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HADRON SCATTERING IN TWO-DIMENSIONAL QCD II :
SECOND ORDER CALCULATIONS, MULTI-REGGE AND INCLUSIVE REACTIONS

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A B S T R A C T

Hadron scattering is studied in higher orders of the $1/N_c$ expansion for quantum chromodynamics in two dimensions. In order $1/N_c^2$, individual Pomeron-like cylinder graphs give a forward amplitude growing as s , but cancellations in their sum remove this embryonic Pomeron. The cancellations reflect colour gauge invariance in a way specific to two dimensions. Otherwise, $1/N_c^2$ diagrams yield the expected "Regge" pole renormalizations and "cut" singularities. Factorizing multi-Regge behaviour is demonstrated for higher point functions. Inclusive reactions have universal quark fragmentation functions and final-state hadron distributions in hadronic and deep inelastic collisions.

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1. - INTRODUCTION

In the last few years a considerable body of folklore ¹⁾ has built up about the high energy behaviour of hadronic scattering amplitudes. Starting from the phenomenological successes of Regge poles at small momentum transfers, Regge cuts ^{2),3)} have been studied, the imposition of a new Pomeron singularity by unitarity has been suggested ⁴⁾, multi-Regge behaviour has been conjectured for production amplitudes ⁵⁾, and Mueller-Regge ideas ⁶⁾ have been developed for inclusive reactions. These conjectured Regge properties have generally been supported by calculations in perturbation theory ⁷⁾ and dual models ⁸⁾. The real world is apparently made of confined quarks, so the relevance of non-confining field theories, or of models with no hard quark structure, is not certain. A candidate theory of the strong interactions is quantum chromodynamics ⁹⁾ (QCD), and it would be nice to know whether QCD is consistent with our hadronic Regge folklore ¹⁾. Calculating with QCD in four dimensions is difficult; in this paper we continue our studies ¹⁰⁾, of high energy scattering ¹¹⁾ in two-dimensional QCD ^{12),13)}.

In our previous paper ^{*)} we studied the leading $1/N_c$ terms in a colour expansion ¹⁴⁾ of meson-meson scattering, found properties characteristic of dual models ⁸⁾ and the ideas of Veneziano ¹⁵⁾ based on topological expansions. Narrow resonances combined to give "Regge" behaviour at high energies, with intercepts and residues exhibiting complete exchange degeneracy. Other interesting properties were found: "Regge intercepts" were additive in the quarks exchanged in the crossed channel and were > -2 even for very massive quarks, while residues factorized in the direct channel. The confirmation of conventional prejudices in two-dimensional QCD in order $1/N_c$ ^{10),11)} encourages one to ask more questions:

- does a new "Pomeron" singularity ⁴⁾ arise in order $1/N_c^2$?
- do Regge pole renormalization and cuts ^{2),3)} appear in order $1/N_c^2$?
- is the Regge interpretation of high energy 2-2 scattering confirmed by "multi-Regge" behaviour ⁵⁾ in 2-n amplitudes ?
- what do inclusive hadronic cross-sections ⁶⁾ look like, and do they resemble inclusive distributions in deep inelastic collisions ¹⁶⁾ ?

^{*)} Hereafter referred to as I. Equations from I will be referred to as e.g., (I.3.21) in this paper.

We find in Section 2¹¹⁾ that a new "Pomeron" singularity does not arise in order $1/N_c^2$ in two-dimensional QCD. While individual contributions of this order to the meson-meson scattering amplitude behave as s^1 as $s \rightarrow \infty$, there is a cancellation in their sum which leads to a net behaviour $O(s^{-1})$. This cancellation results from colour gauge invariance when external states are colour singlets in two dimensions^{*)}. We believe that in four dimensions, where the coloured quarks and anticoloured antiquarks can separate in transverse space, the cancellation among eikonalized gluon exchange diagrams will not be exact as $s \rightarrow \infty$, and a Pomeron singularity near $J = 1$ will result. Another way of looking at this cancellation recalls that bound states of gluons (gluonium) are expected to lie on a Pomeron Regge trajectory, and that no gluonium can exist in two dimensions where gluons are not dynamical degrees of freedom : probably no gluonium means no Regge singularity to support them^{**)} .

In Section 3 we continue our studies of $O(1/N_c^2)$ diagrams, and demonstrate Regge pole renormalization and exchange degeneracy breaking in a pattern reminiscent of the ideas of Chew and Rosenzweig¹⁷⁾. New singularities resembling Regge cuts¹³⁾ also appear in order $1/N_c^2$. Section 4 studies multi-Regge behaviour⁵⁾, supporting our Regge-like interpretation of the high-energy behaviour of the 2-2 amplitude. Multi-Regge residues exhibit direct channel factorization just like the single Regge residues discussed in I.

In Section 5 we discuss the inclusive hadron distributions emanating from Mueller-Regge⁶⁾ s^{α_R-1} terms in inclusive cross-sections. We find that the lowest order contributions to inclusive hadron cross-sections have the expected single, double and triple-Regge behaviours. Furthermore, we demonstrate identity between the hadron distributions in hadronic collisions and the deep inelastic reaction $e^+e^- \rightarrow \text{hadrons}$. This identity reflects a universality of quark $\rightarrow$ hadron fragmentation¹⁸⁾, regardless whether the quark is "asymptotically free" or produced in a "soft" hadronic process.

Finally, in Section 6 we make some speculations about the relevance of our results to four-dimensional physics, emphasizing novel phenomenological possibilities suggested by two-dimensional QCD.

*) This experience emphasizes the difficulties in interpreting perturbative calculations in four-dimensional gauge theories which do not involve scattering colour singlet bound states.

**) Though two dimensions permit no continuation to $t > 0$ in the crossed channel, it was shown in I that Regge and bound state spectra are closely connected, at least when quark masses are small.

2. - HIGH ENERGY FORWARD SCATTERING IN ORDER $1/N_c^2$

In I we set out diagrammatic rules ^{12),13)} for calculating scattering amplitudes in two-dimensional QCD. In order $1/N_c$, meson-meson scattering was given ^{14),15)} by diagrams with the quark boundaries shown in Fig. 2.1. Contributions to meson-meson scattering in order $1/N_c^2$ are characterized by two quark boundaries. The mesons may be attached in any order to either of the boundaries, giving rise to the four-quark diagrams shown in Fig. 2.2 ^{*}). The planar diagram of Fig. 2.2a is expected ¹⁵⁾ to give widths to s channel mesons, and to renormalize t channel Regge exchanges. The diagram of Fig. 2.2b is expected ¹⁵⁾ to give widths to s channel mesons and to have Regge cuts in the t channel. It may also give new direct channel Pomeron (gluonium) bound states in dimensions more than two, as happens in the dual model ⁸⁾. The diagram of Fig. 2.2c is expected to have t channel Regge cuts. Finally, the diagram ⁴⁾ of Fig. 2.2d may contain a new "Pomeron" exchange in the t channel, as well as renormalizing the t channel vacuum Regge exchange through the so-called cylinder mechanism ¹⁷⁾. As shown in Fig. 2.3 by some examples, Fig. 2.2d represents multiple gluon exchange between the constituents of one meson and the other, rather than the exchange of "wee" quarks as in the Regge diagrams of Fig. 2.1, which we discussed in I. From Fig. 2.3a we see clearly that these graphs have the generic topology of a cylinder ⁴⁾ which is typical of dual ⁸⁾ and topological ¹⁵⁾ models of the Pomeron. In this section we study in detail this "Pomeron" graph of Fig. 2.2d : we will return briefly to the other graphs of Fig. 2.2 in Section 3.

The imaginary part of the diagram of Fig. 2.2d is easy to study because it only has contributions from pairs of mesons of masses M_1^2 and M_2^2 . It is therefore given by an integral over the modulus square of the amplitude A^{tu} of Fig. 2.1 :

$$\text{Im } A_{(2.2d)} = \frac{1}{4\pi^2} \int \frac{dM_1^2 dM_2^2}{\lambda^{1/2}(s, M_1^2, M_2^2)} |A^{tu}(s, M_1^2, M_2^2)|^2 \quad (2.1)$$

^{*}) We omit Okubo-Zweig-Iizuka ¹⁹⁾ rule violating insertions on external meson legs.

where $\lambda(s, M_1^2, M_2^2)$ is the usual phase space function. Let us express s and the two masses M_1^2, M_2^2 in terms of rapidity variables :

$$s = m^2 e^Y, \quad M_i^2 = m^2 e^{y_i}, \quad i = 1, 2 \quad (2.2)$$

Then the phase space integral in Eq. (2.1) is approximately the rectangle $0 \leq y_i \leq Y$ of Fig. 2.4. The discussion ^{4),8),15),17)} about the nature of the Pomeron - new singularity or renormalization of the f^0 ? - can be rephrased in terms of the part of phase space whence the integral (2.1) gets its dominant contribution. In the multiperipheral model ¹⁷⁾, where the M_i^2 are clusters ordered in rapidity such as pions or rho mesons, the integral (2.1) is calculated for $y_1 + y_2 \leq Y$. Since this region is dominated by Regge behaviour, it is expected to yield the "cylinder" renormalization of the f^0 or the f^0 extinction model ¹⁷⁾. On the other hand, in dual theory ⁸⁾ the dominant contribution to (2.1) comes from $Y - y_1, Y - y_2$ finite, i.e., M_1^2, M_2^2 both $O(s)$. This region represents two clusters overlapping in rapidity, and may be called a polyperipheral contribution. Since there is no obstacle to exchanging spin 1 gluons across a large rapidity range, one would expect that in QCD "poly-eikonalization" would not be suppressed in Fig. 2.3, and that therefore the $M_1^2, M_2^2 = O(s)$ part of (2.1) would dominate. Dominance by this region of phase space is indeed implicit in the soft gluon exchange ideas of Low ²⁰⁾ and Nussinov ²¹⁾. In this section we will now study the contribution of this region of phase space to the integral (2.1) in two-dimensional QCD. We will study other regions of phase space in Section 3.

