in the Table^{*)}. Since we have in mind the application to the calculation of the exponents, we are primarily interested in the Green functions in E space. In an Appendix, we show that the perturbation expansion for the two point function at fixed rapidity also diverges in much the same way. In Section 5 we discuss the relevance of our results to the problem of the definition and behaviour of RFT for $\alpha(0) > 1$.

2. - ZERO TRANSVERSE DIMENSIONS

In the case of zero transverse dimensions, the action (1.1) becomes

$$A[\bar{\psi}, \psi] = \int dy \left\{ \frac{1}{2} \bar{\psi} \overleftrightarrow{\partial}_y \psi + \Delta_0 \bar{\psi} \psi + \frac{1}{2} i r_0 \bar{\psi} (\bar{\psi} + \psi) \psi \right\} \quad (2.1)$$

*) The values for the indices b we obtain differ by 1 from those quoted in Ref. 4). To within the stated errors, this does not affect our estimates of the critical exponents.

In what follows, we shall be interested in the perturbation expansion in the dimensionless coupling constant $g_0 = r_0/\Delta_0$ for the case $\Delta_0 > 0$. We therefore set $\Delta_0 = 1$ and $r_0 = g_0$ for convenience. The connected Green functions are defined by

$$G^{(n,m)}(\bar{y}_1, \dots, \bar{y}_m; y_1, \dots, y_n) = \frac{\delta^{n+m} W}{\delta \bar{J}(\bar{y}_1) \dots \delta \bar{J}(\bar{y}_m) \delta J(y_1) \dots \delta J(y_n)} \Big|_{J=\bar{J}=0} \quad (2.2)$$

where

$$\exp W[\bar{J}, J] = \int D\bar{\psi} D\psi \exp \left\{ -A[\bar{\psi}, \psi] + \int dy (\bar{J}\bar{\psi} + J\psi) \right\} \quad (2.3)$$

To discuss (vacuum $\rightarrow$ vacuum) Green functions, we can impose the boundary conditions $\psi(-\infty) = \bar{\psi}(+\infty) = 0$.

As discussed in the Introduction, we are interested in the behaviour of (2.3) as $g_0^2 \rightarrow 0^-$. For this purpose we introduce $\lambda_0 = ig_0$ and look for saddle points of the functional integral in the limit of zero sources. These are given by the solutions of the classical equations

$$\begin{aligned} \partial_y \psi + \psi &= -\lambda_0 (\bar{\psi}\psi + \tfrac{1}{2}\psi^2) \\ -\partial_y \bar{\psi} + \bar{\psi} &= -\lambda_0 (\bar{\psi}\psi + \tfrac{1}{2}\bar{\psi}^2) \end{aligned} \quad (2.4)$$

Substituting $\psi = -\phi/\lambda_0$, $\bar{\psi} = -\bar{\phi}/\lambda_0$, these reduce to the parameter-free form

$$\begin{aligned} \partial_y \phi + \phi &= \bar{\phi}\phi + \tfrac{1}{2}\phi^2 \\ -\partial_y \bar{\phi} + \bar{\phi} &= \bar{\phi}\phi + \tfrac{1}{2}\bar{\phi}^2 \end{aligned} \quad (2.5)$$

which can be derived from the action

$$\bar{A}[\bar{\phi}, \phi] = \int dy \left[\tfrac{1}{2} \bar{\phi} \partial_y \phi + \bar{\phi}\phi - \tfrac{1}{2} \bar{\phi} (\bar{\phi} + \phi) \phi \right] \quad (2.6)$$

The action at the saddle point is then

$$A = \bar{A} / \lambda_0^2 \quad (2.7)$$

where $\bar{A}$ is the extremum value of (2.6). To study fluctuations about the saddle point we have to deal with the zero mode problem. This is because the equations (2.5) are translationally invariant in y , and therefore have the solution $\{\bar{\varphi}(y-y_0), \varphi(y-y_0)\}$ where $\{\bar{\varphi}(y), \varphi(y)\}$ is some standard solution. In principle we must integrate over all y_0 . Let us expand $\{\bar{\psi}, \psi\}$ about a particular classical solution $\{\bar{\psi}_c, \psi_c\}$ as

$$\begin{aligned} \bar{\psi} &= \bar{\psi}_c + (\partial \bar{\psi}_c / \partial y) \delta y_0 + \bar{\psi}' \\ \psi &= \psi_c + (\partial \psi_c / \partial y) \delta y_0 + \psi' \end{aligned} \quad (2.8)$$

where $\{\bar{\psi}', \psi'\}$ is orthogonal to $\{\partial \bar{\psi}_c / \partial y, \partial \psi_c / \partial y\}$ in function space. Noting that the imaginary part of the field $\frac{1}{2}(\psi_c - \bar{\psi}_c)$ is in fact purely real at the saddle point, the measure for the functional integration then becomes

