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HADRON SCATTERING IN TWO-DIMENSIONAL QCD I :
FORMALISM AND LEADING ORDER CALCULATIONS

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A B S T R A C T

Two-dimensional quantum chromodynamics in the $1/N_c$ expansion is explored as a model for hadronic interactions. In meson-meson scattering, order $1/N_c$ terms exhibit embryonic "Regge" behaviour, with the crossed channel factorization, signature and exchange degeneracy properties found in dual Regge models. "Regge" intercepts are additive in the quark quantum numbers exchanged, and have a lower limit for large quark masses. Residues also factorize in the direct channel. An analogue for fixed angle scattering is presented which has deviations from dimensional counting rules related to form factors and Regge intercepts.

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1. - INTRODUCTION

It is conceivable that strong interactions may arise from a non-Abelian $SU(3)$ gauge theory of coloured quarks and massless gluons, commonly called quantum chromodynamics (QCD). Interest in QCD has been stimulated by its asymptotic freedom ¹⁾, and by the possibility that it confines quarks. The deepest reason for liking QCD is probably the feeling that since weak, electromagnetic and gravitational forces all seem to have a gauge theoretical origin, why not strong interactions too? Unfortunately, quantitative calculations in QCD are very difficult, and so far we are often reduced to asking whether it agrees qualitatively with general prejudices about strong interaction dynamics.

In these qualitative analyses, we may divide hadronic reactions into three broad classes. First there are "hard" processes where the use of quark parton ideas ²⁾ is supported by clear-cut short-distance or light-cone arguments and asymptotic freedom ¹⁾. These "hard" reactions include the total cross-section for e^+e^- annihilation and the structure functions in deep inelastic scattering. On the other hand there are "soft" processes which are generally described by Regge ³⁾ and duality ⁴⁾ ideas, and short-distance and light-cone ideas are inapplicable. Such reactions include hadron-hadron scattering at small momentum transfers, and inclusive cross-sections at small p_T . In addition, there is a "grey area" of processes which are often thought to be "hard", but where the use of parton ideas is not underwritten by asymptotic freedom. Among such reactions are large angle elastic scattering and hadron production at large p_T ⁵⁾, and the Drell-Yan process for producing lepton pairs ⁶⁾.

In the absence of reliable methods for calculating QCD in four dimensions, a promising model to help us develop qualitative ideas about hadrodynamics is the 't Hooft model ⁷⁾. This is QCD in two dimensions, calculated in an expansion in inverse powers of the number of colours $(1/N_c)$ ⁷⁾⁻⁹⁾. The model is clearly asymptotically free, and has a linearly rising Coulomb potential which confines quarks ⁷⁾. It is therefore ideal for studying how sums over hadronic bound states can act "as if" essentially free quark constituents were involved. Previous authors ^{7),10),11)} have shown that two-dimensional QCD does indeed reproduce the parton ideas of Bjorken and Feynman ²⁾ in "hard" processes like e^+e^- annihilation and deep inelastic scattering, as expected on the basis of asymptotic freedom.

This is the first of two papers in which we apply the 't Hooft model to hadronic scattering, mainly to "soft" processes ³⁾ seeking whether QCD can show embryonic Regge behaviour or a Pomeron in two dimensions. Here success is not guaranteed by short-distance arguments. However, Veneziano ⁹⁾ and others have used $1/N$ expansions in QCD to motivate many aspects of duality and Regge models with narrow resonances, and we will check some of these ideas in a well-defined and relevant model. In addition to "soft" processes, we also study a possible two-dimensional analogue to the "grey" process of fixed angle elastic hadron-hadron scattering, where we can confront hard scattering ideas ⁵⁾. We find that as well as confronting commonly-held prejudices about these areas of hadrodynamics, two-dimensional QCD suggests some novel features and relations between them and "hard" processes, which we discuss at the end of our paper.

We should remember some important limitations of two-dimensional QCD. High energy hadron scattering does look approximately two-dimensional, with transverse momenta limited. But there are important differences between two and four dimensions. From an S matrix point of view, crossing is a much feebler constraint in two-dimensions, there can be no duality ⁴⁾ between direct and crossed channel poles, and there can, of course, be no moving Regge trajectories. From a field theoretical point of view, there are no dynamical degrees of freedom corresponding to gluons in two dimensions, and there are no multi-gluon interactions. The absence of dynamical gluonic degrees of freedom entails the absence of gluonic bound states (gluonium). Despite these defects, we will see that two-dimensional QCD exhibits ¹²⁾ in embryo many features of high energy scattering expected ⁹⁾ on the basis of dual Regge models and topological expansions.

To understand simply why these duality features arise in two dimensions, it is helpful to recall for a moment the $1/N_c$ expansion in four dimensions. What properties do we expect for the leading terms in the expansion of a four-meson scattering amplitude ?

$$A(s,t) = \frac{1}{N_c} A_{(1)} + \frac{1}{N_c^2} A_{(2)} + \dots \quad (1.1)$$

The lowest order $1/N_c$ amplitude is given by a sum over planar graphs, one corresponding to each of the three duality diagrams illustrated in Fig. 1.1. These have $q\bar{q}$ states in the (s,t) , (t,u) and (u,s) channels, respectively. Suppose that colour is confined for the leading term $A_{(1)}$. Since $q\bar{q}$ thresholds are excluded by confinement, and meson-meson thresholds

involve internal quark loops which are suppressed by $1/N_c$, $A_{(1)}$ has no threshold cuts. Hence $A_{(1)}$ must be some sort of hadronic Born term with only zero-width bound state poles.

Experience with planar bound state scattering amplitudes suggests that $A_{(1)}^{st}$ should have Regge behaviour and obey an unsubtracted dispersion relation at fixed t . Hence

$$A_{(1)}^{st}(s, t) = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{\text{Im } A_{(1)}^{st}(s', t)}{s' - s} ds' = \sum_{n=0}^{\infty} \frac{g_n^2(t)}{s - m_n^2} \quad (1.2)$$

where the $g_n(t)$ are polynomials in t . The singularities in t must come entirely from divergences in the sum over s channel poles :

$$A_{(1)}^{st} = \sum_n \frac{g_n^2(t)}{s - m_n^2} = \sum_{n'} \frac{\tilde{g}_{n'}(s)}{t - m_{n'}^2} \quad (1.3)$$

Hence confinement and unsubtracted dispersion relations for the leading order $1/N_c$ amplitude implies generic duality⁴⁾ for meson scattering amplitudes.

We can contrast this situation with a perturbative Born term as in Φ^3 theory :

$$A^{st} = \frac{g^2}{s - m^2} + \frac{\tilde{g}^2}{t - m^2} \quad (1.4)$$

Taking $s \rightarrow \infty$ we see that the pole in t gives a $J=0$ fixed pole and hence a subtraction constant in a fixed t dispersion relation, while the pole in s gives rise to fixed poles at $J=-1, -2, \dots$. On the other hand, for the hadronic Born term $A_{(1)}^{st}$ we expect a Regge power so that

$$A_{(1)}^{st} = \sum_n \frac{g_n^2(t)}{s - m_n^2} \approx \beta(t) (-s)^{\alpha(t)} \quad (1.5)$$

as $|s| \rightarrow \infty$ ^{*)}. This last statement is a purely s channel duality condition : the sum over s channel resonances gives Regge behaviour as $|s| \rightarrow \infty$. It is this form of duality that we verify explicitly for two-dimensional

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*) And $\text{Re } s / \text{Im } s \neq 0$.

QCD : we cannot verify the form (1.3) because we cannot continue to the t channel in two dimensions ^{*}). Although we have not proved unsubtracted dispersion relations for $A_{(1)}^{st}$ in two-dimensional QCD, all the properties we find accord with the above description, which follows from dispersion relations.

As in dual models, the properties of $A_{(2)}$, the next order in $1/N_c$, can be anticipated by considering the first unitarity corrections to the hadronic Born term. The anticipated effects - widths, Regge-Regge cuts, etc. - are all found ^{12),13)} in two-dimensional QCD with the exception of a new "Pomeron" singularity with $J \approx 1$ ¹⁴⁾. These points will be discussed in a second paper ¹³⁾, together with multi-Regge behaviour and inclusive reactions.

The layout of our paper is as follows. In Section 2 we review some features ^{7),10),11)} of two-dimensional QCD, expressed in terms of a first-quantized formalism for the bound states using time-ordered perturbation theory. As an aside, we give a simple argument why the lowest mass bound state decouples ¹⁵⁾ from the model when its mass goes to zero. In Section 3 we discuss meson-meson scattering to first order in $1/N_c$. We demonstrate Regge-like high energy behaviour, with forward amplitudes behaving as non-integral powers of s : $s^{\alpha_{b\bar{d}}}$. The "intercept" $\alpha_{b\bar{d}} = -\beta_b - \beta_d$ is additive in the quark and antiquark exchanged in the t channel and $\beta \rightarrow 0(1)$ when the quark mass $\rightarrow 0(\infty)$. The "Regge" amplitude has the factorization, signature and exchange degeneracy properties found in dual models ^{3),4),9)}. We demonstrate consistency between the Regge limit and the existence of a crossed channel particle which decouples as the exchanged quark masses $\rightarrow 0$. In Appendix B we discuss Compton scattering and exclusive photoproduction in the Regge limit. The results - in particular the difference between these processes and purely hadronic ones, may be very specific to two dimensions, and we hesitate to draw any comparison with the four-dimensional world. In Section 4 we study a possible two-dimensional analogue to fixed angle scattering ⁵⁾ in four dimensions. It involves studying a subamplitude of the $(2-2)0(1/N_c)$ amplitude in a high energy limit where no dynamical variable is kept finite. The high energy behaviour is a sum of terms of the type $s^{-2-\beta_i-\beta_j}$, where

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^{*}) Though we do study the consistency of our "Regge" behaviour with the bound state spectrum : see Section 3.

i and j are any pair of quarks or antiquarks in the external mesons. This result resembles that of dimensional counting¹⁶⁾ for small quark masses ($\beta \rightarrow 0$).

Section 5 summarizes the conclusions of this paper. It also has some speculations about the relevance of our results to four-dimensional QCD. We draw attention to some features of "soft" processes, and relations between these and other processes, which we may speculate to be of phenomenological interest in the real world.

2. - FORMALISM FOR BOUND STATES IN TWO-DIMENSIONAL GAUGE THEORY

In this section we outline the basic mathematical structure of two-dimensional QCD in the large N_c limit ⁷⁾. For further details the reader is referred to the papers of 't Hooft ⁷⁾, Callan, Coote and Gross ¹⁰⁾, and particularly Einhorn ¹¹⁾. Although we follow closely their discussion and notation, we also elaborate on those aspects which are most useful for studying scattering amplitudes for bound states. In particular we emphasize a formulation of the Feynman rules in which all quark propagators are eliminated, which we call the bound state formulation because all integrals involve only the bound state wave functions $\phi_n^{ab}(x)$ and the instantaneous gluon potential V . Although we have derived these rules only in a number of examples relevant to this paper, we are confident they can be made more general.

A. - Basic formalism

In terms of quark and gluon fields the Lagrangian is

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu, i}^2 G^{\mu\nu, i} + \bar{q}^{a, i} (i \not{D}_i - m_a \delta_i^i) q_i^a \quad \left. \begin{array}{l} \text{where} \\ G_{\mu\nu, i}^2 \equiv \partial_\mu A_{\nu, i} - \partial_\nu A_{\mu, i} + g [A_\mu, A_\nu]_i^i \\ \text{and} \\ D_{\mu, i} \equiv \partial_\mu \delta_i^i + g \tilde{A}_{\mu, i} \end{array} \right\} \quad (2.1)$$

The quarks q_i^a come in flavours $a=1, \dots, N_f$ for isospin, strangeness, charm... with masses m_a , and in colours $i=1, \dots, N_c$. The gluons are given by the anti-Hermitian, traceless $N \times N$ matrices $\tilde{A}_{\mu, i}^j$ ($\mu=0,1$) :

$$\tilde{A}_{\mu, i}^j = A_{\mu, i}^j - \frac{1}{N_c} \delta_i^j A_{\mu, k}^k \quad (2.2)$$

The trace of $A_{\mu, i}^j$ is an auxiliary field which is convenient for writing the Feynman rules, but has no effect on the theory because it is a free field in $\mathcal{L}$ (2.1) ^{*)}.

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^{*)} The differences between $1/N_c$ counting for diagrams in $SU(N_c)$ and $U(N_c)$ theories do not appear in the leading orders which we consider.