For $M_1^2, M_2^2 = O(s)$, the kinematics of the A^{tu} amplitude are different from those in I

$$\frac{p_1'}{p_1} \approx \frac{1 - z_1 - z_2 + \lambda^{1/2}(1, z_1, z_2)}{1 - z_1 + z_2 + \lambda^{1/2}(1, z_1, z_2)} \quad (2.3a)$$

and

$$\frac{p_2'}{p_1} \approx \frac{2 z_2}{1 - z_1 + z_2 + \lambda^{1/2}(1, z_1, z_2)} \quad (2.3b)$$

where

$$z_i = \frac{M_i^2}{s} \quad (2.3c)$$

while

$$\frac{p_{2-}}{p_{1-}} = \frac{m_2^2}{s} \quad (2.3d)$$

We see from (2.3a) and (2.3b) that both quarks a and b must carry a finite fraction of p_{1-} , in contrast to the situation in the Regge analysis of I, where one of a or b had to be "wee". There is thus no damping due to the small amplitude (I.2.24) for finding a "cold" quark in a "hot" meson as was the case ^{10),11)} for Regge exchange : we therefore estimate $A^{tu} \sim \text{constant}$ as $s \rightarrow \infty$. Furthermore, the phase space in (2.1) is $O(s)$, so that superficially $\text{Im } A^{(2,2d)} \sim s$, corresponding to the expectations of multiple gluon exchange ^{20),21)}. However, the detailed analysis given below shows that there is a cancellation between the contributions to A^{tu} so that it does not behave as a constant for large M_i^2 and we do not actually obtain a "Pomeron" with unit intercept.

The amplitude A^{tu} is given ^{10),11)} by the three diagrams shown in Fig. 2.5 where $p_i'^2 = M_i^2$ ^{*)}. To evaluate these diagrams we proceed as in I by first doing the internal k_+ integrations or, equivalently, calculating using x_- ("time") ordered perturbation theory. For the diagram of Fig. 2.5a the various x_- orderings of the vertices lead to the contributions shown in Fig. 2.6. The remaining x_- orderings are impossible due to the constraints that the quarks have positive energy ($k_- \geq 0$), that k_- be conserved, and that $p_{1-} > p_{1-}'$ which holds in the limit of interest. The sum of the first four diagrams is proportional to

$$\begin{aligned} & \left(\frac{1}{p_{1+} - \bar{p}_{1+}} + \frac{1}{p_{2+} - \bar{p}_{2+}} \right) \left(\frac{1}{p_{1+} + p_{2+} - (\bar{p}_{1+} + \bar{p}_{2+})} \right) \left(\frac{1}{p_{1+}' - \bar{p}_{1+}'} + \frac{1}{p_{2+}' - \bar{p}_{2+}'} \right) \\ &= \left(\frac{1}{p_{1+} - \bar{p}_{1+}} \right) \left(\frac{1}{p_{2+} - \bar{p}_{2+}} \right) (p_{1+} + p_{2+} - \bar{p}_{1+} - \bar{p}_{2+}) \left(\frac{1}{p_{1+}' - \bar{p}_{1+}'} \right) \left(\frac{1}{p_{2+}' - \bar{p}_{2+}'} \right) \end{aligned} \quad (2.4)$$

where

$$\bar{p}_{1+} = \frac{H_{01}}{2p_{1-}} = \frac{m^2(\gamma_a - 1)}{2k_-} + \frac{m^2(\gamma_b - 1)}{2(p_{1-} - k_-)}$$

*) We consider the case $p_{1-}' > p_{1+}$. The case with a parity inversion of the final state has already been included in Eq. (2.1) by inserting a factor 2.

with $m^2 \equiv (g^2 N_c / \pi)$ and $\gamma_a \equiv (m_a^2 / m^2)$ as in (I.2.19,20), and k_- is the momentum of quark a . These diagrams thus give the contribution

$$A_{(a)}^{tu} = \frac{4\pi m^2}{N_c} \int_0^{p_{2-}} dl_- \phi_1^{a\bar{b}} \left(\frac{p_{1-}' - p_{2-} + l_-}{p_{1-}} \right) \phi_{1'}^{d\bar{a}} \left(\frac{p_{2-} - l_-}{p_{1-}'} \right) \\ \phi_2^{c\bar{d}} \left(\frac{l_-}{p_{2-}} \right) \phi_{2'}^{b\bar{c}} \left(\frac{p_{2-}' - l_-}{p_{2-}'} \right) \left(\frac{2(p_{1+} + p_{2+})}{m^2} - \frac{\gamma_a - 1}{p_{1-}' - p_{2-} + l_-} \right. \\ \left. - \frac{\gamma_b - 1}{p_{2-}' - l_-} - \frac{\gamma_c - 1}{l_-} - \frac{\gamma_d - 1}{p_{2-} - l_-} \right) \quad (2.5)$$

which we represent in Fig. 2.7a.

The contributions of Fig. 2.5b and c containing the quark-antiquark T matrix in the t and u channels give time-ordered perturbation theory graphs which can be obtained from those of Fig. 2.6 by inserting another point in x_- (corresponding to the instantaneous T matrix) in all possible ways. The t channel T matrix can be inserted in Figs. 2.6a through 2.6d in 10 ways which after using identities like Eq. (2.2) as well as Eqs. (I.2.38) and (I.2.39) give the graphs in Figs. 2.7b, d_1 , d_2 and d_3 . The t channel T matrix cannot be inserted in Fig. 2.6f, but can be inserted in Fig. 2.6e in one way which using Eq. (I.2.36) gives the graph of Fig. 2.7d₄. These contributions are

$$A_{(b)}^{tu} = \frac{4\pi m^2}{N_c} \int_0^{p_{2-}} dl_- \int_0^{p_{1-}'} dk_- \phi_1^{a\bar{b}} \left(\frac{k_-}{p_{1-}} \right) \phi_{1'}^{d\bar{a}} \left(\frac{p_{1-}' - k_-}{p_{1-}'} \right) \\ \times \frac{\phi_2^{c\bar{d}} \left(\frac{l_-}{p_{2-}} \right) \phi_{2'}^{b\bar{c}} \left(\frac{p_{2-}' - l_-}{p_{2-}'} \right)}{(p_{2-} - l_- - p_{1-}' + k_-)^2} \quad (2.6)$$

and

$$A_{(a)}^{tu} = \sum_n \frac{G_{11'n} G_{n22'}}{t - m_n^2} \quad (2.7)$$

where G_{ijk} is the three-meson vertex given in Eq. (I.2.50).

Finally, insertion of the u channel T matrix gives Fig. 2.7c :

$$A_{(c)}^{tu} = \frac{4\pi m^2}{N_c} \int_0^{p_{2-}} dl_- \int_0^{p_{2-}'} dk_-' \phi_1^{a\bar{b}}\left(\frac{p_{1-} - p_{2-}' + k_-'}{p_{1-}}\right) \phi_{1'}^{d\bar{a}}\left(\frac{p_{2-} - l_-}{p_{1-}'}\right) \\ \times \frac{\phi_2^{c\bar{d}}\left(\frac{l_-}{p_{2-}}\right) \phi_2^{b\bar{e}}\left(\frac{p_{2-}' - k_-'}{p_{2-}'}\right)}{(l_- - k_-')^2} \quad (2.8)$$

and Fig. 2.7e :

$$A_{(e)}^{tu} = \sum_n \frac{G_{12'n} G_{n1'2}}{n - m_n^2} \quad (2.9)$$

We first evaluate the contribution of the t channel pole terms (2.7) for M_i^2 both $O(s)$. Since both p_{1-} and p_{1-}' are large, we have from (2.3)

$$t = (p_1 - p_{1'})^2 \approx - \frac{2z_1 z_2}{1 - z_1 - z_2 + \lambda^2(1, z_1, z_2)} s. \quad (2.10)$$

Using (I.2.50) for the three-meson vertex we have

$$A_{(a)}^{tu} \approx -\frac{4\pi m^2}{N_c} \sum_n \left\{ (-1)^{n_1'} \left(\frac{p_{1-}'}{p_{1-} - p_{1-}'} \right) \int_0^1 dx \int_0^1 dx' \right. \\ \left. \phi_{1'}^{d\bar{a}}(x) \left[\frac{\phi_1^{b\bar{a}}\left(\frac{(p_{1-} - p_{1-}') (1-x')}{p_{1-}}\right) - \phi_1^{b\bar{a}}\left(\frac{(p_{1-} - p_{1-}') + p_{1-}' x}{p_{1-}}\right)}{\left(x \frac{p_{1-}'}{p_{1-} - p_{1-}'} + y'\right)^2} \right] \phi_n^{d\bar{b}}(x') \right\} \\ \frac{1}{t - m_n^2} \left\{ \left(\frac{p_{1-} - p_{1-}'}{p_{2-}} \right) \int_0^1 dy \int_0^1 dy' \phi_n^{d\bar{b}}(y) \phi_2^{d\bar{e}}(y') \left[\phi_{2'}^{c\bar{b}}\left(\frac{p_{2-} - (1-y')}{p_{2-}'}\right) \right. \right. \\ \left. \left. - \phi_{2'}^{c\bar{b}}\left(\frac{p_{2-} + (p_{1-} - p_{1-}')}{p_{2-}'}\right) \right] / \left(y \frac{p_{1-} - p_{1-}'}{p_{2-}} + y'\right)^2 \right\} \quad (2.11)$$

where we have introduced

$$x \equiv \frac{l_-}{p_{1-}'}, \quad x' \equiv \frac{l_-'}{p_{1-} - p_{1-}'}, \quad y \equiv \frac{k_-}{p_{1-} - p_{1-}'}, \quad y' \equiv \frac{k_-'}{p_{2-}}$$