$$\mathcal{D}\bar{\psi} \mathcal{D}\psi \rightarrow i \left\langle \left\{ \left(\frac{\partial \bar{\psi}_c}{\partial y} \right)^2 + \left(\frac{\partial \psi_c}{\partial y} \right)^2 \right\}^{\frac{1}{2}} \right\rangle_{y_0} dy_0 \mathcal{D}\bar{\psi}' \mathcal{D}\psi' \quad (2.9)$$

where $\langle \rangle_{y_0}$ denotes an average over y_0 . The integral over y_0 gives a factor of the total rapidity which cancels in the evaluation of the Green functions. The Jacobian in (2.9) is of order $1/\lambda_0$. The remaining integral over $\{\bar{\psi}', \psi'\}$ should now be well defined. Retaining only quadratic terms in the exponent, it has the Gaussian form

$$\int \mathcal{D}\bar{\psi}' \mathcal{D}\psi' \exp \{ -A'[\bar{\psi}', \psi'] \} \quad (2.10)$$

where

$$\begin{aligned} A'[\bar{\psi}', \psi'] = \int dy \{ & \frac{1}{2} \bar{\psi}' \partial_y^2 \psi' + \bar{\psi}' \psi' + \lambda_0 (\bar{\psi}_c + \psi_c) \bar{\psi}' \psi' \\ & + \frac{1}{2} \lambda_0 \bar{\psi}_c \psi'^2 + \frac{1}{2} \lambda_0 \psi_c \bar{\psi}'^2 \} \end{aligned} \quad (2.11)$$

and the ψ' , $\bar{\psi}'$ contours run parallel to the imaginary axis.

The important point about (2.11) is that it is independent of λ_0 . Thus the fluctuations about the saddle point contribute only a constant factor to (2.3). Since the fluctuations are $O(1)$, and the fields at the saddle point are $O(1/\lambda_0)$, the use of the method of steepest descent is justified as $\lambda_0 \rightarrow 0$.

The sources $\bar{J}, J$ in (2.3) will couple to both the classical field $\{\bar{\psi}_c, \psi_c\}$ and the fluctuations $\{\bar{\psi}', \psi'\}$. If we consider the two-point function $G^{(1,1)}(\bar{y}, y)$ we would expect the leading contribution as $\lambda_0 \rightarrow 0$ to be proportional to

$$\langle \psi_c(y) \bar{\psi}_c(\bar{y}) \rangle_{y_0} - \langle \psi_c(y) \rangle_{y_0} \langle \bar{\psi}_c(\bar{y}) \rangle_{y_0} \quad (2.12)$$

which is of order $1/\lambda_0^2$. However, in the case of RFT, this leading term vanishes (it does not vanish for ϕ^{2N} theory). The only non-trivial solution of (2.5) which satisfies the boundary conditions $\phi(-\infty) = \bar{\phi}(+\infty) = 0$ is ¹³⁾

$$\begin{aligned} \bar{\phi} &= 1 - \tanh(y/2) \\ \phi &= 1 + \tanh(y/2) \end{aligned} \quad (2.13)$$

For sufficiently large rapidities $(y - \bar{y})$ we must choose $y \gg \gg y_0 \gg \bar{y}$, and we see that $\bar{\psi}_c(\bar{y})$ and $\psi_c(y)$ are effectively constant, independent of y_0 , so that (2.12) vanishes. At fixed rapidity the situation is more complicated - however, we are primarily interested in evaluating the Green functions at $E=0$, so the infinite rapidity limit is relevant.

The dominant term therefore comes from the fluctuations $\langle \psi'(y) \bar{\psi}'(\bar{y}) \rangle_{y_0}$, which is $O(1)$. Since, according to (2.11), the fluctuations have a positive energy gap, this coefficient will decay exponentially as $(\bar{y} - y) \rightarrow \infty$.

If we consider a higher Green function

$$G^{(\bar{m}, m)}(\bar{y}_1, \dots, \bar{y}_m; y_1, \dots, y_m)$$

where

$$y_1 > \dots > y_m > y_m > \dots > \bar{y}_1$$

the leading terms proportional to $\bar{\psi}_c(\bar{y}_1)$ and $\psi_c(y_1)$ will cancel, leaving a dominant term

$$\langle \psi'(y_1) \psi_c(y_2) \dots \psi_c(y_m) \bar{\psi}_c(\bar{y}_m) \dots \bar{\psi}_c(\bar{y}_2) \bar{\psi}'(\bar{y}_1) \rangle_{y_0} \quad (2.14)$$

The intermediate fields are not independent of y_0 , and so do not completely cancel - they remain correlated with the fluctuations $\bar{\psi}'(\bar{y}_1)$ and $\psi'(y_1)$. Equation (2.14) still decays exponentially as $(y_1 - \bar{y}_1) \rightarrow \infty$, and is proportional to $1/\lambda_0^{\bar{m}+m-2}$.