If we now choose the light-cone gauge ⁷⁾

$$A_- = \frac{1}{\sqrt{2}} (A_0 - A_1) = 0 \quad (2.3)$$

we see that in two dimensions there are no gluonic self-interactions ^{*)} :

$$[A_\mu, A_\nu] = 0 \quad (2.4)$$

We are aware of problems ⁷⁾ which arise when the model is formulated in linear gauges other than the light-cone gauge of 't Hooft ⁷⁾. We have no resolution of these problems, but feel that in the absence of any manifest inconsistency in the light-cone gauge, its use in calculations is reasonable. Since the quark-antiquark-gluon vertex involves $\gamma \cdot A = \gamma_+ A_-$, the only contributions of the internal quark propagators are proportional to $\gamma_- p_+$. The entire γ algebra is trivially performed and the matrix quark propagator can be replaced by a non-matrix form :

$$i \frac{\gamma_+ p_- + \gamma_- p_+ + m_a}{p^2 - m_a^2 + i\epsilon} \rightarrow i \frac{p_-}{p^2 - m_a^2 + i\epsilon} \quad (2.5)$$

coupled to constant vertices

$$-i g \gamma_- \rightarrow -2 i g \quad (2.6)$$

The A_+ gluon field is a non-dynamical potential instantaneous in the "time" x_- :

$$\frac{\partial^2}{\partial x_+^2} A_+ = -J \equiv -\sqrt{2} g q_R^+ q_R \quad (2.7)$$

whose Green's function in configuration space is ^{11), **)}:

^{*)} We will be using light-cone co-ordinates

$$x_\pm = x^\mp \equiv \frac{1}{\sqrt{2}} (x_0 \pm x_1), \quad p \cdot q = p_+ q_- + p_- q_+$$

$$\gamma_\pm \equiv \frac{1}{\sqrt{2}} (\gamma_0 \pm \gamma_1), \quad (\gamma_+)^2 = (\gamma_-)^2 = 0, \quad \{\gamma_+, \gamma_-\} = 2$$

^{**)} 't Hooft ⁷⁾ uses an infra-red cut-off with $A = 2/\pi\lambda$ and takes $\lambda \rightarrow 0$ but we prefer the so-called regular cut-off ¹¹⁾ $A = B = 0$.

$$D(x_-, x_+) = i \delta(x_-) \left(\frac{1}{2} |x_+| + B x_+ - A \right) \quad (2.8)$$

Without loss of generality we may take $B=0$, and by appropriate choice of gauge set $A=0$. In momentum space the gluon propagator is defined ¹¹⁾ by a principal value

$$P.V. \left(\frac{-i}{k_-} \right) \equiv \frac{-i}{2} \left[\frac{1}{(k_- - i\epsilon)^2} + \frac{1}{(k_- + i\epsilon)^2} \right] \quad (2.9)$$

The Feynman rules are summarized in Fig. 2.1. They are equivalent to the light-cone quantization of a theory of right-handed quarks $q_R \equiv \frac{1}{2}(1+\gamma_5)q$ with an interaction

$$H_I = g^2 \int dx_+ dy_+ q_R^\dagger(x_+) q_R(x_+) |x_+ - y_+| q_R^\dagger(y_+) q_R(y_+) \quad (2.10)$$

where colour indices have been suppressed.

B. - Bound state wave functions and formalism

To study the theory for large N_c with $g^2 N_c$ fixed, we first consider the quark-antiquark scattering amplitude :

$$T_{\alpha\beta, \gamma\delta} = (\gamma_-)_{\alpha\beta} (\gamma_-)_{\gamma\delta} T(q, q'; p) \quad (2.11)$$

where the momenta q, q' and p are defined in Fig. 2.2a. In the leading order of $1/N_c$, T is a sum over planar diagrams like that indicated in Fig. 2.2b. These have two effects : first the quark lines are renormalized, and second there is a Bethe-Salpeter equation for the exchange of the rungs in the ladder. With the cut-off we have chosen, the effect of renormalizing the quark propagator is just to shift the mass squared m_a^2 of the quark by

$$p\text{-}\bar{\Sigma}(p) = - g^2 \frac{N_c}{\pi} \quad (2.12)$$

so that the renormalized propagator becomes

$$iS^a(p) \equiv \frac{ip_-}{2p_+p_- - \tilde{m}_a^2 + i\epsilon} \quad : \quad \tilde{m}_a^2 = m_a^2 - \frac{g^2 N_c}{\pi} \quad (2.13)$$

The only non-zero self-energy graph is the one-gluon contribution of Fig. 2.3a ^{*)}.

Next we write down the integral equation for T which is indicated in Fig. 2.4 (all quark lines are renormalized) :

$$iT(q, q'; p) = \frac{ig^2}{(q_- - q'_-)^2} + 4g^2 N_c \int \frac{d^2 k}{(2\pi)^2} \frac{S^a(k) S^{\bar{b}}(p-k)}{(k_- - q_-)^2} T(k, q; p) \quad (2.14)$$

If we sit on the n^{th} bound state pole at $s \equiv 2p_+p_- \equiv \mu_n^2 (g^2 N_c / \pi)$ we get the homogeneous equation indicated in Fig. 2.5 for the proper vertex function $\Gamma_n^{a\bar{b}}$:

$$\Gamma_n^{a\bar{b}}(q; p) = -ig^2 \frac{N_c}{\pi^2} \int \frac{d^2 k}{(k_- - q_-)^2} S^a(k) S^{\bar{b}}(p-k) \Gamma_n^{a\bar{b}}(k; p) \quad (2.15)$$

It is convenient to introduce an equal "time" ($x_- = 0$) formalism by defining ⁷⁾ the wave function $\phi_n^{a\bar{b}}(x)$:

$$\phi_n^{a\bar{b}}(x) \equiv \frac{i}{\pi} \int_{-\infty}^{+\infty} dk_+ e^{ix \cdot k_+} S^a(k) S^{\bar{b}}(p-k) \Gamma_n^{a\bar{b}}(k; p) \Big|_{x_- = 0} \quad (2.16)$$

where x is the Feynman parameter for the fraction of p_- carried by the outgoing quark a :

$$x \equiv q_- / p_- \quad (2.17)$$

Doing the integral over k_+ , the vertex equation (2.15) becomes

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^{*)} The logarithmically divergent k_+ integral in Fig. 2.3a can be computed by a symmetric cut-off, and then the integral over k_- performed using the principal value gluon propagator. In all other quark irreducible diagrams with more than one gluon line (Fig. 2.3b), the outside loop k_+ integral is convergent and can be closed in the upper half-plane to give zero.

$$T_n^{a\bar{b}}(q;p) = \frac{g^2 N_c}{\pi p_-} \oint_0^1 \frac{dy}{(x-y)^2} \phi_n^{a\bar{b}}(y) \equiv \frac{1}{p_-} T_n^{a\bar{b}}(x) \quad (2.18)$$

For x inside $[0,1]$, the definitions (2.16) and (2.18) imply that

$$\begin{aligned} \phi_n^{a\bar{b}}(x) &= - \frac{1}{p_+ + \frac{\tilde{m}_a^2}{2q_-} - \frac{\tilde{m}_b^2}{2(p-q_-)} + i\epsilon} \frac{1}{2p_-} T_n^{a\bar{b}}(x) \\ &= - \frac{1}{m^2} \frac{1}{\mu_n^2 - \gamma_{a-1} \frac{1}{x} - \gamma_{b-1} \frac{1}{1-x}} T_n^{a\bar{b}}(x) \end{aligned} \quad (2.19)$$

where $\mu_n^2 = 2p_- p_+ / m^2$, $\gamma_i - 1 \equiv \tilde{m}_i^2 / m^2$ and we have introduced the basic mass scale parameter

$$m^2 \equiv \frac{g^2 N_c}{\pi} \quad (2.20)$$

For x outside $[0,1]$, the k_+ integral in (2.16) can be completed in the upper half plane to show that $\phi_n^{a\bar{b}}(x) = 0$.

Combining Eqs. (2.18) and (2.19) we have ⁷⁾ the wave equation

$$\mu_n^2 \phi_n^{a\bar{b}}(x) = H \phi_n^{a\bar{b}}(x) \quad (2.21)$$

where the "Hamiltonian" $H = H_0 + V$ is given by

$$H_0 \phi_n^{a\bar{b}}(x) = \left(\gamma_{a-1} \frac{1}{x} + \gamma_{b-1} \frac{1}{1-x} \right) \phi_n^{a\bar{b}}(x) \quad (2.22a)$$

$$V \phi_n^{a\bar{b}}(x) = - \oint_0^1 \frac{\phi_n^{a\bar{b}}(y) dy}{(x-y)^2} \quad (2.22b)$$

The wave functions $\phi_n^{a\bar{b}}(x)$ have a "parity" property :

$$\phi_n^{a\bar{b}}(x) = (-1)^n \phi_n^{b\bar{a}}(1-x) \quad (2.23)$$

and obey the boundary conditions

$$\phi_n^{a\bar{b}}(x) \sim \begin{cases} C_n^{a\bar{b}} x^{\beta_a} \\ (-1)^n C_n^{b\bar{a}} (1-x)^{\beta_b} \end{cases} \text{ as } x \rightarrow \begin{cases} 0 \\ 1 \end{cases} \quad (2.24)$$

with

$$\pi \beta_i \omega_i \pi \beta_i + \gamma_i - 1 = 0 \quad (2.25)$$

and we remark that for

$$m_i \rightarrow \begin{cases} 0 \\ \infty \end{cases} \quad \beta_i \rightarrow \begin{cases} 0 \\ 1 \end{cases} \quad (2.26)$$

With these boundary conditions, the wave equation can be solved, albeit numerically, for a complete, orthonormal set of real wave functions

$\phi_n^{a\bar{b}}(x)$:

$$\int_0^1 dx \phi_n^{a\bar{b}}(x) \phi_m^{a\bar{b}}(x) = \delta_{nm}, \quad \sum_n \phi_n^{a\bar{b}}(x) \phi_n^{a\bar{b}}(x') = \delta(x-x') \quad (2.27)$$

Next we wish to solve the Lippmann-Schwinger integral equation (2.14) for $T(q, q'; p) \equiv T(x, x'; p_-)$. It is convenient to do this by introducing operators $\hat{H} = \hat{H}_0 + \hat{V}$, $\hat{T}$, etc., operating on the states $|x\rangle$ (see Appendix A for a construction of these states in terms of the second quantized quark operators) :

$$\langle x | y \rangle = \delta(x-y), \quad I = \int dx |x\rangle \langle x|$$

$$\phi_n^{a\bar{b}}(x) = \langle x | \phi_n^{a\bar{b}} \rangle \quad (2.28)$$

with bound state vectors $|\phi_n^{a\bar{b}}\rangle$ obeying the wave equation

$$\mu_n^2 |\phi_n^{a\bar{b}}\rangle = \hat{H} |\phi_n^{a\bar{b}}\rangle = (\hat{H}_0 + \hat{V}) |\phi_n^{a\bar{b}}\rangle \quad (2.29a)$$

where

$$\left. \begin{aligned} \langle x | \hat{H}_0 | y \rangle &= \left(\gamma \frac{a-1}{x} + \gamma \frac{b-1}{1-x} \right) \delta(x-y) \\ \langle x | \hat{V} | y \rangle &= - \frac{1}{(x-y)^2} \end{aligned} \right\} \quad (2.29b)$$

The integral equation for $\hat{T}$ defined by $T(x, x'; p_-) = (-g^2/p_-^2) \langle x | \hat{T} | x' \rangle$ takes the standard form

$$\hat{T} = \hat{V} + \hat{V} \frac{1}{\mu^2 - \hat{H}_0} \hat{T} \quad (2.30)$$

It is now straightforward to solve the integral equation (2.30) for $\hat{T}$. The solution may be expressed in four convenient and equivalent ways :

$$\hat{T} = -(\mu^2 - \hat{H}_0) + (\mu^2 - \hat{H}_0) \hat{G} (\mu^2 - \hat{H}_0) \quad (2.31a)$$

$$\hat{T} = (\mu^2 - \hat{H}_0) \hat{G} \hat{V} = \hat{V} \hat{G} (\mu^2 - \hat{H}_0) \quad (2.31b)$$

$$\hat{T} = \hat{V} + \hat{V} \hat{G} \hat{V} \quad (2.31c)$$

in terms of the basic Green's function for the wave equation (2.29) :

$$\hat{G}^{ab} = \frac{1}{\mu^2 - \hat{H}} = \sum_{n=0}^{\infty} \frac{|\phi_n^{ab}\rangle \langle \phi_n^{ab}|}{\mu^2 - \mu_n^2} \quad (2.32)$$

Notice that $G^{ab}(x, x'; \mu^2) \equiv \langle x | \hat{G}^{ab} | x' \rangle$ is non-zero only for x and x' in the interval $[0, 1]$. Thus the four expressions (2.31) for $T \equiv -g^2/p_-^2 \langle x | \hat{T} | x' \rangle$ will coincide only when both x and x' are in the interval $[0, 1]$. The four forms of $\hat{T}$ reflect the two expressions (2.18), (2.19) already encountered for the proper vertex $\Gamma_n^{ab}(x) \equiv \langle x | \Gamma_n^{ab} \rangle$. For $x \in [0, 1]$ the expression

$$\frac{1}{\mu^2 - H_0} |\Gamma_n^{ab}\rangle = -m^2 |\phi_n^{ab}\rangle \quad (2.33)$$

can be used for $|\Gamma_n^{ab}\rangle$ [or Eq. (2.19) for $\Gamma_n^{ab}(x)$], but for $x \notin [0,1]$ we need to re-express $|\Gamma_n^{ab}\rangle$ using the wave equation (2.18)

$$|\Gamma_n^{ab}\rangle = -m^2 \hat{V} |\phi_n^{ab}\rangle \quad (2.34)$$

The two expressions (2.33) and (2.34) are both useful : for $x \in [0,1]$, Eq. (2.33) shows that the old-fashioned perturbation theory denominator $^{*)}$ for the $q\bar{q}$ threshold can be absorbed into the proper vertex as indicated in Fig. 2.6a. For $x \notin [0,1]$ the expression (2.34) inserts a gluon exchange $\hat{V}$ to turn around the quark which is moving the "wrong way", such as the quark with $x < 0$ indicated in Fig. 2.6b.