The amplitude is $O(s^{-2})$ unless x, x' and y are small. Introducing η, η' and ξ so that

$$x' = -\frac{m^2}{t} \eta' = \frac{m^2}{M_1^2} \frac{p_{1-}'}{p_{1-} - p_{1-}'} \eta'$$

$$x = \frac{m^2}{M_1^2} \eta, \quad y = -\frac{m^2}{t} \xi$$

and using the analogue of (I.3.35) we have

$$A_{(a)}^{tu} \simeq \frac{4m^2}{\pi N_c} \int_0^\infty \frac{d\lambda}{\lambda+1} \left\{ \int_0^\infty d\eta \int_0^\infty d\eta' (-1)^{\eta'} \phi^d(\eta) \phi^d(\eta'\lambda) \right.$$

$$\left. \frac{[\phi_1^{b\bar{a}}(\frac{p_{1-} - p_{1-}'}{p_{1-}} - \frac{p_{1-}'}{p_{1-}} \frac{m^2}{M_1^2} \eta') - \phi_1^{b\bar{a}}(\frac{p_{1-} - p_{1-}'}{p_{1-}} + \frac{p_{1-}'}{p_{1-}} \frac{m^2}{M_1^2} \eta)]}{((p_{1-}/p_{1-}')\eta + \eta')^2} \right\}$$

$$\times \left\{ \int_0^\infty d\xi \int_0^\infty dy' \phi^d(\xi\lambda) \phi_2^{a\bar{c}}(y') \right.$$

$$\left. \frac{[\phi_2^{c\bar{b}}(\frac{p_{2-}(1-y')}{p_{2-}'}) - \phi_2^{c\bar{b}}(\frac{p_{2-} + \frac{m^2}{M_1^2} p_{1-}' y'}{p_{2-}'})]}{(\frac{m^2}{M_1^2} \frac{p_{1-}'}{p_{2-}'} \xi + \frac{M_1^2}{m^2} \frac{p_{2-}}{p_{1-}'} y')^2} \right\} \quad (2.12)$$

The contribution thus would behave like a constant as $s \rightarrow \infty$ were it not for the fact that the term in the first square bracket vanishes.

The u channel pole terms (2.9) involve a similar integral and have identical behaviour : $O(s^0)$. Finally we examine the first three diagrams in Fig. 2.7. Introducing $x \equiv k_-/p_{2-}$ in Eq. (2.5) gives

$$A_{(a)}^{tu} \simeq \frac{4\pi m^2}{N_c} \int_0^1 dx \phi_1^{a\bar{b}}(\frac{p_{1-}'}{p_{1-}}) \phi_{1'}^{d\bar{a}}(\frac{p_{2-}}{p_{1-}'}(1-x))$$

$$\phi_2^{c\bar{d}}(x) \phi_{2'}^{b\bar{e}}(1 - \frac{p_{2-}}{p_{2-}'} x) \left(\frac{M_2^2}{m^2} - \frac{\gamma_{e-1}}{x} - \frac{\gamma_{d-1}}{1-x} \right)$$

Introducing the scaling forms (I.2.42) for ϕ_1 and ϕ_2' appropriate for large masses gives the asymptotically constant term

$$A_{(a)}^{tu} \simeq \frac{4\pi m^2}{N_c} \int_0^1 dx \phi_1^{a\bar{b}}\left(\frac{p_{1-}'}{p_{1-}}\right) \phi^d\left(\frac{p_{2-}}{p_{1-}'} \frac{M_1^2}{m^2} (1-x)\right) \\ \phi_2^{c\bar{a}}(x) (-1)^{n_2'} \phi^c\left(\frac{p_{2-}}{p_{2-}'} \frac{M_2^2}{m^2} x\right) \left(\frac{M_2^2}{m^2} - \frac{\gamma_{c-1}}{x} - \frac{\gamma_{d-1}}{1-x}\right). \quad (2.13)$$

Making the same change of variables along with $\eta = (p_{1-}' - k_-)/p_{2-}$ in (2.6) gives

$$A_{(b)}^{tu} \simeq \frac{4\pi m^2}{N_c} \int_0^1 dx \int_0^\infty d\eta \phi_1^{a\bar{b}}\left(\frac{p_{1-}'}{p_{1-}}\right) \phi^d\left(\frac{p_{2-}}{p_{1-}'} \frac{M_1^2}{m^2} \eta\right) \\ \frac{\phi_2^{c\bar{a}}(x) (-1)^{n_2'} \phi^c\left(\frac{p_{2-}}{p_{2-}'} \frac{M_2^2}{m^2} x\right)}{(\eta - (1-x))^2} \quad (2.14)$$

The η integral can be done explicitly using the scaling form of the bound state equation (I.2.41)¹³⁾

$$\int_0^\infty d\xi' \frac{\phi^a(\lambda \xi')}{(\xi - \xi')^2} = - \left(\lambda - \frac{\gamma_{a-1}}{\xi}\right) \phi^a(\lambda \xi) \quad (2.15)$$

giving

$$A_{(b)}^{tu} \simeq - \frac{4\pi m^2}{N_c} \int_0^1 dx \phi_1^{a\bar{b}}\left(\frac{p_{1-}'}{p_{1-}}\right) \phi^d\left(\frac{p_{2-}}{p_{1-}'} \frac{M_1^2}{m^2} (1-x)\right) \\ \phi_2^{c\bar{a}}(x) (-1)^{n_2'} \phi^c\left(\frac{p_{2-}}{p_{2-}'} \frac{M_2^2}{m^2} x\right) \left(\frac{p_{2-}}{p_{1-}'} \frac{M_1^2}{m^2} - \frac{\gamma_{d-1}}{(1-x)}\right) \quad (2.16)$$

A similar manipulation gives for (2.8) :

$$A_{(c)}^{tu} \simeq - \frac{4\pi m^2}{N_c} \int_0^1 dx \phi_1^{a\bar{b}}\left(\frac{p_{1-}'}{p_{1-}}\right) \phi^d\left(\frac{p_{2-}}{p_{1-}'} \frac{M_1^2}{m^2} (1-x)\right) \\ \phi_2^{c\bar{a}}(x) (-1)^{n_2'} \phi^c\left(\frac{p_{2-}}{p_{2-}'} \frac{M_2^2}{m^2} x\right) \left(\frac{p_{2-}}{p_{2-}'} \frac{M_2^2}{m^2} - \frac{\gamma_{c-1}}{x}\right) \quad (2.17)$$

The sum of (2.13), (2.16) and (2.17) is easily seen to vanish using the identity

$$\frac{p_{2-}}{p_{1-}} M_1^2 + \frac{p_{2-}}{p_{2-}} M_2^2 = p_{2-} (2p_{1+}' + 2p_{2+}') = m_z^2$$

To understand the origin of this seemingly miraculous cancellation we note that the dominant behaviour s^0 arose from the region where the explicit gluon was very soft ($q_- \approx p_{2-} \ll p_{a-}, p_{b-}$), so we expect it to couple predominantly through colour electric charge. The cancellation occurs between pairs of diagrams like those in Fig. 2.7d₁ and d₂ [2.3(c) and (d) also] which differ only by switching the soft gluon from quark a to quark b . Since meson 1 is a colour singlet, these quarks will have opposite colour charge leading to the cancellation ^{*)}.

Thus although the existence of high mass bound states provides a potential mechanism for generating the Pomeron with intercept $= 1 + O(N_f/N_c)$, it is absent in two dimensions in $O(1/N_c^2)$. The diagram does, of course, not give a vanishing contribution. From our calculations it appears, though we have not been able to prove it rigorously, that for large M_i^2 , the leading behaviour of A^{tu} is s^{-1} which would give $\text{Im } A_{(4d)} \sim s^{-1}$. However, finite M^2 contributions can dominate giving "Regge pole" renormalization contributions $\sim s^{\alpha_{II}(0)} \ell_{ns}$ (see Section 3). These renormalize the vacuum "Regge" intercept upwards, giving a model resembling that of Chew and Rosenzweig ¹⁷⁾. But we believe the Pomeron cancellation will not operate in higher dimensions. The exchanged gluons then have non-zero transverse momentum, and there is ²⁰⁾ no reason to expect a cancellation. Secondly, gluons are not just instantaneous Coulomb potentials but dynamical degrees of freedom which tend to form gluonium in higher dimensions. Since gluonium states should ^{8), 15)} be the particles on the Pomeron trajectory, their absences may be linked ^{**)}.

We close this section with some remarks about the Drell-Yan cross-section for high-mass dilepton production. The quark diagram supposed by Drell and Yan to give the scaling

*) The same cancellation is responsible for the emergence of the parton model result in deep inelastic electron-meson scattering - see Einhorn ¹³⁾.