In conclusion, the behaviour of $G^{(\bar{m},m)}$ as $\lambda_0 \rightarrow 0$ is

$$G^{(\bar{m},m)}(\bar{y}_j, y_i) \sim i \frac{e^{-\bar{A}/\lambda_0^2}}{\lambda_0^{\bar{m}+m-1}} \bar{C}^{(\bar{m},m)}(\bar{y}_j, y_i) \quad (2.15)$$

where $\bar{C}^{(\bar{m},m)} \rightarrow 0$ exponentially as $\max_{i,j}(\bar{y}_j - y_i) \rightarrow \infty$.

In order to evaluate the large n behaviour of the coefficient of $(-g_0^2)^n$ in $G^{(\bar{m},m)}$ we substitute (2.15) in Eqs. (1.4). For $(\bar{m}+m)$ even, we have, putting $u = g'^2$,

$$G_n^{(\bar{m},m)} \sim \bar{C}^{(\bar{m},m)} \int_{-\infty}^0 du (-u)^{-n-1-(\bar{m}+m-1)/2} e^{\bar{A}/u} \quad (2.16)$$

or

$$G_n^{(\bar{m},m)} \sim C^{(\bar{m},m)} \Gamma(n + \frac{\bar{m}+m-1}{2}) \bar{A}^{-n} \quad (2.17)$$

where

$$C^{(\bar{m},m)} = \bar{A}^{-(\bar{m}+m-1)/2} \bar{C}^{(\bar{m},m)} \quad (2.18)$$

For $(\bar{m}+m)$ odd we obtain

$$G_{n+1/2}^{(\bar{m},m)} \sim C^{(\bar{m},m)} \Gamma(n + \frac{\bar{m}+m}{2}) \bar{A}^{-n} \quad (2.19)$$

Equations (2.18) and (2.19) are special cases of (1.6) when $D=0$. Since the $G^{(\bar{m},m)}$ decay exponentially at large rapidities, we can perform the Laplace transform of $G_n^{(\bar{m},m)}$ in the large n limit to obtain the behaviour of the $E=0$ Green functions. For this case we can compare with the exact $D=0$ results of Bronzan et al.¹¹⁾. Noting that $\bar{A}=2$ for the solution (2.13), we have the prediction

$$G_n^{(1,1)}(E=0) \sim \text{constant} \times \Gamma(n+\frac{1}{2}) 2^{-n} [1 + O(1/n)] \quad (2.20)$$

The exact formula is¹¹⁾

$$G^{(1,1)}(E=0) = \int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{1 + i g_0 x / \sqrt{2}} \quad (2.21)$$

Remarkably, the asymptotic estimate (2.20) is seen to be true for all n , with no $O(1/n)$ corrections. This is a striking non-trivial check of the Lipatov method. We have also checked against the exact results for the Green functions

$$\tilde{G}^{(2,1)}(E=0) = \int_0^{\infty} dy \langle 0 | \psi(y)^2 \bar{\psi}(0) | 0 \rangle \quad (2.22)$$

and

$$\begin{aligned} G^{(2,1)}(E_1 = E_2 = 0) \\ = \int_0^{\infty} dy_1 \int_0^{y_1} dy_2 \langle 0 | \psi(y_1) \psi(y_2) \bar{\psi}(0) | 0 \rangle \end{aligned} \quad (2.23)$$

and find agreement.

Our calculation in $D=0$ has several features which are quite general. First, the number $\bar{A}$ in (1.6) is just the action of a solution of the equations of motion with $\Delta_0=1$ and $ir_0=-1$. Second, the number b has contributions both from the Jacobian in the transformation to collective co-ordinates (2.9), and from the number $(\bar{m}+m)$ of external lines. There is, however, a difference from ϕ^{2N} theory in that the relevant contribution is $(\bar{m}+m-2)$ rather than $(\bar{m}+m)$. This can be traced back to the non-relativistic boundary conditions which are relevant for RFT. There is, however, a paradox which we have been unable to resolve. The Green functions $\langle 0 | \psi(y) | 0 \rangle$ and $\langle 0 | \bar{\psi}(y) | 0 \rangle$ are identically zero in perturbation theory. Nevertheless if we calculate their large n behaviour we find a non-zero contribution proportional to $\langle \psi_c(y) \rangle_{y_0}$. We regard this as a fundamental deficiency of our path integral definition (2.2), (2.3).

3. - RENORMALIZATION GROUP FUNCTIONS FOR $0 < D < 4$

When $D < 4$, it is convenient to calculate the renormalization group functions in the theory with $\Delta > 0$, and infer from them the infra-red behaviour of the massless theory ^{6),7)}. The theory is renormalized by using the physical mass gap Δ in the propagator, and including a counter-term $\delta\Delta$ which is adjusted at each order of perturbation theory so that the pole in the propagator remains at Δ . Initially the Green functions are expanded in the dimensionless bare coupling $g_0 = r_0/\Delta^{1-D/4} \alpha_0^{D/4}$. The renormalized truncated one-particle irreducible Green functions $\Gamma_R^{(\bar{m}, m)}$ are related to the bare Green functions by

$$\Gamma_R^{(\bar{m}, m)} = Z^{(\bar{m}+m)/2} \Gamma^{(\bar{m}, m)} \quad (3.1)$$

where the wave function renormalization Z is fixed by

$$\left. \frac{\partial}{\partial E} i \Gamma_R^{(1,1)} \right|_{E=\vec{k}=0} = 1 \quad (3.2)$$