The four expressions (2.31) for $\hat{T}$ are analogous. For both x and $x' \in [0,1]$, the expression (2.31a) is used to cancel $q\bar{q}$ denominators as in Fig. 2.7a :

$$\frac{1}{\mu^2 - \hat{H}_0} \hat{T} \frac{1}{\mu^2 - \hat{H}_0} = -\frac{1}{\mu^2 - \hat{H}_0} + \hat{G} \quad (2.35)$$

follows directly from (2.31a). In terms of $T(x, x'; p_-) = (-g^2/p_-^2) \langle x | \hat{T} | x' \rangle$ Eq. (2.35) becomes ¹¹⁾

$$\frac{1}{\mu^2 - \frac{\gamma_{a-1}}{x} - \frac{\gamma_{b-1}}{1-x}} T(x, x'; p_-) \frac{1}{\mu^2 - \frac{\gamma_{a-1}}{x'} - \frac{\gamma_{b-1}}{1-x'}} = \frac{g^2}{p_-^2} \left[\frac{\delta(x-x')}{\mu^2 - \frac{\gamma_{a-1}}{x} - \frac{\gamma_{b-1}}{1-x}} - \sum_n \frac{\phi_n^{ab}(x) \phi_n^{ab}(x')}{\mu^2 - \mu_n^2} \right] \quad (2.36)$$

Later on the δ function (no gluon exchange) piece in Fig. 2.7a and Eq. (2.36) will be cancelled leaving just a bound state propagator $G^{ab}(x, x'; \mu^2)$.

For $x \notin [0,1]$, $x' \in [0,1]$, we will have only one $q\bar{q}$ denominator for the right-hand side of T , and we can use (2.31b), as indicated in Fig. 2.7b, to give

$$\hat{T} \frac{1}{\mu^2 - \hat{H}_0} = \hat{V} \hat{G} \quad (2.37)$$

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^{*)} In our figures these denominators are represented by dashed lines.

or

$$T(x, x'; p_-) \frac{1}{\mu^2 - \frac{\gamma_{c-1}}{x} - \frac{\gamma_{b-1}}{1-x}} = \frac{g^2}{p_-^2} \int_0^1 dy \frac{G(y, x'; \mu^2)}{(y-x)^2} \quad (2.38)$$

A similar expression holds for $x' \notin [0, 1]$ and $x \in [0, 1]$.

Finally, when both $x, x' \notin [0, 1]$, Eq. (2.31c) implies as in Fig. 2.7c that

$$\begin{aligned} T^{ab}(x, x', p_-) &= \frac{g^2}{p_-^2} \left[\frac{1}{(x-x')^2} - \frac{1}{m^4} \sum_n \frac{T_n^{ab}(x) T_n^{ab}(x')}{\mu^2 - \mu_n^2} \right] \\ &= \frac{g^2}{p_-^2} \left[\frac{1}{(x-x')^2} - \int_0^1 dy \int_0^1 dy' \frac{G(y, y'; \mu^2)}{(y-x)^2 (y'-x')^2} \right] \end{aligned} \quad (2.39)$$

Parenthetically, we note that by going to the poles in the bound state propagator $i/p_-^2 - m_n^2 + i\epsilon$ we can extract the correct normalization for incorporating the bound state quark-antiquark vertex in the Feynman rules of Fig. 2.1 ^{*)}:

$$-2i \left(\frac{\pi}{N_c} \right)^{1/2} \frac{1}{p_-} T_n^{ab}(x) \quad (2.40)$$

We will see later that the result of using the identities (2.36), (2.38) and (2.39) is always to cancel the $q\bar{q}$ continuum and replace it by the bound state propagator $G(x, x'; \mu^2)$.

For further reference, we now catalogue the scaling properties ¹¹⁾ that follow from the wave equation (2.21) for $\phi^{ab}(x)$ and $G^{ab}(x, x'; \mu^2)$. If we substitute into (2.21) the variable $\xi \equiv \mu_n^2 x$ and take the limit as $n \rightarrow \infty$, where $\mu_n^2 \approx \pi^2 n$, we get

$$\phi^a(\xi) = \frac{\gamma_{a-1}}{\xi} \phi^a(\xi) - \int_0^\infty d\eta \frac{\phi^a(\eta)}{(\eta-\xi)^2} \quad (2.41)$$

---

*) The factor 2 arises because the γ trace is already taken into account in the rules of Fig. 2.1. Note that the T matrix (2.39) should be incorporated with a corresponding factor of 4.

where $\phi^a(\xi)$ is the limit of $\phi_n^{a\bar{b}}(\xi/\mu_n^2)$:

$$\phi_n^{a\bar{b}}(\xi/\mu_n^2) \approx \phi^a(\xi), \quad n \rightarrow \infty \quad (2.42)$$

For $\xi \rightarrow 0$, $\phi^a(\xi)$ vanishes analogously to $\phi_n^{a\bar{b}}(x)$ in Eq. (2.24)

$$\phi^a(\xi) \approx C^a \xi^{\beta_a} \quad \text{as } \xi \rightarrow 0 \quad (2.43a)$$

while

$$\phi^a(\xi) \approx \sqrt{2} \sin(\xi/\pi) \quad \text{as } \xi \rightarrow \infty \quad (2.43b)$$

Scaling relations similar to (2.43) immediately apply to the Green's function $G(x, x'; \mu^2)$ (2.32). Particularly useful is the scaling condition

$$G(\xi \frac{m^2}{s}, \xi' \frac{m^2}{s}; s) \approx \frac{1}{\pi^2} \int_0^\infty \frac{d\lambda}{\lambda^{-1} + i\epsilon} \phi^a(\lambda \xi) \phi^a(\lambda \xi') \quad (2.44)$$

as we will see in our subsequent analyses. The reader is referred to Einhorn¹¹⁾ for other scaling properties of $G(x, x')$.

C. - Three meson vertex

Before calculating bound state meson scattering, we need the three bound state vertex shown in Fig. 2.8. We denote it by $G_{123}(p_1, p_2, p_3)$ for $n_1 + n_2 \rightarrow n_3$ with momenta p_1, p_2 and $p_3 = p_1 + p_2$. The vertex iG_{123} is given originally as an integral over a quark loop shown in Fig. 2.8b

$$i G_{123} = N_c (-2i \sqrt{\pi/N_c})^3 \int \frac{dk_- dk_+}{(2\pi)^2} \frac{(-1)^{n_1+n_3} \Gamma_{1-}^{b\bar{a}}(\frac{k_-}{p_{1-}}) \frac{1}{p_{2-}} \Gamma_2^{b\bar{c}}(-\frac{k_-}{p_{2-}}) \frac{1}{p_{3-}} \Gamma_3^{a\bar{e}}(\frac{p_{1-}-k_-}{p_{3-}}) (\frac{i}{2})^3}{[k_+ - \frac{\tilde{m}_b^2 + i\epsilon}{2k_-}] [(p_1-k)_+ - \frac{\tilde{m}_c^2 + i\epsilon}{2(p_1-k)_-}] [(p_2+k)_+ - \frac{\tilde{m}_e^2 + i\epsilon}{2(p_2+k)_-}]} \quad (2.45)$$

This vertex is not necessarily evaluated with the external lines on shell : $p_i^2 \neq m_{n_i}^2$. But when combined with a bound state propagator $i/p^2 - m^2$ and summed over all $n=0,1,2,\dots$ products of iG factors contribute to higher point meson amplitudes. On shell, we will denote the vertex by g_{123} :

$$g_{123} = G_{123} \Big|_{p_i^2 = m_{n_i}^2} \quad (2.46)$$

There is a standard set of procedures to eliminate all the quark propagators and replace the Γ 's by ϕ 's :

- (i) integrate over k_+ to get old-fashioned perturbation theory rules ¹⁸⁾ in light-cone co-ordinates ;
- (ii) absorb the $q\bar{q}$ singularities into the adjacent Γ 's as suggested by Fig. 2.8c ;
- (iii) turn the $x < 0$ quarks around with gluon insertions as shown in Fig. 2.6b.

These rules are adequately illustrated by studying iG_{123} (2.45). When we do the k_+ integral, we get zero for $k_- \notin [-p_{2-}, p_{1-}]$ ^{*}, because all poles are on one side of the contour. For $k_- > 0$, the $\bar{b}$ quark moves forward and we pick up the a quark pole in the upper half-plane as in Fig. 2.8c, getting

$$-T^{3/2} \frac{2\pi}{\sqrt{N_c}} i \int_0^{p_{1-}} \frac{dk_-}{(2\pi)^2} \frac{(-1)^{n_2+n_3} \Gamma_1^{b\bar{a}}(\frac{k_-}{p_{1-}}) \frac{1}{p_{2-}} \Gamma_2^{b\bar{c}}(-\frac{k_-}{p_{2-}}) \frac{1}{p_{3-}} \Gamma_3^{a\bar{c}}(\frac{p_{1-}-k_-}{p_{3-}})} \left[p_{1+} - \frac{\tilde{m}_a^2}{2(p_{1-}-k_-)} - \frac{\tilde{m}_b^2}{2k_-} \right] \left[p_{3+} - \frac{\tilde{m}_a^2}{2(p_{1-}-k_-)} - \frac{\tilde{m}_c^2}{2(p_{2+}+k_-)} \right] \quad (2.47)$$

The two energy denominator singularities in (2.47) correspond to the dashed lines in Fig. 2.8c. The term for $k_- < 0$ shown in Fig. 2.8d with the $\bar{b}$ quark moving backward is similar, except from a crucial minus sign due to picking up the c quark pole in the lower half-plane. The two diagrams are just those found in old-fashioned perturbation theory for light-cone variables. For step (ii) above, we introduce into (2.47) dimensionless variables $\mu_i^2 = 2p_{i+}p_{i-}/m^2$, $\gamma_j - 1 = \tilde{m}_j^2/m^2$ (recall $m^2 \equiv g^2(N_c/\pi)$), and absorb the denominators into Γ_1 and Γ_3 using Eq. (2.19), obtaining

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^{*}) Corresponding to backward moving a or $\bar{c}$ quarks.

$$-2 \sqrt{\frac{\pi}{N_c}} i \int_0^{p_{1-}} dk_- (-1)^{n_1+n_2} \phi_1^{b\bar{a}}\left(\frac{k_-}{p_{1-}}\right) \frac{1}{p_{2-}} \Gamma_2^{b\bar{c}}\left(-\frac{k_-}{p_{2-}}\right) \phi_3^{a\bar{c}}\left(\frac{p_{1-}-k_-}{p_{3-}}\right) \quad (2.48)$$

Step (iii) entails turning around the b quark which emerges from Γ_2 with negative momentum fraction using Eq. (2.18) (Fig. 2.6b). Combining the two "time"-orderings of Figs. 2.8c and d, we obtain

$$G_{123} = \sqrt{\frac{\pi}{N_c}} 2m^2 \int_0^{p_{1-}} dk_- \int_0^{p_{2-}} dk'_- (-1)^{n_1+n_2} \phi_1^{b\bar{a}}\left(\frac{k_-}{p_{1-}}\right) \phi_2^{b\bar{c}}\left(\frac{k'_-}{p_{2-}}\right) / (k_- + k'_-)^2 \quad (2.49)$$

$$\cdot \left[\phi_3^{a\bar{c}}\left(\frac{p_{1-}+k'_-}{p_{3-}}\right) - \phi_3^{a\bar{c}}\left(\frac{p_{1-}-k_-}{p_{3-}}\right) \right]$$

indicated in Figs. 2.9a and b.