**) On the other hand, Henry Tye believes his model ²²⁾ may exhibit a Pomeron even though it has no gluonium.

$$\frac{d\sigma}{dQ^2} = \frac{1}{Q^4} f(Q^2/s) \quad (2.18)$$

contribution to $1+2 \rightarrow \ell^+ \ell^- (Q^2) + X$ is indicated in Fig. 2.8. It has the same structure as A^{tu} in the Pomeron formula (2.1), and the kinematic region $[Q^2 = O(s), M_X^2 = O(s)]$ is similar to that found to experience a cancellation in the Pomeron integration. One might therefore worry that the scaling contribution might also be cancelled. This, however, is not the case : it has been shown ²³⁾ that the two-dimensional analogue of the law (2.18) does hold in QCD, with $f(Q^2/s)$ given by the classical Drell-Yan convolution of quark and antiquark densities. The difference arises in part because in calculating the resonance dominated part of the coupling of the photon in Fig. 2.8 the bound state scaling equation (2.15) is not relevant : instead one must use the equation ¹³⁾

$$\left(1 - \gamma_{\frac{a-1}{\xi}}\right) h_+^a(\xi) + \int_0^\infty \frac{d\eta}{(\eta-\xi)^2} h_+^a(\eta) = 1 \quad (2.19)$$

where

$$h_+^a(\xi) \equiv \lim_{Q^2 \rightarrow \infty} Q^2 G\left(\frac{\xi}{Q^2}; Q^2\right)$$

$$G(x, Q^2) \equiv \int_0^1 dy G(x, y; Q^2) \quad (2.20)$$

The term on the right-hand side of (2.19) ensures that the Pomeron cancellation is not complete in the case of the Drell-Yan process. It is also necessary to include in the two-dimensional QCD calculation diagrams where the photon couples directly to the quarks, which have no Pomeron-like cancellation. For details of how the Drell-Yan cross-section eventually emerges, we refer the reader to Ref. 23).

3. - REGGE CUTS AND REGGE POLE RENORMALIZATION

In this section we discuss higher order "Regge" phenomena - Regge cuts and Regge pole renormalization - which occur in the four-particle amplitude in order $(1/N_c)^2$. These arise from the diagrams shown in Fig. 2.2 and we outlined our expectations in the beginning of Section 2. We begin by completing our discussion of Fig. 2.2d which we expect ^{8),15)} to give t channel Regge pole renormalization. On the other hand, we do not expect Regge cut ^{2),3)} effects (asymptotic powers $s^{-2\beta_b-2\beta_d-1}$) due ³⁾ to the absence of left-hand singularities in the Reggeon particle scattering amplitudes.

We recall that the imaginary part of Fig. 2.2d is expressed in terms of the square of $A^{tu}(s, M_1^2, M_2^2)$, according to Eq. (2.1). If we write

$$H_i^2 = m^2 e^{Y_i}, \quad S = m^2 e^Y \quad (3.1)$$

then the phase space integral in (2.1) is approximately the rectangle shown in Fig. 2.4. In Section 2 we found that the small portion of phase space with $y_1 \approx Y$, while potentially behaving like s^1 , was subject to a cancellation. We now show that this cancellation actually operates in the much larger region

$$y_1 + y_2 \geq Y + \Delta \quad (1 \ll \Delta \ll Y) \quad (3.2)$$

i.e., in the entire phase space to the right of the diagonal in Fig. 2.4. Consider $A_{(d)}^{tu}$ given by Eq. (2.11). In order to have a cancellation it is necessary that

$$x' \ll 1, \quad x \ll \frac{p_{1-} - p_{1-}'}{p_{1-}'}$$

The variables x and x' are required to be small if M_1^2 is large (then $x \sim 1/M_1^2$) and m_n^2 is large (then $x' \sim 1/m_n^2$). The latter condition is forced if t is large which from (2.10) is

$$\frac{M_1^2 M_2^2}{s} \gg m^2$$

which is just (3.2). For the cancellation of $A_{(a)}^{tu}$ with $A_{(b)}^{tu}$ and $A_{(c)}^{tu}$, and of $A_{(e)}^{tu}$, only the weaker condition $M_2^2 \gg m^2$ is necessary.

The dominant contribution to Eq. (2.1) therefore arises from the diagonal shaded region in Fig. 2.4. It is easy to check that here $|A^{tu}|^2$ has the double "triple-Regge" behaviour (Fig. 3.1) :

$$|A^{tu}|^2 \sim \left(\frac{M_1^2}{s}\right)^{2\beta_a} \left(\frac{M_2^2}{s}\right)^{2\beta_b} = (M_1^2)^{\alpha_1} \left(\frac{s}{M_1^2 M_2^2}\right)^{2\alpha_0} (M_2^2)^{\alpha_2} \quad (3.3)$$

Substituting in (2.1) we have

$$\begin{aligned} \text{Im } A_{(4d)} &\sim \int_R dy_1 dy_2 e^{y_1(1-\alpha_2)} e^{y_2(1-\alpha_1)} e^{Y(\alpha_1+\alpha_2-1)} \\ &\sim \frac{s^{\alpha_1} - s^{\alpha_2}}{\alpha_1 - \alpha_2} \end{aligned} \quad (3.4)$$

which renormalizes the residues of the poles α_1 and α_2 and the location as well when $\alpha_1 = \alpha_2$ ^{*}). There may also be contributions from the remaining part of phase space. In particular the inner shaded corner in Fig. 2.4 generally gives rise to Regge cuts due to the Regge behaviour of A^{tu} for fixed M_i^2 . Generally these are found to vanish if either the upper or lower Reggeon particle amplitude in Fig. 3.1 has only right-hand (or only left-hand) singularities in M_i^2 . So since Fig. 2.2d has only right-hand M_i^2 singularities we expect no Regge cut ³⁾. The usual method of proof involves closing M_i^2 contours in $A_{(4d)}$ away from the singularities ^{3), 24)}. Since we have not studied the complex M_i^2 behaviour we use a trick which enables us to show directly that the Regge cut vanishes in Eq. (2.21) and only pole renormalization remains.

We begin by considering the contribution of $A_{(d)}^{tu}$ to Eq. (2.1) for a fixed M_2^2

$$\text{Im } A_{(22d)} \approx \frac{1}{4\pi^2} \sum_{n_1'=0}^{s_{\pi^2 m^2}} \lambda^{-1/2} |A_{(d)}^{tu}(s, M_1^2, M_2^2)|^2 \quad (3.5)$$

To this we add and subtract

$$\frac{1}{4\pi^2} \sum_{n_1'=0}^{\infty} (\lambda^{-1/2} |A_{(d)}^{tu}(s, M_1^2, M_2^2)|^2) \Big|_{M_1^2 = \bar{M}_1^2} \quad (3.6)$$

^{*}) $s^{\alpha_{\text{hns}}}$ is characteristic of a double pole in the J plane which shifts the original pole's position.

where

$$\bar{M}_1^2 = \frac{1}{\Delta} m^2 \frac{s}{M_2^2} \equiv \bar{z}_1 s \quad (\Delta \gg 1) \quad (3.7)$$

and $M_1^2 = \bar{M}_1^2$ everywhere except in Φ_{11}^{da} . We thus obtain

$$\begin{aligned} \ln A(2.2a) \approx & \frac{1}{4\pi^2} \sum_{n_1' = \frac{M_1^2}{\pi^2 m^2}}^{\frac{s}{\pi^2 m^2}} \left[(\lambda^{-1/2} |A_{(d)}^{tu}|^2) - (\lambda^{-1/2} |A_{(d)}^{tu}|^2) \right]_{M_1^2 = M_1^2} \\ & - \frac{1}{4\pi^2} \sum_{n_1' = \frac{s}{\pi^2 m^2}}^{\infty} \left(\lambda^{-1/2} |A_{(d)}^{tu}|^2 \right)_{M_1^2 = \bar{M}_1^2} + \frac{1}{4\pi^2} \sum_{n_1' = 0}^{\infty} \left(\lambda^{-1/2} |A_{(d)}^{tu}|^2 \right)_{M_1^2 = \bar{M}_1^2} \end{aligned} \quad (3.8)$$

which we will see has the effect of replacing the small n_1' contribution in (3.5), which apparently behaves like a Regge cut, by terms which behave like pole renormalizations. To see this consider the $A_{(d)}^{tu}$ given by Eq. (2.11) which occurs in (3.8) :

$$\begin{aligned} A_{(d)}^{tu}(s, \bar{M}_1^2, M_2^2) \approx & -\frac{4\pi m^2}{N_c} \sum_n \frac{s}{M_2^2} \left\{ \int_0^1 dx \int_0^1 dx' \right. \\ & \left. \left[\phi_1^{b\bar{a}}\left(\frac{M_2^2}{s} + x\right) - \phi_1^{b\bar{a}}\left(\frac{M_2^2}{s}(1-x')\right) \right] \frac{\phi_{1'}^{d\bar{a}}(x) \phi_n^{d\bar{b}}(x')}{\left(x \frac{s}{M_2^2} + x'\right)^2} \right\} \\ & \frac{1}{\bar{t} - m_n^2} \left\{ \int_0^{\frac{p_{1-} - p_{1-}'}{p_{2-}}} d\eta \int_0^1 dy' \phi_n^{d\bar{b}}\left(\frac{p_{2-}}{p_{1-} - p_{1-}'} \eta\right) \phi_2^{d\bar{c}}(y') \times \right. \\ & \left. \times \frac{\left[\phi_{2'}^{c\bar{b}}\left(\frac{p_{2-}}{p_{2-}'}(1-y')\right) - \phi_{2'}^{c\bar{b}}\left(\frac{p_{2-}}{p_{2-}'}(1+\eta)\right) \right]}{(\eta + y')^2} \right\} \end{aligned} \quad (3.9)$$

where

$$M^2 \equiv \frac{M_2^2}{1 - \bar{z}_1} - m^2, \quad \bar{t} = t(s, \bar{M}_1^2, M_2^2) = \text{finite} \quad (3.10)$$

In the last term in (3.8) the only dependence on the state $1'$ is in

$$\sum_{n_1'=0}^{\infty} \phi_{1'}^{d\bar{a}}(x) \phi_{1'}^{d\bar{a}}(\tilde{x}) = \delta(x-\tilde{x}) \quad (3.11)$$

where we have used the completeness relation (I.2.27). The dominant contribution arises for $x \sim M^2/s$ which gives

$$\sum_{n_1'=0}^{\infty} (\lambda^{-1/2} |A_{(d)}^{tu}|^2) \Big|_{M^2=M_1^2} \sim \frac{1}{M^2} \left(\frac{M^2}{s}\right)^{2\beta_b} \left(\frac{m_2^2}{M^2}\right)^{\beta_d}$$

For finite M_2^2 this gives a term $s^{-2\beta_b} = s^{\alpha_1}$ and for large M_2^2 a term

$$\int_{\Delta m^2}^{\frac{1}{\Delta s}} dM_2^2 \frac{1}{M_2^2} \left(\frac{M_2^2}{s}\right)^{2\beta_b} \left(\frac{m_2^2}{M_2^2}\right)^{\beta_d}$$

which has the form (3.4). There is thus no cut arising from small n_1 . The other terms in Eq. (3.8) are easily seen to have similar behaviour ^{*)}. Finally the contributions from the remaining graphs in Fig. 2.7 can be handled in a similar way. Since they vanish for large M_2^2 they contribute only terms s^{α_1} . In conclusion we see the effect of Fig. 2.2d is to selectively renormalize the vacuum Regge trajectory.