Thus

$$Z^{-1} = \left. \frac{\partial}{\partial E} i \Gamma^{(1,1)} \right|_{E=\vec{k}=0} = -[G^{(1,1)}]^{-2} \left. \frac{\partial}{\partial E} G^{(1,1)} \right|_{E=\vec{k}=0} \quad (3.3)$$

The renormalized slope is defined by

$$\left. \frac{\partial}{\partial k^2} i \Gamma_R^{(1,1)} \right|_{E=\vec{k}=0} = -\alpha' \quad (3.4)$$

and the renormalized dimensionless coupling by

$$g = \frac{(2\pi)^{(D+1)/2}}{\Delta^{1-D/4} \alpha_0^{D/4}} \left. \Gamma_R^{(1,2)} \right|_{E=\vec{k}=0} \quad (3.5)$$

Once we know the expansion coefficients of $G^{(1,1)}$ and $G^{(1,2)}$ in g_0^2 , Eqs. (3.1)-(3.5) determine the expansions of Z , α' and g . Of these quantities, only the expansion of g involves knowledge of $G^{(1,2)}$. The renormalization group functions are

$$\beta(g) = \Delta \frac{\partial}{\partial \Delta} g \Big|_{r_0, \alpha'_0} \quad (3.6)$$

$$\gamma(g) = \Delta \frac{\partial}{\partial \Delta} \ln Z \Big|_{r_0, \alpha'_0} \quad (3.7)$$

$$\zeta(g) = \frac{\Delta}{\alpha'_0} \frac{\partial}{\partial \Delta} \alpha' \Big|_{r_0, \alpha'_0} \quad (3.8)$$

which can be expressed as power series in g^2 by inverting Eq. (3.5). If g_1 is a zero of $\beta(g)$ such that $\beta'(g_1) > 0$, the critical exponents are given by

$$\gamma = \gamma(g_1), \quad z = 1 - \zeta(g_1), \quad \lambda = \beta'(g_1) \quad (3.9)$$

We wish to determine the large order behaviour of the expansions of the renormalization group functions $\beta(g)$, $\gamma(g)$, $\zeta(g)$. First we repeat the argument of Section 2, to obtain the large n behaviour of $G_n^{(\bar{m}, m)}$. The main difference will be that the fluctuations about the saddle point will have $(1+D)$ zero modes, corresponding to translational invariance in rapidity and impact parameter, so that the Jacobian will now be of order $1/\lambda_0^{1+D}$. $\bar{A}$ will be given by the extremum of

$$\bar{A}[\bar{\phi}, \phi] = \int dy db \left\{ \frac{1}{2} \bar{\phi} \Delta_y \phi + \nabla \bar{\phi} \cdot \nabla \phi + \bar{\phi} \phi - \frac{1}{2} \bar{\phi} (\bar{\phi} + \phi) \phi \right\} \quad (3.10)$$

We can neglect the counter terms in the computation since they are $O(\lambda_0^2)$ as $\lambda_0 \rightarrow 0$. As a result

$$G_n^{(\bar{m}, m)} \sim C^{(\bar{m}, m)} \Gamma\left(n + \frac{\bar{m} + m - 1 + D}{2}\right) \bar{A}^{-n} \quad (3.11)$$

We define any power series in g_0^2 for which the coefficient of $(-g_0^2)^n$ for large n behaves like $\Gamma(n+b)\bar{A}^{-n}$ to have index b . Thus $G^{(1,1)}$ has index $(1+D)/2$, and $(G^{(1,2)}/g_0)$ has index $(3+D)/2$. It is a remarkable fact (which we leave to the reader to demonstrate) that when we perform simple algebraic manipulations on such series, the resulting series has the same large n behaviour, with an index equal to the maximum of the

indices of the participating series. Thus we see that, as power series in g_0^2 , Z and α' have index $(1+D)/2$, while (g/g_0) has index $(3+D)/2$. Since the renormalization group functions (3.6)-(3.8) involve a differentiation with respect to Δ , they each pick up an extra factor of n . Therefore, as power series in g_0^2 , γ and ζ have index $(3+D)/2$, and β has index $(5+D)/2$. When we invert (g_0/g) as a power series in g^2 , it will have index $(3+D)/2$. Therefore the renormalization group functions have the same indices as power series in g^2 as they do in g_0^2 . We conclude that

$$\beta(g)/g = \dots + \sum_{n \text{ large}} C_\beta \Gamma(n + \frac{5+D}{2}) (-g^2/\bar{A})^n \quad (3.12)$$

$$\gamma(g) = \dots + \sum_{n \text{ large}} C_\gamma \Gamma(n + \frac{3+D}{2}) (-g^2/\bar{A})^n \quad (3.13)$$

$$\zeta(g) = \dots + \sum_{n \text{ large}} C_\zeta \Gamma(n + \frac{3+D}{2}) (-g^2/\bar{A})^n \quad (3.14)$$

where $C_\beta, C_\gamma, C_\zeta$ are constants.