Thus we have a new set of graph rules which apply to Fig. 2.9 :

- (a) all quark lines have propagator 1 ;
- (b) gluon lines have propagators $-i/k_-^2$;
- (c) the bound state vertices are $\phi_n^{a\bar{b}}(k_{a-}/p_-)$.

Diagrams like Fig. 2.9 are integrated over all k_- loop momenta where the $\phi_n^{a\bar{b}}$ have support. These rules are general, but we have not done the combinatorics necessary to enumerate the weight given to each diagram for a general process.

In subsequent calculations we will often write the vertex using Feynman variables $x = k_-/p_{1-}$, $x' = k'_-/p_{2-}$ and interchanging quark and antiquark labels using (2.23). These give the expression used in Section 3 ^{*)} :

$$G_{123} = \sqrt{\frac{\pi}{N_c}} 2m^2 \frac{p_{1-}}{p_{2-}} \int_0^1 dx \int_0^1 dx' \frac{(-1)^{n_2}}{(x \frac{p_{1-}}{p_{2-}} + x')^2} \phi_1^{b\bar{a}}(x) \phi_2^{b\bar{c}}(x')$$

$$\cdot \left[\phi_3^{c\bar{a}}\left(\frac{1-x'}{1+p_{1-}/p_{2-}}\right) - \phi_3^{c\bar{a}}\left(\frac{1+x \frac{p_{1-}}{p_{2-}}}{1+p_{1-}/p_{2-}}\right) \right] \quad (2.50)$$

---

^{*)} Note that in this and all subsequent integrals gluon propagators are understood to have a principal value prescription (2.9), which renders all integrals finite.

Since p_{1-}/p_{2-} is Lorentz invariant, the entire vertex is also manifestly Lorentz invariant.

Finally we consider a few properties of the vertex G , others will be investigated in subsequent sections. If we examine the limit of a zero mass meson state ($m_a, m_b \rightarrow 0, n=0$) $p_1^2 = 2p_{1-}p_{1+} = m_1^2 \rightarrow 0$, we find that it decouples. This is straightforward to see from (2.50) in the frame $p_z > 0$: $p_{1-} = (p_0 - p_1)/\sqrt{2} < p_{1+}$, since $m_1^2 \rightarrow 0$ then requires $p_{1-} \rightarrow 0$ while the integral remains finite because of the principal value prescription. The vanishing is due to the vanishing of the range of integration for k_- in (2.49). In the parity inverted frame $p_z < 0$ this zero can be demonstrated, but requires more extensive manipulations ^{15),*)}. This decoupling of the zero mass meson can be generalized to higher point functions, by working in a similar frame : $p_- \rightarrow 0$, scaling integrals with respect to m_1^2 , and noting that all integrals are finite because of the principal value prescription for gluon propagators. By decoupling the zero mass state we avoid having a Goldstone boson in two dimensions, as instructed by Coleman ¹⁹⁾. There is no need for a breakdown in the $1/N$ expansion as noted in the model of Gross and Neveu ²⁰⁾.

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*) The lack of manifest parity invariance is a consequence of the gauge choice $A_- = 0$. The use of parity leads to a large number of identities that have often not been proved directly ^{10),11)} : hence the model suffers from an "identity crisis".

3. - HIGH ENERGY FORWARD SCATTERING - FIRST ORDER IN $1/N_c$

In this section we begin our study of high energy scattering in two dimensional QCD by studying the simplest process : meson-meson scattering in the forward direction. All contributions to meson-meson scattering in first order in $1/N_c$ are characterized by having only a single quark boundary. The mesons may be attached in any order leading to the three quark-antiquark exchange diagrams shown in Fig. 1.1. The A^{st} contribution, for example, represents the sum of all gluon exchange diagrams which are planar with the given ordering of mesons. Given confinement, A^{st} will have only singularities which correspond to bound mesons in the s ($a\bar{c}$) and t ($d\bar{b}$) channels. Similarly A^{tu} and A^{us} will have only poles in the two channels specified by their labels.

From experience with dual models which, like two-dimensional QCD in leading order in $1/N_c$, have a linearly rising spectrum of bound states, we expect A^{st} and A^{tu} to dominate over A^{us} as $s \rightarrow \infty$ with t fixed^{*}). Thus we turn first to these two amplitudes, leaving the discussion of A^{us} to Section 4. In two-dimensional QCD the contributions to A^{st} fall into three classes : those containing no exchanged gluons, those with gluons exchanged between quark a and antiquark $\bar{c}$, and those with gluons exchanged between quark d and antiquark $\bar{b}$. The sum of all the gluon exchanges in a given colour singlet channel like the s or t channel is given by the quark-antiquark T matrix [Eq. (2.11)] and we are thus led to the expression of Fig. 3.1¹⁰⁾.

A. - Imaginary part of forward amplitude

We turn now to forward scattering in the s channel (i.e., $\Theta_s = 0$). We start by discussing the behaviour of the imaginary part since it has contributions only from the meson bound states in the s channel T matrix in Fig. 3.1b and is thus rather simpler than the full amplitude. The imaginary part is then given by the diagram of Fig. 3.2 and can be written down immediately using the results of Section 2 :

$$\text{Im } A^{st} = \sum_n g_{12n} \pi \delta(s - m_n^2) g_{1'2'n} \quad (3.1)$$

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^{*}) Of course, in two dimensions t is not exactly fixed since we are restricted to $\Theta_s = 0$.

where G_{ijk} is the three-meson vertex [Eq. (2.45)]

$$G_{12n} = \sqrt{\frac{\pi}{N_c}} 2m^2 \frac{p_{1-}}{p_{2-}} \int_0^1 dx \int_0^1 dx' \frac{(-1)^{n_2}}{(x \frac{p_{1-}}{p_{2-}} + x')^2} \phi_{1-}(x) \phi_{2-}^{b\bar{a}}(x') \quad (3.2)$$

$$\left[\phi_n^{c\bar{a}}\left(\frac{1-x'}{1+p_{1-}/p_{2-}}\right) - \phi_n^{c\bar{a}}\left(\frac{1+x \frac{p_{1-}}{p_{2-}}}{1+p_{1-}/p_{2-}}\right) \right]$$

Since $\text{Im } A^{st}$ consists of a series of delta functions some averaging procedure must be applied to make sense of the high energy $s \rightarrow +\infty$ limit. The natural procedure, long employed in dual models, is to smear out the contribution of a delta function over a distance in s much greater than the spacing between resonances ^{*)}. Since for large n ⁷⁾

$$m_{n+1}^2 - m_n^2 \simeq \pi^2 m^2 \quad (3.3)$$

we have

$$\text{Im } A^{st} \simeq \frac{1}{\pi m^2} g_{12n} g_{1'2'n} \quad (3.4)$$

where

$$n \simeq \frac{s}{\pi^2 m^2}$$

Therefore to obtain the high energy limit we need the behaviour of the vertex for one mass very large ^{**))}.

Let us consider the three-meson vertex G_{123} for $m_3^2 \rightarrow \infty$. [For the present application, $m_3^2 = s$.] We note that

$$m_3^2 = m_1^2 + m_2^2 + 2(p_{1-} p_{2+} + p_{1+} p_{2-}) \quad (3.5)$$

^{*)} It has not been formally proved that this gives the same behaviour as obtained by going along a ray away from the positive real axis in the complex s plane.

^{**))} See also Einhorn, Ref. 11).

and

$$2p_{1+}p_{1-} = m_1^2, \quad 2p_{2+}p_{2-} = m_2^2$$

and hence

$$m_3^2 = m_1^2 + m_2^2 + m_2^2 \left(\frac{p_{1-}}{p_{2-}} \right) + m_1^2 \left(\frac{p_{2-}}{p_{1-}} \right) \quad (3.6)$$

Therefore $m_3^2 \rightarrow \infty$ requires either

$$p_{1-}/p_{2-} \approx m_3^2/m_2^2 \quad (3.7a)$$

or

$$p_{1-}/p_{2-} \approx m_1^2/m_3^2 \quad (3.7b)$$

These two choices correspond to p_1 along the $+z$ axis or p_2 along the $+z$ axis, respectively, and are therefore related by parity. We adopt (3.7a) for definiteness. Substituting into (2.50), we see that due to the gluon denominator the vertex behaves like s_3^{-1} unless $x \approx 0$. We therefore write

$$x = \eta \frac{p_{2-}}{p_{1-}} = \eta \frac{m_2^2}{m_3^2} \quad (3.8)$$

so that

$$\begin{aligned} G_{123} \simeq & 2 \sqrt{\frac{\pi}{N_c}} m^2 (-1)^{n_2} \int_0^\infty d\eta \int_0^1 dx' \phi_1^{b\bar{a}} \left(\eta \frac{m_2^2}{m_3^2} \right) \phi_2^{b\bar{c}}(x') / (\eta+x')^2 \\ & \times \left[\phi_3^{c\bar{a}} \left((1-x') \frac{m_2^2}{m_3^2} \right) - \phi_3^{c\bar{a}} \left((1+\eta) \frac{m_2^2}{m_3^2} \right) \right] \quad (3.9) \end{aligned}$$

From (2.24) and (2.42) we have

$$\phi_1^{b\bar{a}}(x) \underset{x \rightarrow 0}{\simeq} C_1^{b\bar{a}}(x) \beta_b \quad (3.10)$$

and

$$\phi_3^{c\bar{a}}\left(\xi \frac{m^2}{m_3^2}\right) \underset{m_3^2 \rightarrow \infty}{\simeq} \phi^c(\xi) \quad (3.11)$$

Substituting in (3.9) gives

$$G_{123} \simeq \left(\frac{m_3^2}{m^2}\right)^{-\beta_b} 2 \sqrt{\frac{\pi}{N_c}} m^2 C_1^{b\bar{a}} I_2^{b\bar{c}} \quad (3.12)$$

where ^{*})

$$I_2^{b\bar{c}} \equiv (-1)^{n_2} \left(\frac{m_2^2}{m^2}\right)^{\beta_b} \int_0^\infty d\eta \int_0^1 dx' \eta^{\beta_b} \phi_2^{b\bar{c}}(x') / (\eta + x')^2 \\ \times \left[\phi^c((1-x') \frac{m_2^2}{m^2}) - \phi^c((1+\eta) \frac{m_2^2}{m^2}) \right] \quad (3.13)$$

Had we made the choice of axis corresponding to (3.7b) we would have obtained an expression similar to (3.12) but with C and I interchanged. Since the theory has parity invariance these expressions must be equal ^{**}) and hence

$$\frac{C_1^{b\bar{a}}}{I_1^{b\bar{a}}} = \frac{C_2^{b\bar{c}}}{I_2^{b\bar{c}}} = \tau^b \quad (3.14)$$

Since the ratio is independent of everything except b we can evaluate it, say, as $m_2^2 \rightarrow \infty$. Then

$$C_2^{b\bar{c}} \underset{m_2^2 \rightarrow \infty}{\simeq} \left(\frac{m_2^2}{m^2}\right)^{\beta_b} C^b \quad (3.15)$$

where from Eq. (2.43a)

$$\phi^b(\xi) \underset{\xi \rightarrow 0}{\simeq} \xi^{\beta_b} C^b \quad (3.16)$$

^{*}) There is an apparent divergence for $\eta \rightarrow \infty$ if $\beta_b = 1$. In this case the whole region of integration $0 \leq x \leq 1$ contributes to the dominant asymptotic behaviour and the limiting form (3.11) cannot be used. The original integral (3.2) is finite. There is also an apparent divergence for $\eta \approx \lambda x$ and $x' \rightarrow 0$ if $\beta_b = 0$ since the integrand behaves like $(x')^{2\beta_b - 1}$ aside from the factor in square brackets. However, this factor is $\approx x'$ also and the integral is well behaved.

^{**}) We would like to verify such identities explicitly but the consequences of parity are not at all trivial to check in the light-cone gauge. See Refs. 10) and 11).