Finally we turn briefly to the other diagrams in Fig. 2.2. The diagram of Fig. 2.2c is very similar to the one we have just discussed except for the additional twists in the Reggeon particle amplitudes. These introduce a factor $(-1)^{n_1+n_2}$ [see Eq. (I.2.24)] whose oscillations cause the M_1^2 and M_2^2 integrals to become very convergent. Thus only small M_1^2 contribute and the diagram has only a Regge cut ^{**)}. Similarly Fig. 2.2b gives a Regge cut while Fig. 2.2a gives only Regge pole renormalization.

We have seen that Regge pole renormalizations come from Figs. 2.2a and 2.2d. The first of these clearly respects exchange degeneracy, and reflects the proposals of Veneziano ¹⁵⁾ and of Chew and Rosenzweig ¹⁷⁾ that the ensemble of planar diagrams should be exchange degenerate. On the other

^{*)} For large M_2^2 we do not need to employ the above device since $A_{(d)}^{tu} \sim (M_2^2)^{-\beta_d}$ and we have a term $s^{-2\beta_d} = s^{\alpha_2}$.

^{**)} Note (3.11) cannot now be used.

hand, Fig. 2.2d clearly violates exchange degeneracy, and has the effect of promoting the analogue of the f^0 trajectory. Since this diagram also gives no new Pomeron singularity, as was shown in Section 2¹¹⁾, two-dimensional QCD is an example of the proposal of Chew and Rosenzweig¹⁷⁾ and others that the Pomeron and f^0 singularities are not distinct. However, we emphasized already in Section 2 that we believe the Pomeron cancellation is specific to two dimensions. In four-dimensional QCD we expect to find two distinct singularities⁴⁾, as favoured by Veneziano¹⁵⁾.

4. - HIGH ENERGY MULTIPARTICLE SCATTERING

In this section we study the high energy behaviour of multiparticle amplitudes. We find that their structure is that characteristic of factorizable multiple Regge exchanges⁵⁾. This lends further support to the interpretation^{10),11)} of forward high energy meson-meson scattering as embryonic Regge behaviour. We first discuss the $2 \rightarrow 3$ production amplitude. A typical quark-antiquark exchange diagram contributing in leading order in $1/N_c$ is shown in Fig. 4.1. In the high energy limit

$$\begin{aligned} S, S_1, S_2 &\rightarrow \infty \\ S_1 S_2 &\approx S m_4^2 \end{aligned} \quad (4.1)$$

which is the usual double Regge limit in two dimensions, we have

$$\frac{p_{1-}}{p_{2-}} \sim \frac{S}{m_2^2}, \quad \frac{p_{5-}}{p_{4-}} \sim \frac{S_1}{m_4^2}, \quad \frac{p_{4-}}{p_{3-}} \sim \frac{S_2}{m_3^2} \quad (4.2)$$

with the usual choice of frame. There will be several contributions to Fig. 4.1 corresponding to quark-antiquark T matrices inserted in various ways. However, after the steps outlined in Section I.2 are carried out all contributions will be proportional to a multiple integral over the product of five wave functions, one for each external particle. As usual, if the gluon exchanges are not to give terms $O(1/S_1)$ the momenta carried by the quarks b, d and e must be small:

$$p_{b-} = O(p_{2-}), \quad p_{d-} = O(p_{3-}), \quad p_{e-} = O(p_{4-}).$$

The wave functions for mesons 1, 4 and 5 then give "dribble factors" so that

$$\begin{aligned} A^5 &\sim \left(\frac{p_2}{p_1}\right)^{-\beta_b} \left(\frac{p_3}{p_4}\right)^{-\beta_d} \left(\frac{p_4}{p_5}\right)^{-\beta_e} \\ &\sim (s_1)^{\alpha_1} (s_2)^{\alpha_2} \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \alpha_1 &= \alpha_1^{b\bar{e}}(0) = -\beta_b - \beta_e \\ \alpha_2 &= \alpha_2^{b\bar{a}}(0) = -\beta_b - \beta_d \end{aligned} \quad (4.4)$$

This is just the usual double-Regge behaviour.

Let us now compute the amplitude of Fig. 4.1 in detail in the limit (4.1). It is easiest to calculate the double discontinuities of the amplitude shown in Fig. 4.2 and use the smearing procedure of Section I.3. We have then

$$\begin{aligned} \text{disc}_s \text{ disc}_{s_1} A^5 &= \sum_{n, n_1} G_{1,2,n} 2\pi \delta(s - m_n^2) G_{n, n_1, 3} \\ &\times 2\pi \delta(s_1 - m_{n_1}^2) G_{n_1, 4, 5} \end{aligned} \quad (4.5)$$

where the G_{ijk} are the three-meson vertices given in (I.2.49). Using the averaging procedure one obtains

$$\text{disc}_s \text{ disc}_{s_1} A^5 \approx G_{1,2,n} \frac{2i}{\pi m^2} G_{n, n_1, 3} \frac{2i}{\pi m^2} G_{n_1, 4, 5} \quad (4.6)$$

where

$$n \approx \frac{s}{\pi^2 m^2}, \quad n_1 \approx \frac{s_1}{\pi^2 m^2}$$

For the first and last meson vertices we borrow Eq. (I.3.20) :

$$G_{1,2,3} \underset{m_3^2 \rightarrow \infty}{\approx} 2 \sqrt{\frac{\pi}{N_c}} m^2 \left(\frac{m_3^2}{m^2}\right)^{-\beta_b} V_1^{b\bar{a}} V_2^{b\bar{e}} \quad (4.7)$$

For the vertex with two large masses we need to take a further limit. This is easily done using $C_1^{ba} \rightarrow (m_1^2/m^2)^{\beta_b} C^b$ from Eq. (I.3.15) :

$$\begin{aligned} V_1^{b\bar{a}} &= \sqrt{\frac{\pi}{C^b \sin \pi \beta_b}} C_1^{b\bar{a}} \approx \sqrt{\frac{\pi C^b}{\sin \pi \beta_b}} \left(\frac{m_1^2}{m^2}\right)^{\beta_b} \\ &\equiv \left(\frac{m_1^2}{m^2}\right)^{\beta_b} V^b \end{aligned} \quad (4.8)$$

so that

$$G_{123} \underset{m_3 \gg m_1 \gg m^2}{\approx} \left(2 \sqrt{\frac{\pi}{N_c}} m^2\right) \left(\frac{m_3^2}{m_1^2}\right)^{-\beta_b} V^b V_2^{b\bar{c}} \quad (4.9)$$

Substituting this result in (4.6) gives

$$\begin{aligned} \text{disc}_s \text{disc}_{s_1} A^5 &\simeq -m^2 \frac{2^5}{\pi^{1/2} N_c^{3/2}} \left(\frac{m_4^2}{m^2}\right)^{-\alpha_1} V_1^{b\bar{a}} V_2^{b\bar{c}} V^d V_3^{d\bar{e}} V_5^{e\bar{a}} V_4^{e\bar{d}} \\ &\quad \times \left(s_1/m^2\right)^{\alpha_1} \left(s_2/m^2\right)^{\alpha_2} \end{aligned} \quad (4.10)$$

so A^5 has the expected double-Regge behaviour and can be written in a factorized form.

To obtain the two-Regge-one-meson vertex, we note that for an amplitude (like the one under consideration) which has no simultaneous singularities in s_1 and s_2 , the contribution of a factorizable Regge pole has the form $5), 25), *)$

$$\begin{aligned} A^5 &\sim m^2 V_{15} \Gamma(-\alpha_1) \left[(-s_{m_4})^{\alpha_1} (-s_2)^{\alpha_2 - \alpha_1} \bar{V}_1 + (-s_{m_4})^{\alpha_1} (-s_1)^{\alpha_1 - \alpha_2} \bar{V}_2 \right] \\ &\quad \times \Gamma(-\alpha_2) V_{23} \end{aligned} \quad (4.11)$$

The discontinuity under consideration comes only from the second term

$$\begin{aligned} \text{disc}_s \text{disc}_{s_1} A^5 &\sim m^2 V_{15} \left(\frac{2\pi i (s_1/m^2)^{\alpha_1}}{\Gamma(\alpha_1+1)} \right) \left(\frac{m^2 \pi (\alpha_2 - \alpha_1)}{2\pi i \Gamma(-\alpha_1)} \bar{V}_2 \right) \\ &\quad \times \left(\frac{2\pi i (s_2/m^2)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right) V_{23} \end{aligned} \quad (4.12)$$

*) A factor of m^2 has been extracted to show explicitly the dimensions of the amplitude.