We conclude this section with a discussion of the calculation of $\bar{A}$. Unfortunately for $D > 0$ no exact solutions of the equations of motion are available. Indeed, it is not even clear that they exist, because of the lack of positivity of $\bar{A}[\vec{\phi}, \vec{\phi}]$. However, assuming existence, the most efficient way to estimate $\bar{A}$ is variationally. Since $\bar{A}$ is an extremum, the use of rather crude trial functions should suffice. In order to eliminate the lack of positivity resulting from the term $\vec{\phi} \vec{\phi}_y$ in the Lagrangian, we chose trial functions of the form

$$\phi(y, \vec{b}) = f(y) u(\vec{b}) \quad , \quad \bar{\phi}(y, \vec{b}) = \bar{f}(y) u(\vec{b}) \quad (3.15)$$

The rapidity and impact parameter dependence then decouple, and the variational problem for f and $\bar{f}$ (being equivalent to a $D=0$ RFT) can be solved exactly. As described in Ref. 4) this leads to a well-defined variational problem for u :

$$\bar{A}[u] = 2 \left\{ \frac{\int [(\nabla u)^2 + u^2] d^D b}{\int u^3 d^D b} \right\}^2 (\int u^2 d^D b) \quad (3.16)$$

which, for $D=2$, yields $\bar{A}/4\pi \approx 4.47$.

4. - D = 4 AND THE LARGE ORDER BEHAVIOUR OF THE ϵ EXPANSION

In $4-\epsilon$ dimensions it is conventional ⁸⁾ to calculate the renormalization group functions in the massless theory ($\Delta = 0$). For $D = 4$ the theory is just renormalizable and there are many more counter terms. An expansion in r_0 is only legitimate in the cut-off theory. However, as before, the counter terms will not change the leading order in n calculation. We have to assume that the limits (cut-off $\rightarrow \infty$) and $n \rightarrow \infty$ are interchangeable, as is done by Lipatov ¹⁾ and Brezin et al. ²⁾ for ϕ^{2N} theory. With this assumption the calculation goes as in Sections 2, 3. The only additional complication is that the $\Delta = 0$ theory is dilatation invariant in $D = 4$, giving rise to an extra zero mode. We then find that in $D = 4$, $\beta(g)/g$ has index 5, and $\gamma(g)$ and $\zeta(g)$ have index 4. The parameter $\bar{\Lambda}$ can be estimated as before, remembering that $\Delta = 0$. The equivalent variational calculation gives

$$\bar{\Lambda}[u] = 2 \left\{ \frac{\int (\nabla u)^2 d^4b}{\int u^2 d^4b} \right\}^2 (\int u^2 d^4b) \quad (4.1)$$

from which we obtain $\bar{\Lambda}/8\pi^2 \approx 9.22$.

We now turn to the calculation of the exponents in the ϵ expansion. As emphasized by Bronzan and Dash ⁸⁾, the ϵ expansion of the renormalization group functions away from the fixed point g_1 depends on the renormalization prescription. We find it convenient to choose a different definition from the conventional one ⁸⁾. The normalization conditions, instead of being imposed in $4-\epsilon$ dimensions, will be imposed after continuing the Green functions back to $D = 4$. Thus, we impose at the normalization point $E = -E_N$, $\vec{k} = 0$,

$$\left. \frac{\partial}{\partial E} i \Gamma_R^{(4,1)} \right|_{D=4} = 1 \quad (4.2)$$

$$\left. \frac{\partial}{\partial k^2} i \Gamma_R^{(4,1)} \right|_{D=4} = -\alpha' \quad (4.3)$$

$$\left. \Gamma_R^{(1,2)} \right|_{D=4} = r / (2\pi)^{D+D/2} \quad (4.4)$$

The dimensionless renormalized coupling is defined as before by

$$g(E_N) = r E_N^{D/4-1} \alpha'^{-D/4} \quad (4.5)$$

Note that the only D dependence in g will come from these trivial factors. Z and α' are independent of D. Therefore we have

$$\beta(g, \epsilon) = -(\epsilon/4)g + \beta_4(g) \quad (4.6)$$

$$\gamma(g, \epsilon) = \gamma_4(g) \quad \zeta(g, \epsilon) = \zeta_4(g) \quad (4.7)$$

where $\beta_4, \gamma_4, \zeta_4$ are the D=4 RG functions. Thus the only ϵ dependence is in the first term of $\beta(g)$. We can write down the power series expansions of the RG functions by setting $\epsilon=0$ in the results of Bronzan and Dash⁸⁾. The zero of $\beta(g, \epsilon)$ then satisfies

$$\beta_4(g_1)/g_1 = \epsilon/4 \quad (4.8)$$

To first order⁸⁾

$$g_1^2 = (32\pi^2/3)\epsilon + O(\epsilon^2) \quad (4.9)$$

Therefore (4.9) can be written

$$g_1^2 = (32\pi^2/3)\epsilon + a_2 g_1^4 + \dots \\ \dots + \sum_{n \text{ large}} C_\beta \Gamma(n+5) (-g_1^2/\bar{A})^n \quad (4.10)$$