On the other hand, using (3.11) in Eq. (3.13) gives

$$I_2^{b\bar{c}} \underset{m_2 \rightarrow \infty}{\simeq} \left(\frac{m_2^2}{m^2}\right)^{\beta_b} \int_0^\infty d\eta \int_0^1 dx' \eta^{\beta_b} \phi^c\left((1-x')\frac{m_2^2}{m^2}\right) \left[\phi^c\left((1-x')\frac{m_2^2}{m^2}\right) - \phi^c\left((1+\eta)\frac{m_2^2}{m^2}\right) \right] / (\eta+x')^2 \quad (3.17)$$

Since for large arguments the scaling functions oscillate [Eq. (2.43b)]

$$\phi^c(\xi) \simeq \sqrt{2} \sin(\xi/\pi) \quad (3.18)$$

the second term vanishes and the first averages to unity. We then have ^{*})

$$I_2 \underset{m_2 \rightarrow \infty}{\simeq} \left(\frac{m_2^2}{m^2}\right)^{\beta_b} \int_0^\infty d\eta \int_0^1 dx' \frac{\eta^{\beta_b}}{(\eta+x')^2} = \left(\frac{m_2^2}{m^2}\right)^{\beta_b} \frac{\pi}{\sin \pi \beta_b} \quad (3.19)$$

Therefore

$$\tau^b = \frac{C_b \sin \pi \beta_b}{\pi}$$

and we can write

$$G_{123} \underset{m_3 \rightarrow \infty}{\simeq} 2 \sqrt{\frac{\pi}{N_c}} m^2 \left(\frac{m_3^2}{m^2}\right)^{-\beta_b} V_1^{b\bar{a}} V_2^{b\bar{c}} \quad (3.20)$$

where

$$V_i^{e\bar{f}} \equiv C_i^{e\bar{f}} \sqrt{\frac{\pi}{C_e \sin \pi \beta_e}} \quad (3.21)$$

We may now evaluate the high energy limit of A^{st} easily :

$$\text{Im } A^{st} \approx \frac{4}{N_c} m^2 \left(\frac{s}{m^2}\right)^{-\beta_b - \beta_d} V_1^{b\bar{a}} V_{1'}^{d\bar{a}} V_2^{b\bar{c}} V_{2'}^{d\bar{c}} \quad (3.22)$$

---

^{*}) This expression holds only for $0 < \beta_b < 1$ corresponding to $0 < m_b < \infty$: see (2.26). The limit $\beta_b \rightarrow 0$ and $m_b^2 \rightarrow \infty$ is non-uniform so the case $\beta_b = 0$ must be considered separately. This is done at the end of this section.

We note the following properties : (i) the asymptotic behaviour in s is a power which depends only on the "exchanged" quarks b and d ; (ii) the coefficient of the asymptotic power factorizes into a part depending on the upper part of the diagram (mesons 1 and 1') and a part depending on the lower part (mesons 2 and 2'). These properties are characteristic of a t channel Regge pole exchange in four dimensions. Two-dimensional QCD therefore exhibits an embryonic Regge behaviour ^{*)}

$$\text{Im } A^{st} \approx m^2 V_{11'} V_{22'} \frac{\pi}{\Gamma(\alpha^{b\bar{d}}(0)+1)} \left(\frac{s}{m^2}\right)^{\alpha^{b\bar{d}}(0)} \quad (3.23a)$$

where

$$V_{ij} \equiv \frac{2\sqrt{\Gamma(\alpha^{b\bar{d}}(0)+1)}}{\sqrt{\pi N_c}} V_i^{b\bar{e}} V_j^{d\bar{e}} \quad (3.23b)$$

and we have extracted a factor m^2 to give the dimensions explicitly and a factor $\pi/\Gamma(\alpha^{b\bar{d}}(0)+1)$ which is appropriate for the imaginary part of a Reggeon propagator. The "Regge" intercept $\alpha^{b\bar{d}}(0)$ is given by

$$\alpha^{b\bar{d}}(0) = -\beta_b - \beta_d \quad (3.24)$$

Since $0 \leq \beta_i \leq 1$, $\alpha(0) \leq 0$ as is reasonable in a model with scalar bound states. We also note : (iii) the Regge residue V_{ij} in Eq. (3.23b) exhibits direct channel factorization, i.e., between the incoming and outgoing particles. This property reflects the existence of just one bound state at any given mass, and carries over into direct channel factorization in multi-Regge and inclusive reactions, as we will see and discuss in II ¹³⁾.

B. - The full amplitude

The behaviour of its imaginary part leads us to expect that A^{st} behaves like $(-s)^{\alpha^{b\bar{d}}(0)}$ as $|s| \rightarrow \infty$ which is typical of an amplitude with Regge behaviour and only right-hand singularities. However, it is possible that A^{st} has terms behaving like positive integral powers of s since these would not show up in the imaginary part. If these terms are present A^{st} does not satisfy an unsubtracted dispersion relation and

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^{*)} In retrospect, the asymptotic behaviour (3.20) of G_{123} for large m_3^2 can be seen as "Reggeized" quark exchange.

therefore cannot be expressed entirely in terms of the s channel bound states, i.e., it would not have the primitive duality discussed in Section 1 which we hope to find in QCD. We now turn to a study of the full amplitude to investigate this important point. We will find that A^{st} indeed has the "Regge" behaviour $|s|^{\alpha_{\bar{d}\bar{b}}(0)}$ as $s \rightarrow \pm \infty$ with no subtractions. Before turning to this we make one further remark. According to Eq. (3.24) the "Regge intercept" can be less than -1 for sufficiently large quark masses. In this case, if A^{st} satisfies an unsubtracted dispersion relation we have

$$A^{st}(s) = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im} A^{st}(s')}{s' - s} \sim -\frac{1}{s} \left(\frac{1}{\pi} \int_{s_0}^{\infty} ds' \text{Im} A^{st}(s') \right) \quad (3.25)$$

The integral is finite and non-vanishing (at least for elastic scattering) and therefore the real part of A^{st} must contain a term s^{-1} which dominates the Regge term ^{*)}. We find that this is indeed the case.

The graph of Fig. 3.1a is given by

$$i A_{(a)}^{st} = -N_c \int \frac{d^2 k}{(2\pi)^2} \left(2i \sqrt{\frac{\pi}{N_c}} \frac{1}{p_{1-}} \Gamma_1^{a\bar{b}} \left(\frac{k_-}{p_{1-}} \right) \right) \left(2i \sqrt{\frac{\pi}{N_c}} \frac{1}{p_{2-}} \Gamma_2^{b\bar{c}} \left(\frac{k_- - p_{1-}}{p_{2-}} \right) \right) \\ (-1)^{n_1+n_2+n_1'+n_2'} \left(2i \sqrt{\frac{\pi}{N_c}} \frac{1}{p_{2'-}} \Gamma_{2'}^{d\bar{e}} \left(\frac{p_{1'-} - k_-}{p_{2'-}} \right) \right) \left(2i \sqrt{\frac{\pi}{N_c}} \frac{1}{p_{1'-}} \Gamma_{1'}^{a\bar{d}} \left(\frac{k_-}{p_{1'-}} \right) \right) \quad (3.26) \\ \frac{i k_-}{k_-^2 - \tilde{m}_a^2 + i\epsilon} \frac{i(k_- - p_{1-})}{(k_- - p_{1-})^2 - \tilde{m}_a^2 + i\epsilon} \frac{i(k_- - r_-)}{(k_- - r_-)^2 - \tilde{m}_c^2 + i\epsilon} \frac{i(k_- - p_{1-})}{(k_- - p_{1-})^2 - \tilde{m}_b^2 + i\epsilon}$$

where

$$p_1 + p_2 = p_{1'} + p_{2'} \equiv r \quad (3.27)$$

If p_{1z} is chosen to lie along the negative z axis in the centre-of-mass frame, the minus components are given by

$$p_{2-}/p_{1-} = \frac{2m_2^2}{(s - m_1^2 - m_2^2 + \lambda^{1/2}(s, m_1^2, m_2^2))} \quad (3.28)$$

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^{*)} We thank G. Veneziano for emphasizing this point to us.

and

$$p_{1-}'/p_{1-} = \frac{(s - m_1'^2 + m_2^2 + \lambda^{1/2}(s, m_1'^2, m_2^2))(s - m_1'^2 - m_2'^2 + \lambda^{1/2}(s, m_1'^2, m_2'^2))}{(s - m_1'^2 - m_2^2 + \lambda^{1/2}(s, m_1'^2, m_2^2))(s - m_1'^2 + m_2'^2 + \lambda^{1/2}(s, m_1'^2, m_2'^2))} \quad (3.29)$$

For s large and $m_2'^2 > m_2^2$,

$$p_{1-}'/p_{1-} < 1 \quad (3.30)$$

For definiteness, we shall assume this to be the case in what follows.

Following the steps outlined in Section 2, we first carry out the k_+ integral in (3.26) or, equivalently, calculate the contribution using x_- ("time") ordered perturbation theory. The four-meson quark-antiquark vertices can be ordered in four ways as shown in Fig. 3.3 other orderings vanish due to $p_{1-} > p_{1,-}$. The dashed lines represent the $q\bar{q}$ denominators discussed in Section 2 ; for example, for the first diagram in Fig. 3.3 they are

$$\frac{1}{\left(\frac{m_1^2}{m^2} - \frac{\gamma_{a-1}}{k-/p_{1-}} - \frac{\gamma_{b-1}}{1-k-/p_{1-}}\right)} \frac{1}{\left(\frac{s}{m^2} - \frac{\gamma_{a-1}}{k-/r_-} - \frac{\gamma_{c-1}}{1-k-/r_-}\right)} \frac{1}{\left(\frac{m_1'^2}{m^2} - \frac{\gamma_{a-1}}{k-/p_{1-}'} - \frac{\gamma_{d-1}}{1-k-/p_{1-}'}\right)}$$

The graph of Fig. 3.1b can be similarly obtained using "time"-ordered perturbation theory. The four possible orderings are shown in Fig. 3.4. In each case we can substitute for the T matrix Eq. (2.36). For the first, third and fourth terms the disconnected piece exactly cancels the corresponding term in Fig. 3.2. For the second term the disconnected piece vanishes as the delta function cannot be satisfied.

Finally there are four classes of time-orderings possible for the graph of Fig. 3.1c as shown in Fig. 3.5. For the first term we use Eq. (2.39), for the second and third terms, Eq. (2.38), and for the fourth term, Eq. (2.36). The disconnected piece in the last cancels the third term in Fig. 3.3

Collecting together the above results, we find nine contributions to A^{st} which are shown in Fig. 3.6. The gluon exchange graph of Fig. 3.6a which arises from the first term in Fig. 3.5 is :

$$A_{(a)}^{st} = \frac{4\pi m^2}{N_c} \int_{p_1-p_1'}^{p_1-} dk_- \int_0^{p_2-} dl_- (-1)^{n_2} \phi_1^{b\bar{a}}\left(\frac{k_-}{p_1-}\right) \phi_1^{d\bar{a}}\left(\frac{p_1'-p_1+k_-}{p_1'}\right) \phi_2^{b\bar{c}}\left(\frac{l_-}{p_2-}\right) \times \phi_2^{c\bar{a}}\left(\frac{p_1-l_-}{p_2'}\right) / (k_-+l_-)^2 \quad (3.31)$$

The s channel pole graphs of Fig. 3.6b are

$$A_{(b)}^{st} = \sum_n \frac{G_{12n} G_{n1'2'}}{s - m_n^2} \quad (3.32)$$

and the t channel pole graphs of Fig. 3.6c are ^{*)}

$$A_{(c)}^{st} = \sum_n \frac{G_{11'n} G_{n22'}}{t - m_n^2} \quad (3.33)$$

We now take the asymptotic limit as $s \rightarrow +\infty$ with $t = O(1/s)$. As is the case for the imaginary part, the amplitudes are enhanced when small momenta are carried by quarks b and d. For the gluon exchange graph we make the change of variables

$$k = (p_1 - p_1') + \eta p_2- \\ l = x' p_2-$$

which leads to

$$A_{(a)}^{st} \simeq \frac{4\pi}{N_c} m^2 \left(\frac{s}{m^2}\right)^{-\beta_b - \beta_d} (-1)^{n_2} C_1^{b\bar{a}} C_1^{d\bar{a}} \times \left(\frac{m^2}{m^2}\right)^{\beta_b + \beta_d} \int_0^\infty d\eta \int_0^1 dx' \left(\eta + \frac{m_2'^2 - m^2}{m^2}\right)^{\beta_b} \eta^{\beta_d} \times \phi_2^{c\bar{a}}\left((1-x')\frac{m^2}{m_2'^2}\right) \phi_2^{b\bar{c}}(x') / \left(\eta + x' + \frac{m_2'^2 - m^2}{m^2}\right)^2 \quad (3.34)$$

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^{*)} These terms vanish for $m_2^2 = m_2'^2$.