Comparing with (4.10) and using (I.3.24) for V_{ij} allows us to identify

$$\bar{V}_2 = \frac{2}{\sqrt{\pi N_c}} \left(\frac{m_4^2}{m^2} \right)^{-\alpha_2} \frac{\sin \pi \alpha_1}{\sin \pi (\alpha_1 - \alpha_2)} \sqrt{\Gamma(\alpha_1+1) \Gamma(\alpha_2+1)} v^d v_\psi e^{\bar{d}} \quad (4.13)$$

Similarly studying $\text{disc}_s \text{disc}_{s_2} A^5$ (Fig. 4.2b) allows us to identify

$$\bar{V}_1 = \frac{2}{\sqrt{\pi N_c}} \left(\frac{m_4^2}{m^2} \right)^{-\alpha_1} \frac{\sin \pi \alpha_2}{\sin \pi (\alpha_2 - \alpha_1)} \sqrt{\Gamma(\alpha_1+1) \Gamma(\alpha_2+1)} v^e v_\psi d^{\bar{e}} \quad (4.14)$$

We should note one interesting property of the two-Regge-one-meson vertices (4.13), (4.14), namely that apart from trivial factors arising from the trajectories α_1 and α_2 , they are independent of the quark b . This direct channel factorization parallels that for the one-Regge-two-meson vertex found in I. As usual since the analytic properties of amplitudes are not evident in our representations we cannot check explicitly that the full amplitude has the form (4.11). If the amplitude has normal analyticity properties as expected then (4.11) will follow^{5),25)} so we need not concern ourselves with it.

The $2 \rightarrow 4$ amplitude (see Fig. 4.3) is also easily seen to have "triple-Regge" behaviour in the limit

$$s, s_{12}, s_{23}, s_1, s_2, s_3 \rightarrow \infty$$

$$s_1 s_2 \simeq m_5^2 s_{12}, \quad s_2 s_3 \simeq m_4^2 s_{23}, \quad s_1 s_2 s_3 = m_4^2 m_5^2 s \quad (4.15)$$

by the arguments given above. Factorizability of the Regge pole residues has the consequence that all multiple discontinuities of A^6 factorize when there are no singularities in overlapping energy variables^{5),26)}. We have verified that all triple discontinuities of A^6 factorize as required. As an illustration we compute the triple discontinuity shown in Fig. 4.3b which is the most difficult one to calculate. The contribution of factorizable Regge poles should have the form⁵⁾

$$\text{disc}_s \text{ disc}_{s_1} \text{ disc}_{s_3} A^6 \sim m^2 V_{16} \frac{2\pi i (S_1/m^2)^{\alpha_1}}{\Gamma(\alpha_1+1)} \left(\frac{\sin \pi(\alpha_1-\alpha_2)}{\sin \pi \alpha_1} \bar{V}_2 \right) \\ \times \frac{2\pi i (S_2/m^2)^{\alpha_2}}{\Gamma(\alpha_2+1)} \left(\frac{\sin \pi(\alpha_3-\alpha_2)}{\sin \pi \alpha_3} \bar{V}_2 \right) \frac{2\pi i (S_3/m^2)^{\alpha_3}}{\Gamma(\alpha_3+1)} V_{23}. \quad (4.16)$$

In two-dimensional QCD we have

$$\text{disc}_s \text{ disc}_{s_{12}} \text{ disc}_{s_{23}} A^6 \simeq G_{1,2,n} \frac{2i}{\pi m^2} G_{n,n_1,n_2} \frac{2i}{\pi m^2} \\ G_{n,S,6} \frac{2i}{\pi m^2} G_{n_2,3,4}, \quad (4.17)$$

where

$$n \simeq \frac{S}{\pi^2 m^2}, \quad n_1 \simeq \frac{S_1}{\pi^2 m^2}, \quad n_2 \simeq \frac{S_3}{\pi^2 m^2}$$

To compute this we need the behaviour of the three-meson vertex for three large masses. This is easily obtained from (4.9) using again (4.8),

$$G_{123} \simeq_{m_3^2 \gg m_1^2, m_2^2 \gg m^2} \left(2 \sqrt{\frac{\pi}{N_c}} m^2 \right) \left(\frac{m_3^2 m^2}{m_1^2 m_2^2} \right)^{-\beta_b} V^b V^b \quad (4.18)$$

Substituting (4.7) and (4.18) in (4.17) gives

$$\text{disc}_s \text{ disc}_{s_{12}} \text{ disc}_{s_{23}} A^6 \sim m^2 \frac{(i)^3 2^7}{\pi N_c^2} \left(\frac{S}{m^2} \right)^{-\beta_b} V_1^b \bar{V}_2^b \bar{b} \bar{c} \\ \left(\frac{S_1 m^2}{S_1 S_3} \right)^{-\beta_e} V^e V^e \left(\frac{S_1}{m^2} \right)^{-\beta_f} V_6^f \bar{V}_5^f \bar{f} \bar{e} \left(\frac{S_3}{m^2} \right)^{\beta_d} V_4^d \bar{V}_3^d d \bar{e} \bar{c} \quad (4.19)$$

It is easy to check that this is indeed of the form (4.16) using (4.13) and (4.14).

The general $2 \rightarrow n$ production amplitude can be treated in exactly the same manner using (4.7), (4.9) and (4.18) and it is clear that the factorization of discontinuities will hold although we have not gone through the algebra. For the reasons discussed above it is difficult to treat the full amplitudes but if normal analyticity holds the results for the factorization of the $(n-1)$ fold discontinuities will imply the usual form for it. We therefore have the remarkable result that discontinuities of production amplitudes exhibit factorizable "multi-Regge" behaviour in two-dimensional QCD.

5. - INCLUSIVE CROSS-SECTIONS

We now turn to inclusive hadronic cross-sections. The simplest diagrams contributing to the inclusive cross-section for $1+2 \rightarrow 3+X$ are those where X is a single resonance. The amplitude for such a process has the same form as that shown in Fig. 2.1, which we have redrawn in Fig. 5.1. From the study of dual models we expect ^{5),8),27)} the amplitudes (a) and (b) to contribute in the fragmentation region for particle 1, amplitudes (b) and (c) to contribute in the fragmentation region for particle 2, and the central region to receive a contribution from amplitude (b) only. We will find that the diagrams of Fig. 5.1 in fact have interpretations in terms of quark fragmentation, and that the quark fragmentation functions are the same as those found by Einhorn ¹⁶⁾ in e^+e^- annihilation.

The inclusive cross-section is given by the discontinuity of the six particle amplitude shown in Fig. 5.2a :

$$2\pi \cdot 2E_1 \frac{d\sigma}{dp_3} \sim \frac{1}{s} \frac{1}{2i} \text{disc}_{M_X^2} A^6(s, s^*, M_X^2) \Big|_{s=R(s)+i\epsilon} \quad (5.1)$$

From general analyticity arguments we expect ^{5),28)} A^6 to have the generic form :

$$A^6 \approx (-M_X^2)^{\alpha_{b\bar{b}}} \left(\frac{s}{M_X^2}\right)^{\alpha_{b\bar{a}}} \left(\frac{s^*}{M_X^2}\right)^{\alpha_{\bar{b}a}} \sum_{i,j=0}^{\infty} a_{ij}(\alpha_{b\bar{b}}, \alpha_{b\bar{a}}, \alpha_{\bar{b}a}; \\ ; (p_1+p_2+p_3)^2, (p_1+p_3+p_2)^2, (p_2+p_3)^2) \left(\frac{M_X^2}{s}\right)^i \left(\frac{M_X^2}{s}\right)^j \quad (5.2) \\ + \text{ terms with no discontinuity in } M_X^2.$$

From this form it follows that ^{*)}

$$\text{disc}_{M_X^2} A^6 = \text{disc}_s \text{disc}_{s^*} \text{disc}_{M_X^2} A^6 \frac{e^{-i\pi\alpha_{b\bar{a}}} e^{i\pi\alpha_{\bar{b}d}}}{\sin\pi\alpha_{b\bar{a}} \sin\pi\alpha_{\bar{b}d}} \quad (5.3)$$

so that we can calculate the inclusive cross-section from the triple discontinuity illustrated in Fig. 5.2b. The triple discontinuity can in turn be written completely in terms of the three meson vertices G_{ijk} :

$$\begin{aligned} \text{disc}_s \text{disc}_{s^*} \text{disc}_{M_X^2} A^6 &= \sum_n \sum_{n^*} \sum_{n_x} \frac{G_{12n} (2im_n \Gamma_n)}{(s-m_n^2)^2 + m_n^2 \Gamma_n^2} G_{3n_x n} \\ &\quad \frac{2im_{n_x} \Gamma_{n_x}}{(m_x^2 - m_{n_x}^2)^2 + m_{n_x}^2 \Gamma_{n_x}^2} G_{3n_x n^*} \frac{-2im_{n^*} \Gamma_{n^*}}{(s^* - m_{n^*}^2)^2 + m_{n^*}^2 \Gamma_{n^*}^2} G_{12n^*} \quad (5.4) \end{aligned}$$

In the fragmentation limit

$$s \rightarrow +\infty, \quad \frac{M_X^2}{s} = 1 - x_F \quad \text{fixed} \quad (5.5)$$

we need (for G_{12n}, G_{n^*12}) the asymptotic form of the vertex for one meson mass large, which has already been obtained in I :

$$G_{12n} \approx_{m_n^2 \rightarrow \infty} 2 \sqrt{\frac{\pi}{N_c}} m^2 \left(\frac{m_n^2}{m^2} \right)^{-\beta_b} V_1^{b\bar{a}} V_2^{b\bar{c}} \quad (5.6)$$