If we solve (4.10) iteratively, the dominant term proportional to ϵ^n will arise from substituting the lowest order expression (4.9) into the term proportional to $(-g_1^2)^n$ in (4.11). Thus

$$g_1^2 = (32\pi^2/3)\epsilon + \dots + \sum_{n \text{ large}} C_\beta \Gamma(n+5) (-\epsilon/R)^n \quad (4.11)$$

where

$$R = 3\bar{A} / 32\pi^2 \approx 6.91$$

Similar considerations now show that when we substitute (4.11) in the expansions for $\gamma(g_1)$, $\zeta(g_1)$, since they have index 4, the dominant term comes from the $O(\epsilon^n)$ term in g_1^2 .

$$\gamma = -\frac{\epsilon}{12} + \dots + \sum_{n \text{ large}} (-C_\beta / 128\pi^2) \Gamma(n+5) (-\epsilon/R)^n \quad (4.12)$$

$$\zeta = -\frac{\epsilon}{24} + \dots + \sum_{n \text{ large}} (-C_\beta / 256\pi^2) \Gamma(n+5) (-\epsilon/R)^n \quad (4.13)$$

The exponent $\lambda = \beta'(g_1)$ has a similar character

$$\lambda = \frac{\epsilon}{2} + \dots + \sum_{n \text{ large}} (3C_\beta / 64\pi^2) \Gamma(n+5) (-\epsilon/R)^n \quad (4.14)$$

5. - DISCUSSION

We have shown how to derive the asymptotic behaviour of the perturbation series in RFT, with a view to the practical calculation of the exponents ⁴⁾. It is also interesting to study the general implications. The perturbation expansion in RFT is always divergent (even at fixed rapidity - see Appendix), but, at least for $\Delta_0 > 0$, it is Borel summable. If one repeats our $D=0$ calculation for $\Delta_0 < 0$ one obtains the same result but with $\bar{A} = -2$. This implies that the Borel transform has a singularity on the positive real axis, and so the full Green functions cannot be reconstructed from the perturbation expansion ³⁾. This is consistent with the work of Bronzan et al. ¹¹⁾, who find, for $\Delta_0 < 0$,

$$G^{(n)}(E=0) = (8\pi)^{\frac{1}{2}} r_0^{-1} \exp(-2\Delta_0^2/r_0^2) + \text{perturbation theory} \quad (5.1)$$

The additional term is zero in every order of perturbation theory, but its Borel transform has a singularity at $+2$, as expected. Its presence is associated with the existence of a low-lying excited state as $r_0 \rightarrow 0$ with $\Delta_0 < 0$, with energy $\sim \exp(-2\Delta_0^2/r_0^2)$. In fact, it is possible to argue¹⁴⁾, along the same lines as Polyakov¹⁵⁾ for ϕ^4 theory, that this small energy gap arises because of tunnelling between the two field configurations $(\psi = 2i\Delta_0/r_0, \bar{\psi} = 0)$ and $(\psi = 0, \bar{\psi} = 2i\Delta_0/r_0)$. The "instanton" which connects these two configurations at $y = -\infty$ and $y = +\infty$ is just the analytic continuation into $\Delta_0 < 0$ of the classical solution we used in our saddle point calculation in Section 2¹⁶⁾.

When $D > 0$, one cannot argue the existence of an instanton solution for $\Delta_0 < 0$ on the basis of a classical solution for $\Delta_0 > 0$, r_0 imaginary, because of the additional parameter α'_0 . A variational approximation would break down because of the lack of positivity. For example, the analogue of (3.16) would have a minus sign in front of the u^2 . This is as it should be, since the existence of an instanton for $\Delta_0 < 0$, r_0 real, would argue against the presence of a phase transition³⁾, which we believe exists for $D \geq 1$.

We note that in order to obtain the correct continuation of the theory into $\Delta_0 > 0$, r_0 imaginary, we had to consider the path integral representation for the connected Green functions, even though they are identically equal to the complete Green functions at any finite order in perturbation theory. If we had considered only the complete Green functions, in the limit $r_0^2 \rightarrow 0^-$ the two-point function would be proportional to

$$\langle \psi_c(y) \bar{\psi}_c(\bar{y}) \rangle_{y_0} \quad (5.2)$$

Since $\psi_c(+\infty) = \bar{\psi}_c(-\infty) \neq 0$, this would have unphysical singularities as $E \rightarrow 0$. A similar situation seems to occur for the continuation into $\Delta_0 < 0$, r_0 real¹⁴⁾: one must consider the connected Green functions in order to reproduce the tunnelling effect for $D = 0$.