For the s channel pole graphs we use the high energy limit of the Green's function [Eq. (2.44)]

$$G\left(\xi \frac{m^2}{s}, \xi' \frac{m^2}{s}; \frac{s}{m^2}\right) = m^2 \sum_n \frac{\phi_n^{c\bar{a}}\left(\xi \frac{m^2}{s}\right) \phi_n^{c\bar{a}}\left(\xi' \frac{m^2}{s}\right)}{s - m_n^2} \quad (3.35)$$

$$\sim \frac{1}{\pi^2} \int_0^1 \frac{d\lambda}{\lambda - 1 + i\epsilon} \phi^c(\lambda \xi) \phi^c(\lambda \xi')$$

along with (3.3) to obtain ^{*})

$$A_{(b)}^{st} \simeq - \frac{4m^2}{\pi N_c} \left(\frac{s}{m^2}\right)^{-\beta_b - \beta_d} C_{,b\bar{a}} C_{,d\bar{a}} \int_0^1 \frac{d\lambda}{\lambda - 1 + i\epsilon}$$

$$\left\{ (-1)^{n_2} \left(\frac{m_2^2}{m^2}\right)^{\beta_b} \int_0^\infty d\eta \int_0^1 dz \, \eta^{\beta_b} \phi_2^{b\bar{c}}(z) / (\eta + z)^2 \right.$$

$$\left. \times \left[\phi^c\left((1-z)\lambda \frac{m_2^2}{m^2}\right) - \phi^c\left((1+\eta)\lambda \frac{m_2^2}{m^2}\right) \right] \right\}$$

$$\cdot \left\{ b \rightarrow d, \, 2 \rightarrow 2', \, \eta \rightarrow \eta', \, z \rightarrow z' \right\} \quad (3.36)$$

For the t channel pole terms we use (2.50) with the change of variables

$$l = \eta (p_i - p_i')$$

$$l' = x' (p_i - p_i')$$

to obtain

$$A_{(c)}^{st} \simeq - \frac{4\pi m^2}{N_c} \left(\frac{s}{m^2}\right)^{-\beta_b - \beta_d} C_{,b\bar{a}} C_{,d\bar{a}} \sum_n \frac{1}{t - m_n^2}$$

$$\times \left\{ \left(\frac{m_2'^2 - m_2^2}{m^2}\right)^{\beta_b + \beta_d} \int_0^\infty d\eta \int_0^1 dx \, \eta^{\beta_d} \phi_n^{b\bar{a}}(1-x') \right.$$

$$\left. \frac{\left[(1+\eta)^{\beta_b} - (1-x')^{\beta_b} \right]}{(x' + \eta)^2} \frac{1}{2} \sqrt{\frac{\pi}{N_c}} G_{n22'} \right\} \quad (3.37)$$

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^{*}) See footnote on p. 22.

Comparing Eqs. (3.34), (3.36) and (3.37), we see they all have the same power behaviour $s^{\alpha^{b\bar{d}}(0)}$. Furthermore the coefficient of this power factorizes into a part depending on particles 1 and 1' and a part depending on particles 2 and 2'. The dependence on 1 and 1' is proportional to the "Regge" vertex $V_{11'}$ defined in Eq. (3.24). The dependence on 2 and 2', however, involves a number of complicated integrals. Again parity invariance implies these are proportional to $V_{22'}$ although we have not been able to explicitly check this complicated identity. The full amplitude thus has a Regge like behaviour as $s \rightarrow +\infty$.

The above calculations lead to a simple heuristic description of the high energy behaviour of A^{st} . The dominant asymptotic behaviour comes when the gluon propagator carries a small momentum $q_- [q_- = 0(p_{2-}, p_{2' -})$ for s channel graphs and $q_- = 0(p_{1-}, p_{1' -})$ for t channel graphs]. This then forces the a quark to be "hot" ($k_- \approx p_{1-} + 0(q_-)$). The "phase space" is thus $O(q_-^2)$ and cancels against the gluon propagator. The "Regge" behaviour comes from the small amplitudes (2.24) $\sim x^{\beta_b}, x^{\beta_d}$ for the dribbling of "cold" quarks ($x \rightarrow 0$) by the meson bound states. This picture is a concrete realization of Feynman's parton description of Regge behaviour²⁾.

We now return to the question raised above of possible s^{-1} behaviour of A^{st} . While Eq. (3.36) is perfectly well behaved for $\beta_b, \beta_d \neq 1$ one sees that (3.34) and (3.37) diverge as $\eta \rightarrow \infty$ for $\beta_b + \beta_d - 2 \geq -1$; i.e., $\alpha^{b\bar{d}}(0) \leq -1$. Considered as analytic functions of $\alpha^{b\bar{d}}(0)$, these expressions have a simple pole at $\alpha^{b\bar{d}}(0) = -1$. This is not surprising since (3.23) corresponds to

$$A^{st} \approx m^2 V_{11'} V_{22'} \Gamma(-\alpha^{b\bar{d}}(0)) \left(\frac{-s}{m^2}\right)^{\alpha^{b\bar{d}}(0)}$$

which has a simple pole at $\alpha^{b\bar{d}}(0) = -1$ due to the singularities in V_{ij} , Eq. (3.23b). The original expressions Eq. (3.31) and (3.33) do not have such a singularity. This type of behaviour is a signal for the existence of an additional term in the asymptotic behaviour which dominates the "Regge" term. Indeed for $\alpha^{b\bar{d}}(0) < -1$ we see that the region

$$k = (p_{1-} - p_{1' -}) + x p_{1' -}$$

dominates. This gives

$$A_{(a)}^{st} \approx \frac{4\pi}{N_c} m^2 \left(\frac{s}{m^2}\right)^{-1} \int_0^1 dx \int_0^1 dx' \frac{\phi_1^{b\bar{a}}(x) \phi_1^{d\bar{a}}(x) \phi_2^{c\bar{d}}((1-x')\frac{m^2}{m'^2}) \phi_2^{c\bar{b}}(1-x')}{x^2} \quad (3.38)$$

and

$$A_{(c)}^{st} \approx \frac{4\pi}{N_c} m^2 \left(\frac{s}{m^2 - m'^2}\right)^{-1} \sum_n \frac{1}{t - m_n^2} \left\{ \int_0^1 dx \int_0^1 dx' \phi_1^{b\bar{a}}(x) \phi_1^{d\bar{a}}(x) \right. \\ \left. \times \frac{\phi_n^{b\bar{d}}(1-x')}{x^2} \frac{1}{2} \sqrt{\frac{\pi}{N_c}} G_{n22'} \right\} \quad (3.39)$$

These terms behave as s^{-1} , like the "fixed-pole" terms anticipated. Equations (3.38) and (3.39) diverge for $x \rightarrow 0$ if $\alpha^{b\bar{d}}(0) \geq -1$ since then the "Regge pole" dominates. Considered as analytic functions of $\alpha^{b\bar{d}}(0)$ they also have a pole at $\alpha^{b\bar{d}}(0) = -1$. This singularity precisely cancels against the pole in the "Regge" term giving a finite amplitude for all $\alpha^{b\bar{d}}(0)$ as required. The amplitude thus behaves asymptotically as the sum of the "Regge" and "fixed-pole" terms appropriately continued,

$$A^{st} \underset{s \rightarrow \infty}{\approx} m^2 \left[V_{11}, V_{22}, \Gamma(-\alpha^{b\bar{d}}(0)) \left(\frac{-s}{m^2}\right)^{\alpha^{b\bar{d}}(0)} + R \left(\frac{-s}{m^2}\right)^{-1} \right] \quad (3.40)$$

The behaviour at $\alpha^{b\bar{d}}(0) = -1$ is characteristic of a multiplicative fixed pole. It should be remarked that this sort of behaviour is quite unlike the real fixed poles in angular momentum found in four-dimensional theories in one respect. The s^{-1} behaviour of A^{st} can appear in physical amplitudes, e.g., those for mesons with flavours such that only the A^{st} term contributes, whereas real fixed poles in four dimensions are always wrong signature and thus give no s^{-1} behaviour in physical amplitudes. There is no contradiction in two-dimensions, of course, since we cannot exploit analyticity in complex angular momentum. It is tempting to speculate that the "fixed poles" found in two dimensions may be related to the sister trajectories found in dual models²¹⁾ since these always collide with the leading trajectories at $J = -1$ *).

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*) The collision produces singular residues only in higher point amplitudes, however.

We now briefly discuss the analytic properties of A^{st} . If A^{st} is analytic in s with only simple poles at $s = m_n^2$ and is power bounded, in complex s , the asymptotic behaviour given in (3.40) will hold for all complex s . However, as remarked by previous authors ^{10),11)}, it is very difficult to prove analyticity starting from the formulae derived in the light-cone gauge. Indeed the individual contributions $A_{(a)}^{st}$, $A_{(b)}^{st}$ and $A_{(c)}^{st}$ do not separately have the above-mentioned analytic properties - they can have left-hand singularities in s due to the vanishing of the gluon propagators. Thus, for example, they do not individually have the ratio of real to imaginary parts characteristic of $(-s)^{\alpha_{b\bar{d}}(0)} : A_{(a)}^{st}$ and $A_{(c)}^{st}$ are purely real for $s \rightarrow +\infty$, in particular. In fact we have not explicitly verified that the sum has the expected behaviour given in (3.40), and have not attempted to explicitly prove analyticity in complex s . We note, however, that for $s \rightarrow -\infty$, the asymptotic behaviour again has the terms $s^{\alpha_{b\bar{d}}(0)}$ and s^{-1} (although we have not shown their coefficients are the same) ^{*}). The existence of an s^{-1} term is further support for the usual analytic properties although one should also show that R (3.40) as obtained in (3.38) and (3.39) agrees with that given by (3.25); this is a further "identity crisis". In conclusion we believe that the behaviour (3.40) holds for all $|s| \rightarrow \infty$ and thus two-dimensional QCD exhibits primitive duality, as expressed by Eq. (1.5) ^{**)}.

If the behaviour $s^{\alpha_{b\bar{d}}(0)}$ is indeed an embryonic Regge behaviour $\alpha_{b\bar{d}}(0)$ should be connected with the t channel bound states. Since β_i increases with quark mass m_i^2 , $\alpha_{b\bar{d}}(0)$ decreases with m_i^2 . The mass $m_{b\bar{d}^2}$ of the lowest bound state increases with m_i^2 ⁷⁾ and so qualitatively $\alpha_{b\bar{d}}(0)$ and m_0^2 are related as we would expect them to be if they were joined by a Regge trajectory ^{***)}.

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^{*}) This is equivalent to taking the behaviour of A^{tu} as $s \rightarrow +\infty$ and is studied in Section 4.

^{**}) It should be remarked that this is not so apparent from Fig. 3.1 which seems to express A^{st} as a sum of s channel and t channel poles. However, this is not an analytic decomposition like the usual Feynman graph decomposition.

^{***}) If we make the bold assumption that the states of $n > 0$ are spin 0 daughters of a leading state with $J = n$ we can ask if the m_n^2 vs $J = n$ plots ⁷⁾ extrapolate to $J = \alpha_{b\bar{d}}(0)$ at $m^2 = 0$. We have studied it numerically and found it not to be the case. Such an assumption is unjustifiable: since n is not linearly related to m_n^2 we cannot expect the daughters to be integrally spaced.

For small quark masses we find

$$\alpha^{d\bar{b}}(0) \approx -\frac{1}{m} \frac{\sqrt{3}}{\pi} (m_b + m_d) \quad (3.41)$$

while ²²⁾

$$m_0^{d\bar{b}2} \approx m \frac{\pi}{\sqrt{3}} (m_b + m_d) \quad (3.42)$$

so both vanish linearly with m_1 . We can therefore define a "slope" for light quark trajectories near $t=0$:

$$\alpha'_0 \equiv -\frac{\alpha^{d\bar{b}}(0)}{m_0^{d\bar{b}2}} \approx \frac{3}{\pi g^2 N_c} = 3/\Delta m_{\infty}^2 \quad (3.43)$$

where $\Delta m_{\infty}^2 = \pi g^2 N_c$ is the separation between adjacent high mass excited states ^{*)}.