We also need (for $G_{3n_x n}, G_{n_x 3n^*}$) the asymptotic form of the vertex for two masses large and proportional. Inserting the scaling forms (I.2.42) for Φ_{n_x} and Φ_n gives :

^{*)} The phase factor is unity in the present case. We have not verified explicitly that this phase obtains in two-dimensional QCD, but (5.3) can be directly verified in the dual model.

$$G_{3n \times n} \approx 2 \sqrt{\frac{\pi}{N_c}} m^2 \frac{p_{3-}}{p_{x-}} \int_0^1 dx \int_0^1 dx' \phi_3^{b\bar{a}}(x) \phi^c((1-x') \frac{M_x^2}{m^2})$$

$$\cdot \frac{[\phi^c((1-x') \frac{M_x^2}{m^2}) - \phi^c(\frac{M_x^2 + x(s-M_x^2)}{m^2})]}{(x \frac{p_{3-}}{p_{x-}} + x')^2} \quad (5.7)$$

where

$$\frac{p_{3-}}{p_{x-}} \approx \frac{s - M_x^2}{M_x^2} = \frac{x_F}{1 - x_F} \quad (5.8)$$

From (I.2.43b) we see that only the first term in the square brackets will survive integration, and indeed averages to unity, so that ¹⁶⁾

$$G_{3n \times n} \approx_{M_x^2, s \gg m^2} 2 \sqrt{\frac{\pi}{N_c}} m^2 \frac{p_{x-}}{p_{1-}} \int_0^1 dx \int_0^1 dx' \frac{\phi_3^{b\bar{a}}(x)}{(x + \frac{p_{x-}}{p_{3-}} x')^2}$$

$$= 2 \sqrt{\frac{\pi}{N_c}} m^2 G_3^{a\bar{b}}(1 - \frac{m_x^2}{s}) \quad (5.9)$$

where

$$G_h^{a\bar{b}}(x_F) \equiv (-1)^{n_h} \frac{1}{x_F} \int_{x_F}^1 dy \int_0^1 dx \frac{\phi_h^{a\bar{b}}(x)}{(x - y/x_F)^2} \quad (5.10)$$

A diagrammatic representation of $G_h^{a\bar{b}}(x_F)$ is given in Fig. 5.3. We can infer directly from (5.1), (5.3), (5.4) and (5.10) that

$$E_3 \frac{d\sigma}{dp_3} \propto [G_3^{a\bar{b}}(x_F)]^2 \quad (5.11)$$

which gives the inclusive cross-section an interpretation in terms of quark fragmentation $[a \rightarrow b + 3 \text{ in } (5.11)]$. If we compare (5.10) with the expression of Einhorn ¹⁶⁾ for the cross-section for $e^+e^- \rightarrow \text{hadron} + \text{anything}$, we see that the quark fragmentation function $[G_3^{a\bar{b}}]^2$ is universal. The final

state x_F distribution is independent of whether the initial quark is asymptotically free (as in e^+e^- annihilation) or a hadronic constituent (as in hadron-hadron collisions). This universality arises because no matter where it comes from, the quark is rescattered through intermediate bound state resonances, as seen in (5.4) and Fig. 5.4 - reminiscent of Watson's theorem? This universality is considerably stronger than that generally suggested in parton models, which usually restrict themselves to deep inelastic lepton-induced properties.

In order to determine the normalization in (5.11) we need a prescription for evaluating the sums over bound states in (5.4). We will need to average over both the direct channel and the missing mass bound states. Two prescriptions are available.

- (i) The classic dual model technique ²⁹⁾ of averaging the poles in s , s^* and m_x^2 , separately : then (5.4) becomes

$$\text{disc}_s \text{disc}_{s^*} \text{disc}_{m_x^2} A^b \approx G_{12n} \left(\frac{2i}{\pi m^2} \right) G_{3n_x n} \left(\frac{2i}{\pi m^2} \right) G_{3n_x n^*} \left(\frac{-2i}{\pi m^2} \right) G_{12n} \quad (5.12)$$

where

$$n \approx \frac{S}{\pi^2 m^2}, \quad n^* \approx \frac{S^*}{\pi^2 m^2}, \quad n_x \approx \frac{M_x^2}{\pi^2 m^2} \quad (5.13)$$

Using (5.1), (5.3), (5.4), (5.6), (5.9) and (5.12) we get :

$$2\pi \cdot 2E_3 \frac{d\sigma}{dp_3} \sim \frac{m^2}{s} \frac{2^6}{\pi N_c^2} [V_1^{b\bar{a}} V_2^{b\bar{c}}] [G_3^{a\bar{b}}(x_F)]^2 \left(\frac{s}{m^2} \right)^{-2\beta_b} / (\sin \pi \alpha_{b\bar{a}})^2 \quad (5.14)$$

which has the appropriate Mueller-Regge ⁶⁾ power behaviour, as well as the universal quark fragmentation (5.11). This method is most suitable when the resonance widths are not much narrower than their separations in mass :

$$\Gamma_n \text{ not } \ll 2\pi^2 m^2 \quad (5.15)$$

In QCD the resonance widths Γ_n are $O(1/N_c)$, so that the averaging procedure (i) may not be optimal. Indeed we see from (5.14) that it gives an inclusive cross-section $O(1/N_c^2)$, while there is an energy momentum sum rule

$$\sigma_R(1+2 \rightarrow X) = \sum_3 \int_0^1 dx_F x_F E_3 \frac{d\sigma_R}{dp_3} \quad (5.16)$$

reflecting unitarity, where $\sigma_R(1+2 \rightarrow X) \sim (1/s) \text{Im} A^{st}$

$$\approx \frac{m^2}{s} \frac{4}{N_c} [V_1^{b\bar{a}} V_2^{b\bar{c}}]^2 \left(\frac{s}{m^2}\right)^{-2\beta_b} \quad (5.17)$$

from (I.3.22). However, (5.14) might be a reasonable phenomenological approximation for moderate values of N_c , such as $N_c = 3$.

An alternative prescription for evaluating (5.4) was suggested by Einhorn ¹⁶⁾ :

(ii) Average over the products of the poles in s and s^* , and average separately in M_x^2 . This method takes account of the fact that the poles in s and s^* pinch together as $N_c \rightarrow \infty$, enhancing the inclusive cross-section by the power of N_c required by unitarity (5.16). Rewriting (5.4) as

$$\begin{aligned} \text{disc}_s \text{disc}_{s^*} \text{disc}_{M_x^2} A^6 &\approx \sum_n (G_{12n})^2 \frac{4m_n^2 \Gamma_n^2}{[(s-m_n^2)^2 + m_n^2 \Gamma_n^2]^2} \times \\ &\times \sum_{n_x} (G_{3n_x n})^2 \frac{2im_{n_x} \Gamma_{n_x}}{[(m_x^2 - m_{n_x}^2)^2 + m_{n_x}^2 \Gamma_{n_x}^2]} \end{aligned} \quad (5.18)$$

and using

$$\left\langle \frac{m_{n_x} \Gamma_{n_x}}{[(m_x^2 - m_{n_x}^2)^2 + m_{n_x}^2 \Gamma_{n_x}^2]} \right\rangle \approx \left(\frac{1}{\pi^2 m^2}\right) \pi, \quad \left\langle \frac{m_n^2 \Gamma_n^2}{[(s-m_n^2)^2 + m_n^2 \Gamma_n^2]^2} \right\rangle \approx \left(\frac{1}{\pi^2 m^2}\right) \left(\frac{\pi}{2m_n \Gamma_n}\right) \quad (5.19)$$

we find

$$2\pi 2E_3 \frac{d\sigma}{dp_3} \sim \frac{m^2}{s} \frac{2s}{N_c^2} \left(\frac{m^2}{m_n \Gamma_n}\right) [V_1^{b\bar{a}} V_2^{b\bar{c}}]^2 [G_3^{a\bar{d}}]^2 \left(\frac{s}{m^2}\right)^{\beta_b} \quad (5.20)$$

Since $\Gamma_n = O(1/N_c)$, and $m_n \Gamma_n \rightarrow \text{constant}$ ¹⁶⁾ as $m_n \rightarrow \infty$ in the approximation where only the diagrams of Fig. 5.1 are used, the expression (5.20) has the correct (5.16) power of $1/N_c$, as well as the correct Mueller-Regge ⁶⁾ power behaviour and the universal quark fragmentation property (5.11). In this averaging procedure, diagrams with two or more resonances in the missing mass are actually of the same order in $1/N_c$. However, the universal quark fragmentation property will still hold, because of the direct channel factorization in e^+e^- and hadron-hadron collisions, expressed diagrammatically in Fig. 5.4.