Finally, we showed in the Appendix that, unlike the case of ϕ^4 theory, the perturbation expansion in RFT for the Green functions in configuration space (that is at fixed rapidity Y) for negative mass ($\Delta_0 < 0$) is Borel summable, and the Green functions can in principle be recovered.

This suggests that the theory is in fact well defined for $\Delta_0 < 0$ without the introduction of external sources, by analytic continuation in Δ_0 at fixed Y . However, the radius of convergence of the Borel transform goes to zero like $e^{\Delta_0 Y}$, as $Y \rightarrow \infty$ for $\Delta_0 < 0$, indicating the non-commutativity of the continuation in Δ_0 and the limit $Y \rightarrow \infty$.

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TABLE

	$\bar{A}$	$G^{(\bar{m},m)}(g_0)$	$\beta(g)/g$	$\gamma(g)$	$\zeta(g)$	$\lambda(g)$
$D = 0$	2	$(\bar{m}+m-1)/2$	5/2	3/2	3/2	7/2
$0 < D < 4$	*4.47(4 π) ($D = 2$)	$(\bar{m}+m-1+D)/2$	$(5+D)/2$	$(3+D)/2$	$(3+D)/2$	$(7+D)/2$
$D = 4$	*9.22(8 π^2)	$(\bar{m}+m+4)/2$	5	4	4	6
ϵ expansion	*6.91	-	-	5	5	5

A typical series diverges like $\Gamma(n+b)(-g^2/\bar{A})^n$. The values of $\bar{A}$ and b are listed above. The first column gives the value of $\bar{A}$ for each expansion; the remaining columns list the indices b . Asterisks denote approximate values. For $(\bar{m}+m)$ even, n takes on integer values; for $(\bar{m}+m)$ odd, half-integer values. For the renormalization group functions, n is an integer. In the last row we give the ϵ expansion of $\gamma(g_1)$, $\zeta(g_1)$ and $\lambda(g_1)$.

APPENDIX

We consider the propagator $\langle 0 | \psi(Y) \bar{\psi}(0) | 0 \rangle$ at fixed Y and zero transverse momentum. The analysis can be extended to non-zero transverse momentum. To make every diagram finite, we impose a cut-off in the transverse momentum integrations, by choosing a propagator

$$\exp - [(\alpha'_0 y + R^2) \vec{k}^2 + \Delta_0 y] \quad (A.1)$$

Since each internal rapidity gap satisfies $0 \leq y \leq Y$, we have

$$\begin{aligned} \exp - [(\alpha'_0 Y + R^2) \vec{k}^2 + \Delta_0 Y] &\leq \exp - [(\alpha'_0 y + R^2) \vec{k}^2 + \Delta_0 y] \\ &\leq \exp - [R^2 \vec{k}^2 + \Delta_0 y] \end{aligned} \quad (A.2)$$

Consider a diagram of $O(r_0^{2n})$ with n loops. With a propagator proportional to $e^{-R^2 \vec{k}^2}$, the $\vec{k}$ integrations give

$$\pi^{Dn/2} R^{-Dn} C^{-D/2} \quad (A.3)$$

where C is the determinant of the diagram, and is the number of distinct ways of placing n cuts on the internal lines so that the diagram becomes a tree diagram¹⁷⁾. Since the diagram has $(3n-1)$ internal lines, we have

$$1 < C \leq \binom{3n-1}{n} \quad (A.4)$$

For $n \rightarrow \infty$, the right-hand side behaves like $(3\pi n)^{-\frac{1}{2}} (27/4)^n$. Therefore, if $F_n^{(D)}$ is any diagram of order r_0^{2n} in D dimensions, we have the bounds

$$(3\pi)^{D/4} n^{D/4} \left[\frac{4\pi}{27(\alpha'_0 Y + R^2)} \right]^{nD/2} \leq \frac{F_n^{(D)}}{F_n^{(0)}} \leq (\pi/R^2)^{nD/2} \quad (A.5)$$

We now concentrate on the case $D=0$. The diagrams are generated by expanding¹¹⁾

$$G^{(4)}(Y) = \langle 0 | a e^{-HY} a^\dagger | 0 \rangle \quad (A.6)$$

where

$$H = \Delta_0 a^\dagger a + \frac{1}{2} i r_0 (a^{\dagger 2} a + a^\dagger a^2)$$

and

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0 \quad (A.7)$$

Writing

$$G^{(n)}(Y) = \sum_{n=0}^{\infty} (-1)^n f_n(\Delta_0, Y) r_0^{2n} \quad (A.8)$$

$$f_n = \int \pi dy_i \langle 0 | P(Y - y_{2n}) \times P(y_{2n} - y_{2n-1}) \dots \times P(y_1) | 0 \rangle \quad (A.9)$$

where the integration is over $0 \leq y_1 \leq \dots \leq y_{2n} \leq Y$, and

$$x = \frac{1}{2}(a^\dagger + a) \\ P(y) = a e^{-\Delta_0 y a^\dagger a} a^\dagger = (N+1) e^{-\Delta_0 y (N+1)} \quad (A.10)$$

where N is the number operator $a^\dagger a$.