From (3.41) and (3.42) we see that for zero mass quarks ($m_b = m_d = 0$) the "Regge" power coincides with that of the t channel exchange of the zero-mass bound state. It is therefore of great interest to see if the Regge residues coincide with the coupling of the zero-mass bound state. This bound state is known to decouple from all states ^{10), 15)}. This would correspond to a vanishing "Regge residue" β and thus a finite and real asymptotic behaviour in (3.40). We now show that this is indeed the behaviour found in the model. Let us consider the case of equal mass scattering for simplicity. Then the t channel bound state terms of Fig. 3.6c do not contribute. The gluon exchange diagram for equal-mass scattering becomes

$$A_{(a)}^{st} \approx \frac{4\pi}{N_c} m^2 \left(\frac{s}{m^2}\right)^{-\beta_b - \beta_d} \int_0^1 d\eta \int_0^1 dx' \frac{C_1^{b\bar{c}} C_1^{d\bar{e}} \eta^{\beta_b + \beta_d} \phi_2^{b\bar{c}}(x') \phi_2^{d\bar{e}}(x')}{(\eta + x')^2}$$

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*) One ¹¹⁾ can do some amusing numerology with (3.41)-(3.43). If we put $\alpha'_0 \approx 0.9 \text{ GeV}^{-2}$ and use [from (3.42)] $m_u \approx m_d \approx \frac{1}{2} \sqrt{\alpha'_0} m_\pi^2$, $m_s \approx \sqrt{\alpha'_0} \times (m_K^2 - \frac{1}{2} m_\pi^2)$ then we get $m_u \approx m_d \approx 9 \text{ MeV}$, $m_s \approx 0.23 \text{ GeV}$. We then find $\beta_u \approx \beta_d \approx 0.01$ and $\beta_s \approx 0.21$. This gives a difference of 0.2 between strange and non-strange Regge "intercepts", which is not phenomenologically unreasonable. For a heavy quark Q , $\beta_Q \approx 1 - (3/\pi^2 \alpha'_0 m_Q^2)$: for a charmed quark with $m_c^2 \approx 3 \text{ GeV}^2$, $\beta_c \approx 0.9$.

and the η integral can be done explicitly. For small β_b and β_d we have

$$A_{(a)}^{st} \approx \frac{4\pi}{N_c} m^2 \int_0^1 dx' C_1^{b\bar{c}} C_1^{d\bar{e}} \phi_2^{b\bar{c}}(x') \phi_2^{d\bar{e}}(x') / x' \quad (3.44)$$

This behaves like $(\beta_b + \beta_d)^{-1}$ due to the divergence of the integral for $x' \rightarrow 0$. To calculate the s channel bound state contribution we first follow the manipulations of Smith¹⁵⁾. The brackets in (3.36) thus can be written

$$\begin{aligned} & \int_0^\infty d\eta \int_0^1 dz \eta \beta_b \phi_2^{b\bar{c}}(z) \left[\frac{\phi^c((1+\eta)\lambda \frac{m_2^2}{m^2}) - \phi^c((1-z)\lambda \frac{m_2^2}{m^2})}{(\eta+z)^2} \right] \\ & \approx \int_0^1 dz \phi_2^{b\bar{c}}(z) \left[\int_0^\infty d\eta \frac{\phi^c((1+\eta)\lambda \frac{m_2^2}{m^2})}{(\eta+z)^2} - \frac{\phi^c((1-z)\lambda \frac{m_2^2}{m^2})}{z} \right] \end{aligned} \quad (3.45)$$

The bound state equation can be used on the last term to write

$$\begin{aligned} & - \int_0^1 dz \frac{\phi_2^{b\bar{c}}(z)}{z} \phi^c((1-z)\lambda \frac{m_2^2}{m^2}) \\ & = \int_0^1 dz \left[\left(\frac{m_2^2}{m^2} - \frac{\tilde{m}_c^2}{m^2} \frac{1}{1-z} \right) \phi_2^{b\bar{c}}(z) + \int_0^1 dx \frac{\phi_2^{b\bar{c}}(x)}{(x-z)^2} \right] \phi^c((1-z)\lambda \frac{m_2^2}{m^2}) \\ & = \int_0^1 dz \phi_2^{b\bar{c}}(z) \left[\frac{m_2^2}{m^2} - \frac{\tilde{m}_c^2}{m^2} \frac{1}{1-z} + \int_{-1}^0 d\eta \frac{\phi^c((1+\eta)\lambda \frac{m_2^2}{m^2})}{(\eta+z)^2} \right] \end{aligned} \quad (3.46)$$

where $z \rightarrow -\eta$ and $x \rightarrow z$ in the last step. Substitution of (3.46) into (3.45) and writing $\xi = (m_2^2/m^2)(1+\eta)\lambda$ gives

$$\begin{aligned} \int_0^\infty d\eta \int_0^1 dz \dots & \approx \int_0^1 dz \phi_2^{b\bar{c}}(z) \left[\left(\frac{m_2^2}{m^2} - \frac{\tilde{m}_c^2}{m^2} \frac{1}{1-z} \right) \phi^c((1-z)\lambda \frac{m_2^2}{m^2}) \right. \\ & \quad \left. + \frac{m_2^2}{m^2} \lambda \int_0^\infty d\xi \frac{\phi^c(\xi)}{(\xi - \frac{m_2^2}{m^2}(1-z)\lambda)^2} \right] \end{aligned}$$

We then use the scaling form ¹¹⁾ of the bound state equation

$$\int_0^\infty d\xi \frac{\phi^c(\xi)}{(\xi-\eta)^2} = -\left(1 - \frac{\tilde{m}_c^2}{m^2} \frac{1}{\eta}\right) \phi^c(\eta) \quad (3.47)$$

to obtain

$$\int_0^\infty d\eta \int_0^1 dz \dots \simeq (1-\lambda) \frac{m^2}{m^2} \int_0^1 dz \phi_2^{b\bar{c}}(z) \phi^c((1-z)\lambda \frac{m^2}{m^2}) \quad (3.48)$$

Substituting (3.48) in (3.36) gives

$$\begin{aligned} A_{(b)}^{st} &\simeq -\frac{4m^2}{\pi N_c} \left(\frac{m^2}{m^2}\right)^2 c_1^{b\bar{c}} c_1^{d\bar{e}} \int_0^\infty d\lambda (\lambda-1) \left(\int_0^1 dz \phi_2^{b\bar{c}}(z) \right. \\ &\quad \left. \times \phi^c((1-z)\lambda \frac{m^2}{m^2}) \right) \left(\int_0^1 dz' \phi_2^{d\bar{e}}(z') \phi^c((1-z')\lambda \frac{m^2}{m^2}) \right) \quad (3.49) \end{aligned}$$

The λ integral can be done using the identity

$$\frac{1}{\pi^2} \int d\lambda (1-\lambda) \phi^a(\xi\lambda) \phi^a(\xi'\lambda) = \left(1 - \frac{\tilde{m}_a^2}{m^2 \xi}\right) \delta(\xi-\xi') + \frac{1}{(\xi-\xi')^2} \quad (3.50)$$

which follows from

$$(\mu^2 - H)^2 G(x, y; \mu^2) = (\mu^2 - H) \delta(x-y)$$

We therefore obtain

$$\begin{aligned} A_{(b)}^{st} &\simeq +\frac{4\pi}{N_c} m^2 \left[\int_0^1 dz \phi_2^{b\bar{c}}(z) \phi_2^{d\bar{e}}(z) \left(\frac{m^2}{m^2} - \frac{\tilde{m}_c^2}{m^2(1-z)} \right) \right. \\ &\quad \left. + \int_0^1 dz \int_0^1 dz' \phi_2^{b\bar{c}}(z) \phi_2^{d\bar{e}}(z') / (z-z')^2 \right] \\ &\simeq -\frac{4\pi}{N_c} m^2 \int_0^1 dz \phi_2^{b\bar{c}}(z) \phi_2^{d\bar{e}}(z) / z \quad (3.51) \end{aligned}$$

The singularity in (3.51) precisely cancels that in the gluon exchange term (3.44) with the result that A^{st} is finite in the limit. It is easy to see also from (3.49) that $\text{Im } A^{st}$ vanishes. The Regge term therefore has the behaviour corresponding to the exchange of a t channel bound state which decouples ^{*}).

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^{*}) In some sense what we are actually checking here is the uniformity under interchange of the limits $s \rightarrow \infty$ and $m_b, m_d \rightarrow 0$.

4. - AN ANALOGUE FOR FIXED ANGLE SCATTERING

In our studies of scattering amplitudes at leading order in $1/N_c$ we have so far only examined A^{st} and A^{tu} (see Fig. 1.1) in the limit $s \rightarrow \infty$, $u \rightarrow -\infty$, t fixed. We found these diagrams gave Regge-like behaviour with factorization and exchange degeneracy properties. However, we should check that these are not masked by the behaviour of the A^{us} amplitude in the high energy scattering limit before we can fairly claim to have verified "Regge" properties for the scattering amplitudes.

There is another reason for studying A^{us} in the $s \rightarrow \infty$, $u \rightarrow -\infty$ limit. If we contemplate the fixed angle limit of meson-meson scattering in four dimensions, then the three subamplitudes A^{st} , A^{tu} and A^{us} presumably all contribute. However, each of the subamplitudes only has dynamical singularities in two of the three Mandelstam variables. It therefore seems reasonable to conjecture that the power behaviour of an individual subamplitude may be identical in any limit such that the two variables where it has dynamical singularities become infinite. In this case it would be reasonable to use the behaviour of A^{us} in two dimensions in the limit $s \rightarrow \infty$, $u \rightarrow -\infty$ as an indicator of the possible fixed angle behaviour of scattering amplitudes in four dimensions.

For convenience and economy of figures, we will in fact examine A^{st} in the limit $s \rightarrow \infty$, $t \rightarrow -\infty$, corresponding to $\sqrt{2}p_{1-} \approx \sqrt{s}$, $\sqrt{2}p_{2-} \approx m_2^2/\sqrt{s}$, $\sqrt{2}p_{2'} \approx m_2^2/\sqrt{s}$ and $\sqrt{2}p_{1'} \approx \sqrt{s}$. We first consider the piece

$$A_{(a)}^{st} = 4g^2 \int_0^{p_{1-}} dk_- \int_0^{p_{2-}} dk'_- \frac{\phi_1^{a\bar{b}}(\frac{k_-}{p_{1-}}) \phi_2^{b\bar{c}}(\frac{k_- - k'_-}{p_{2-}}) \phi_{2'}^{d\bar{c}}(\frac{p_{2'} - k'_-}{p_{2'}}) \phi_{1'}^{a\bar{d}}(\frac{k_-}{p_{1'}})}{(p_{1-} + p_{2-} - k_- - k'_-)^2} \quad (4.1)$$

corresponding to Fig. 3.6a.