The possibility that this universality of quark fragmentation holds in four dimensions seems worth pursuing both phenomenologically and theoretically. It is often said to be a phenomenological fact ²¹⁾ although usually one takes the scaling hadronic distribution, i.e., the Pomeron exchange contribution ($\sim s^0$) while here we are discussing the Regge-exchange contribution ($\sim s^{-\frac{1}{2}}$). (We turn below to the Pomeron contribution.) Should we expect this universality to hold in four dimensions? Firstly, it must be borne in mind that the valence quark approximation where the bound states have no gluon constituents or quark-antiquark sea will presumably break down. It should also be noted that there is an important peculiarity of two dimensions. There is only one bound state of a given mass so that we have factorization between initial states (e.g., e^+e^- , $h_1 h_2$, etc.) and final states (e.g., $h+X$). This leads to the direct channel factorization of Regge amplitudes discussed in Section I.3 and Section 4 of this paper as well as the universality under discussion. This universality can in principle lead to a clash between triple-Regge behaviour ⁵⁾ of inclusive distributions (see Fig. 5.5)

$$\frac{d\sigma}{dx_F} \propto (1-x_F)^{\alpha_1 - \alpha_2 - \alpha_3} \quad (5.22)$$

and the behaviour of the e^+e^- annihilation cross-section as $x \rightarrow 1$

$$\frac{d\sigma}{dx} \propto (1-x)^{2\gamma} \quad (5.23)$$

which is related to form factors

$$F(q^2) \propto (q^2)^{-\gamma-1} \quad (5.24)$$

In two dimensional QCD no such clash exists since

$$\alpha_1 - \alpha_2 - \alpha_3 = (-2\beta_b) - 2(-\beta_b - \beta_d) = 2\beta_d$$

and $\gamma = \beta_d$ ^{*)}. The formula (5.22) makes explicit reference to the partial waves α_i in the various crossed channels indicated in Fig. 5.5. Therefore we might expect that in four dimensions universality which relates the infinite sum over crossed channel partial waves in hadron induced reactions to the single partial wave in e^+e^- annihilation might break down when the triple-Regge formula (5.21) applies. For x outside this range, probably $x \leq 0.8$, there seems to be more chance that universality might hold.

So far we have not discussed explicitly the diagram of Fig. 5.1b. We did not need to compute its contribution in the fragmentation region of particle 1 because we were able to use the analytic structure (5.2) ^{**)} to express the inclusive cross-section in terms of the triple discontinuity (5.3) which is only possessed by Fig. 5.1a. However, it is easy to check that Fig. 5.1b has the appropriate triple-Regge limit. As for the double Regge central region limit, using the relations

$$t \sim s - M_x^2, \quad u \sim \frac{m_3^2}{1 - M_x^2/s}$$

from which it follows that

$$p_3/p_- \sim -m_3^2/t, \quad p_x/p_- \sim (1 - \frac{u}{M_x^2})$$

etc., it is easy to see that Eqs. (4.5), (4.6) and (4.7) give a contribution

$$\text{disc}_{M_x^2} A_6 \sim t^{-2\beta_a} u^{-2\beta_d} \quad (5.25)$$

^{*)} The implicit connection this gives between Regge intercepts and form factor powers is not the same as that advocated by Bjorken and Kogut ³⁰⁾ on the basis of their correspondence principle.

^{**)} Recall that like most analyticity properties of two-dimensional QCD, this has not been verified explicitly.

This is the typical Mueller double-Regge ⁶⁾ behaviour.

What of the scaling inclusive hadron distributions ? Although the Pomeron amplitude does not lead to a constant cross-section in two-dimensional QCD, as shown in Section 4, the diagram 2.2d suggests a very simple picture. Each of the incoming mesons emits quarks with finite momentum fractions, which then recombine to yield two high mass bound states. We expect the dominant contribution to the inclusive cross-section to come from processes like that shown in Fig. 5.6. We are therefore led to speculate that the scaling contribution to the inclusive cross-section may take the approximate form ^{*)}

$$\frac{d\sigma}{dx_F} \propto \sum_{qq'} \int_0^1 dy P_q(y) G_n^{q \rightarrow q'}(x/y) \quad (5.26)$$

where $P_q(y)$ is the quark probability density, as measured in deep inelastic scattering and given by $[\Phi(y)]^2$ in the two-dimensional model, and the sum runs over all quarks and antiquarks in the incoming hadron.

6. - CONCLUSIONS AND PHENOMENOLOGICAL SPECULATIONS

Although some people find a two-dimensional theory too unrealistic to be a useful model for strong interactions, it remains true that two-dimensional QCD ¹²⁾ reflects many of our prejudices about quark confinement, bound state spectra ¹²⁾, deep inelastic processes ^{13),16),23)} and hadronic scattering ^{10),11)}. In I we exhibited "Regge" behaviour for forward scattering in order $1/N_c$, and in this paper we have confirmed this interpretation by exhibiting pole renormalization and "cuts" in order $1/N_c^2$ ¹⁵⁾, together with multi-Regge behaviour ⁵⁾ and Mueller-Regge ⁶⁾ formulae for inclusive cross-sections. A big theoretical defect of the model - the absence of dynamical gluon degrees of freedom - lies behind the failure of the model to produce a new "Pomeron" vacuum singularity in order $1/N_c^2$ ⁴⁾. Apart from this, it seems that QCD is consistent with dual and topological expansion ideas ^{8),14),15)}, at least in two dimensions.

^{*)} This equation cannot be exactly correct since it makes no reference to transverse momenta and these must play an essential rôle in giving a non-vanishing Pomeron, as discussed in Section 2.

In addition to confirming familiar ideas, the model suggests interesting phenomenological possibilities. Some of these ¹¹⁾ (additivity and a lower limit for Regge intercepts, direct channel factorization of Regge residues, and connections between Regge, form factor and fixed angle power behaviours) we discussed at the end of Paper I. We now would like to emphasize some interesting phenomenological properties found in this analysis of higher order scattering amplitudes.

(a) A Pomeron intercept near 1 seems a natural expectation for QCD in four dimensions. Individual diagrams in two dimensions have $J=1$, and the cancellation in the sum seems peculiar to two dimensions. The result $J=1$ corresponds to eikonalized exchange of spin 1 gluons, as in the models of Low ²⁰⁾ and Nussinov ²¹⁾, and is not an obvious reflection of multi-peripheral exchanges pumping up a new singularity through s channel unitarity ³¹⁾, but it is not clear the two pictures are incompatible. It is not obvious the Pomeron in four-dimensional QCD will be exactly at 1 at $t=0$: quark loops may modify ¹⁵⁾ the intercept by $O(N_F/N_C)$, and multi-gluon interactions may also have an effect.

(b) Direct channel factorization of multi-Regge vertices, as shown in Eqs. (4.13) and (4.14), occurs in a similar way to the direct channel factorization of single Regge vertices discussed in I. It would be interesting to devise tests of this possibility for the double- and triple-Regge vertices accessible experimentally.

(c) Universality of quark fragmentation into hadrons, independently whether the quark is produced "asymptotically free" in a deep inelastic process or in a "soft" hadronic collision, is a property of the model. Often favoured phenomenologically, we are not aware that this possibility has been tested in the precise form (5.12) found here, namely a strict proportionality between S^{α_R-1} Mueller-Regge corrections to inclusive cross-sections in hadronic collisions, and the hadron distributions in $e^+e^- \rightarrow h + X$.

(d) A convolution model for Feynman scaling distributions (5.26) is suggested by the model's cylinder "pseudo-Pomeron" indicated in Fig. 2.2. The conjecture (5.13) is speculative because the model actually has no Pomeron at $J=1$, and transverse degrees of freedom (e.g., the p_T of partons) must play an essential rôle. Still, other people have suggested it as well, and it is a natural consequence of the two-fireball picture of Fig. 5.6 which underlies many models of the bare Pomeron.

Despite the defects of two-dimensional QCD, we hope it may serve as a starting point for new developments in building four-dimensional models for strong interactions. Perhaps one can develop a phenomenological parton model with properties abstracted from the two-dimensional model. Perhaps new approaches to constructing an order $1/N_c$ meson-meson scattering Born term will be suggested. Maybe more sophisticated justifications will be found for some of the relations between "hard" and "soft" hadronic processes found in this model. A strong interaction phenomenology of QCD may exist.

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FIGURE CAPTIONS

- Figure 2.1 The three quark-antiquark exchange diagrams contributing to the meson-meson scattering amplitude in order $1/N_c$.
- Figure 2.2 Contributions to meson-meson scattering in order $1/N_c^2$.
- Figure 2.3 Multigluon exchange graphs included in Fig. 2.2d.
- Figure 2.4 Rectangle showing phase space in Eq. (2.1). The outer shaded corner is the Pomeron contribution considered in Section 2. The inner shaded region might have given a Regge cut, while the shaded diagonal stripe gives Regge pole renormalization : see Section 3.
- Figure 2.5 Three contributions to A^{tu} .
- Figure 2.6 Possible orderings for the meson-quark-antiquark vertices in Fig. 2.5a.
- Figure 2.7 Diagrams contributing to A^{tu} .
- Figure 2.8 The Drell-Yan diagram for dilepton production.
- Figure 3.1 Double triple-Regge behaviour.
- Figure 4.1 A quark-antiquark exchange diagram contributing to the (2-3) meson production amplitude in leading order in $1/N_c$.
- Figure 4.2 Double discontinuities of the (2-3) amplitude of Fig. 4.1.
- Figure 4.3 (a) A typical piece of a (2-4) meson production amplitude, and (b) one of its triple discontinuities.
- Figure 5.1 Leading quark diagrams for the inclusive cross-section for $1 + 2 \rightarrow 3 + X$.
- Figure 5.2 (a) Contribution of Fig. 5.1a to the inclusive cross-section, and (b) the related triple discontinuity (5.4).

- Figure 5.3 A diagrammatic representation of the quark fragmentation function $G_h^{a\bar{b}}(x_F)$ (5.10).
- Figure 5.4 Scattering through intermediate resonances gives universal quark fragmentation functions and inclusive hadron cross-sections.
- Figure 5.5 Triple-Regge behaviour for the inclusive hadron cross-section.
- Figure 5.6 A possible model for the scaling inclusive hadron cross-section arising from the Pomeron.

FIG. 2.1

FIG. 2.2

FIG. 2.3

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FIG. 2.8

FIG. 3.1

FIG. 4.1

FIG.4.2

FIG.4.3

FIG. 5.1

FIG. 5.2

FIG. 5.3

FIG. 5.4

FIG. 5.5

FIG. 5.6