Taking either a or $a^\dagger$ at the different times y_i in (A.9) we obtain 2^{2n} contributions, each of which is positive and corresponds to a sum of a subset of diagrams. Consider the contribution B_n from taking $a^\dagger$ at $y_1, \dots, y_n$ and a at $y_{n+1}, \dots, y_{2n}$. These are the diagrams considered by Bronzan¹⁸⁾, which can be divided into two trees by a cut through the $(n+1)$ particle intermediate state. They always give a lower bound to f_n . We get an upper bound by noting that each of the 2^{2n} contributions is smaller than B_n , evaluated at $\Delta_0 = 0$ for $\Delta_0 \geq 0$, and for the true Δ_0 for $\Delta_0 < 0$. That is

$$B_n(\Delta_0) < f_n(\Delta_0) < 2^{2n} B_n(0) \quad (\Delta_0 \geq 0) \quad (A.11)$$

$$B_n(\Delta_0) < f_n(\Delta_0) < 2^{2n} B_n(\Delta_0) \quad (\Delta_0 < 0) \quad (A.12)$$

For $D=0$ it turns out that B_n can be evaluated exactly. Remembering that $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and $a |n\rangle = \sqrt{n} |n-1\rangle$, we get

$$B_n = 2^{-2n} (n!)^3 (n+1) \times$$

$$\int \Pi dy_i \exp -\Delta_0 [y_1 + 2(y_2 - y_1) + \dots + n(y_n - y_{n-1}) + (n+1)(y_{n+1} - y_n) + n(y_{n+2} - y_{n+1}) + \dots + 2(y_{2n} - y_{2n-1}) + (Y - y_{2n})] \quad (A.13)$$

$$= 2^{-2n} (n!)^3 (n+1) \times \int \Pi dy_i \exp \Delta_0 (y_1 + \dots + y_n) \cdot \exp -\Delta_0 (y_{n+1} + \dots + y_{2n} + Y) \quad (A.14)$$

$$= 2^{-2n} \Delta_0^{-2n+1} n(n+1) \cdot n! \int_0^Y dy e^{-\Delta_0 y} (e^{\Delta_0 y} - 1)^n (e^{-\Delta_0 y} - e^{-\Delta_0 Y})^{n-1} \quad (A.15)$$

This integral can be estimated as $n \rightarrow \infty$ using the method of steepest descents. We obtain

$$B_n(\Delta_0) \underset{n \rightarrow \infty}{\sim} \pi^{1/2} e^{-3\Delta_0 Y/4} \Gamma(n + \frac{5}{2}) \left[\frac{1 - e^{-\Delta_0 Y/2}}{2\Delta_0} \right]^{2n} \quad (A.16)$$

Putting these results together, for large n , the coefficient $G_n^{(1,1)}$ of $(-x_0^2)^n$ in $G^{(1,1)}(Y)$ satisfies, for $\Delta_0 \geq 0$,

$$\begin{aligned} & \pi^{1/2} (3\pi)^{D/4} e^{-3\Delta_0 Y/4} \Gamma(n + \frac{5}{2} + \frac{D}{4}) \left[\left(\frac{4\pi}{27(\alpha' Y + R^2)} \right)^{D/2} \left(\frac{1 - e^{-\Delta_0 Y/2}}{2\Delta_0} \right)^2 \right]^n \\ & \leq G_n^{(1,1)} \leq \pi^{1/2} \Gamma(n + \frac{5}{2}) \left[\left(\frac{\pi}{R^2} \right)^{D/2} \frac{Y^2}{4} \right]^n \end{aligned} \quad (A.17)$$

For $\Delta_0 < 0$ the upper bound is replaced by

$$G_n^{(1,1)} \leq \pi^{1/2} e^{-3\Delta_0 Y/4} \Gamma(n + \frac{5}{2}) \left[\left(\frac{\pi}{R^2} \right)^{D/2} \left(\frac{1 - e^{-\Delta_0 Y/2}}{2\Delta_0} \right)^2 \right]^n \quad (A.18)$$

For $D=0$, $\Delta_0 > 0$, (A.17) shows that the radius of convergence $\bar{A}$ of the Borel transform satisfies, as $Y \rightarrow \infty$,

$$O(Y^{-2}) \leq \bar{A} \leq 4 \quad (\text{A.19})$$

which is consistent with the exact result $\bar{A}=2$. This implies that the Bronzan diagrams B_n do not dominate the perturbation expansion as $n \rightarrow \infty$. For $\Delta_0 < 0$ we have

$$O(e^{\Delta_0 Y}) \leq \bar{A} \leq O(Y^{D/2} e^{\Delta_0 Y}) \quad (\text{A.20})$$

so that $\bar{A} \rightarrow 0$ as $Y \rightarrow \infty$.

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