In this limit, we clearly need to route a large momentum $\sim \sqrt{s}$ across the diagram for $A_{(a)}^{st}$. The only way to do this is through the gluon whose propagator therefore is $\sim O(1/s)$ in this limit. The top and bottom quark lines are constrained to carry a fraction $O(1/s)$ of the momenta $p_{1'}$ and $p_{2'}$, respectively. This means that the k_- and k'_- integrals are constrained to ranges $O(1/\sqrt{s})$ each. Also, the vanishings of the 1 and 1' meson wave functions as momentum fractions $\rightarrow 0$ induce extra suppressions $s^{-\beta_a}$, $s^{-\beta_c}$. Putting together all these powers of s , we find that

$$A^{st} \sim s^{-2-\beta_a-\beta_c} \quad (4.2)$$

in the limit $s \rightarrow \infty$, $t \rightarrow -\infty$. This is not the whole story, as we find when we turn to the other diagrams in Fig. 3.6. For each diagram we must study all possible routings of the large momentum $\sqrt{s}$ across the diagram. In the case of Fig. 3.6b, a routing across the gluon linking quarks b and c is possible, as is a routing across the $a\bar{c}$ bound state sum. The gluonic routing has phase space integration factors $(\pi dk) \sim (1/\sqrt{s})^4$, the $b \rightleftharpoons c$ gluon gives a factor $\sim 1/s$, the $c \rightleftharpoons d$ gluon gives a factor $\sim s$, and there are "dribble factors" of $s^{-\beta_a}$ and $s^{-\beta_d}$ for the slow quarks a and d . Hence this routing gives a contribution to A^{st}

$$\sim s^{-2-\beta_a-\beta_d} \quad (4.3)$$

The routing across the bound state sum is only slightly more complicated to analyze. In this case we need to know the asymptotic behaviour of the $\phi\phi$ sum in the limit

$$\lim_{|\mu^2| \rightarrow \infty} G(s/\mu^2, 1 - s/\mu^2, \mu^2)$$

In this crossed limit the sum over intermediate states alternates. Because it is a large momentum transfer limit we would expect the asymptotic freedom of the theory to ensure naive power counting for the sum over multi-gluon exchange. To within external propagator factors G is essentially the T matrix, and the off-shell T matrix obeys the integral equation of Fig. 4.1. In the limit $\ell_{1-}, \ell_{2-}' = O(\sqrt{s})$; $\ell_{2-}, \ell_{1-}' = O(1/\sqrt{s})$ the iteration of the gluon term does not build up logarithms (we have verified this explicitly for two-gluon exchange, and see no reason to doubt its generality) and the solution of the integral equation is a behaviour

$$T \sim \frac{1}{(\ell_1 + \ell_2)^2} \quad \text{as } (\ell_1 + \ell_2)^2 \rightarrow \infty \quad (4.4)$$

which in turn implies that

$$G(s/\mu^2, 1 - s/\mu^2, \mu^2) = O\left(\frac{1}{\mu^4}\right) \quad \text{as } |\mu^2| \rightarrow \infty \quad (4.5)$$

With this behaviour, the momentum integration and "dribble factors" are easily seen to give an asymptotic behaviour

$$s^{-2-\beta_b-\beta_d} \quad (4.6)$$

for this routing of the large momentum. Another way of getting the result (4.6) is to use the fact that the T matrix for $q\bar{q}$ scattering ($0 \leq x, y \leq 1$) with exactly one hard gluon exchange can be rewritten with soft initial and final state interactions corresponding to the usual scaling G kernels (2.44) as shown in Fig. 4.1^{*}). Turning to the third diagram in $A_{(b)}^{st}$ we see that three large momentum routings are possible. Two of them give the asymptotic behaviours (4.3), (4.6) : the third possible routing, through the gluon connecting $a \rightleftharpoons d$, gives an asymptotic behaviour

$$s^{-2-\beta_b-\beta_c} \quad (4.7)$$

Analyzing all the other diagrams for A^{st} just gives more terms with behaviours like (4.2), (4.3), (4.6) and (4.7), and we conclude that A^{st} behaves as

$$\sum_{i \neq j} s^{-2-\beta_i-\beta_j} C_{ij} \quad (4.8)$$

where the i, j indices run over all quarks in the diagrams, and we were too lazy to calculate the coefficients C_{ij} . The result (4.8) and the fact that $0 < \beta_i < 1$ mean that in the $s \rightarrow \infty, u \rightarrow -\infty, t$ fixed limit A^{us} is always $\ll A^{st} + A^{tu}$, so that we can legitimately claim Regge-like behaviour for meson-meson scattering amplitudes.

If we accept the arguments given earlier in this section, what salient conclusions can we draw about fixed angle scattering from the results (4.8) ? First off, we see that the dimensional counting law result of Brodsky and Farrar (BF), Mateev, Muradyan and Tavkhelidze¹⁶⁾ for fixed angle scattering is almost reproduced, but is modified by two $s^{-\beta}$ factors. This recalls the deviation from dimensional counting found by Einhorn¹¹⁾ in the case of meson form factors : $(q^2)^{-1-\beta}$. The power law $s^{-2-\beta_i-\beta_j}$ is the same as that for a product of two form factors, but this model

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^{*}) The turning of quark lines outside $x, y \in [0, 1]$ is again performed by outer gluons as in Fig. 2.7.

cannot be written as a simple crossed-channel product of form factors, and differs from the constituent interchange model ⁵⁾ where the product of three form factors is advocated. We agree with the "dimensional counting" diagram of Brodsky and Farrar ¹⁶⁾ in the sense that we also obtain a routing of large momenta through a minimum number of gluon lines. In our case this is one gluon line, whereas there are three for meson-meson scattering in the BF approach. But this is only a formal distinction since two of the BF hot gluons can be thought to be part of the far off-shell Bethe-Salpeter wave function, which is given explicitly by $\phi_n(x)$ in the two-dimensional case. These wave functions near $x=0,1$ are the sources of the $s^{-\beta}$ factors in our result (4.8). In his form factor analysis, Einhorn ¹¹⁾ pointed out that the $(q^2)^{-1-\beta}$ behaviour was not generated by short-distance effects alone, as connections of the photon both directly to quarks and through mesonic states gave similar asymptotic behaviours. Similarly, in our "fixed angle" case, there are $s^{-2-\beta_i-\beta_j}$ contributions where the large momenta are routed through bound state sums as well as through single gluons, and we have no simple short-distance picture for the scaling. In particular, our scattering amplitude is not real at fixed angles, in contrast to what usually happens in constituent models ⁵⁾.

If any aspects of our calculation can be abstracted for fixed angle scattering in four dimensions, it would be amusing to see if fixed angle processes involving heavy (strange, charmed,...) quarks fall more rapidly than those involving the light up and down quarks alone. Unfortunately, if the light quarks occur anywhere at all in the external bound states, they will yield a fixed-angle contribution dominating bits due to heavy quarks. We have failed to come up with a single clean test of the fascinating $s^{-\beta_i-\beta_j}$ factors in (4.8), or of the suggestive correlations between the power behaviours of form factors, Regge and fixed angle limits.

It has previously been pointed out ²³⁾ that multiple quark-anti-quark scattering diagrams like Fig. 4.2 could dominate over the dimensional counting contributions to fixed angle scattering. These enter in $O(1/N_c^2)$ in our QCD approach, and correspond to new limits of the "Pomeron" cylinder diagrams ^{9),14)} to be studied in II ¹³⁾. Suffice to say here that in the presence of two hard scatterings ^{*}) (S_1 and S_2 in Fig. 4.2) with large momentum transfers $O(\sqrt{s})$ these diagrams are suppressed

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^{*}) There are other regions of phase space which just renormalize the power behaviour (4.8) in a way similar to that found for Regge behaviour : see Ref. 13).

by $O(1/s^2)^2$ relative to the $O(s)$ behaviour of the naive pre-cancellation Pomeron ¹²⁾. Thus Landshoff diagrams ²³⁾ do not dominate the results (4.8) for light quarks, as they have asymptotic behaviours at most $O(s^{-3})$ ^{*)}.

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^{*)} Landshoff diagrams may also have a Pomeron-like cancellation ^{12),13)} in this two-dimensional model, which would suppress their high energy behaviour still further.

5. - CONCLUSIONS AND PHENOMENOLOGICAL SPECULATIONS

Despite the defects inherent to its dimensionality, such as the absence of transverse degrees of freedom implying the absence of dynamical gluons and multi-gluon couplings, two-dimensional QCD is emerging as a very powerful and elegant model for strong interactions. Taken together with previous results ^{10),11)} on deep inelastic processes, the results of this paper suggest that QCD and hadronic folklore may not be incompatible. In addition to parton model ideas, many properties of dual models and topological expansions ⁹⁾ are recovered in two-dimensional QCD. The model also possesses many refinements of the conventional wisdom, and we would like to speculate how these properties may carry over to the real four-dimensional world.

In addition to "dual" crossed channel factorization, signature and exchange degeneracy properties, we found three intriguing properties of our "Regge" amplitudes. They were :

- (a) additivity of Regge "intercepts" in the quarks exchanged in the crossed channel : $\alpha^{b\bar{d}}(0) = -\beta_b - \beta_d$;
- (b) a lower limit to Regge "intercepts" : $\alpha^{b\bar{d}}(0) > -2$, arising because $\beta_q > 1$ even for very massive quarks ;
- (c) direct channel factorization of Regge residues :

$$(\bar{V}_{ij})^2 = (\bar{V}_{ii})(\bar{V}_{jj}) \quad (5.1)$$

Property (a) has been speculated before in the context of linearly rising Regge trajectories. However, two-dimensional QCD looks as if it is trying to have non-linear trajectories : witness the spectrum investigations of 't Hooft ^{*)} and the property (b) above. Our results suggest (a) may be a good approximation independently of the earlier arguments. Is it checkable for trajectories involving charmed quarks ?

Property (b) is antithetical to conventional dual models : it is clearly of vital phenomenological importance to searchers for charmed particles in hadronic reactions. It would be interesting to see if such a limiting behaviour for Regge trajectories obtains in four-dimensional QCD.

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^{*)} See particularly his Figs. 5 and 6.

Property (c) seems novel, but we can motivate it from extreme local duality. If there is a range in s where meson-meson scattering amplitudes are dominated by a single multiplet of direct channel resonances, but nevertheless average out to give leading Regge behaviour, then direct channel factorization will be forced on the Regge residues. This situation might arise in pseudoscalar meson scattering in the region of the tensor meson nonet, for example. Another context where direct channel factorization would hold would be in a valence quark approximation, where the amplitude for emitting a slow quark determines the whole bound state. Otherwise we would expect (5.1) to be replaced by a Schwarz inequality.

As for fixed angle scattering, we found :

- (d) a finite renormalization of dimensional counting, related to Regge and form factor powers :

$$A \sim s^{-2-\beta_i-\beta_j} : 0 < \beta_i < 1 \text{ as usual} \quad (5.2)$$

This renormalization arises because the relevant regions of internal loop integrations are infra-red, with quarks close to their mass shells, and just one "hot" off-shell gluon. The direct connection between Regge, form factor and fixed angle powers in two dimensions reflects the valence quark property which intimately relates "soft" and "hard" parts of the meson wave functions. Hence the precise amount of renormalization (5.2) of dimensional counting rules may be rather model-dependent. The suppression of multiple scattering (Landshoff) diagrams at fixed angle may also be an artefact of the two dimensional kinematics.

This ends our discussion of the properties of hadronic interactions in two-dimensional QCD in first order in $1/N_c$. We have seen that in addition to confirming many ideas about strong interactions such as partons, duality and topological expansions ⁹⁾, the model can go further and suggest new ideas for strong interaction phenomenology. Examples above include suggestions about Regge and fixed angle limits. We would like to understand our results from a more fundamental point of view, perhaps in configuration space. Another interesting direction may be to develop a phenomenological four-dimensional parton model with properties abstracted from our two-dimensional experience. A high priority is to study how many of the properties of the two-dimensional $1/N_c$ meson-meson scattering "Born term" discussed in this paper may be expected to carry over to four dimensions.

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APPENDIX A - CONSTRUCTION OF STATES $|x\rangle$

In this Appendix we give a more physical interpretation of the vector $|x\rangle$ introduced in Eq. (2.28) in Section 2A. Clearly, $|x\rangle$ is a kind of plane wave state for the quark-antiquark system, and we will return to the second quantized quark theory to construct it. The bound state wave function (2.16) is given by an equal "time" ($x_- = y_-$) matrix element

$$e^{-\frac{i}{2}p_-(x_++y_+)} \phi_n^{a\bar{b}}(x_+-y_+) = \langle n, p_- | q_R^{a+}(x_+) q_R^b(y_+) | 0 \rangle \quad (A.1)$$

where $|n, p_- \rangle$ is the n^{th} bound state with momentum p_- . The wave function $\phi_n^{a\bar{b}}(x)$ is the Fourier transform in the relative co-ordinate $z_+ \equiv x_+ - y_+$:

$$\phi_n^{a\bar{b}}(x) = \langle x | \phi_n^{a\bar{b}} \rangle = \int dz_+ e^{-\frac{i}{2}z_+r_-} \phi_n^{a\bar{b}*}(z_+) \quad (A.2)$$

where $r_- \equiv q_- - (p_- - q_-) = (2x-1)p_-$ is the difference in momentum between the quark a and the antiquark $\bar{b}$.

We can expand $q_R(q_R^+)$ in plane waves :

$$q_R(x_+) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk_-}{\sqrt{2k_-}} \{ a_k^+ e^{-ik_-x} + b_k e^{ik_-x} \} \quad (A.3)$$

and Fourier transform (A.1) in both x_+ and y_+ to obtain

$$\phi_n^{a\bar{b}}(x) p_- \delta(p'_- - p_-) = \langle x, p'_- | n, p_- \rangle \quad (A.4)$$

where the quark-antiquark plane wave state is

$$|x, p_- \rangle = \frac{a_{p_-}^+}{\sqrt{2x}} \frac{b_{p_-(1-x)}^+}{\sqrt{2(1-x)}} | 0 \rangle \quad (A.5)$$

so that

$$\langle x', p'_- | x, p_- \rangle = \delta(x' - x) \delta(p'_- - p_-) p_- \quad (A.6)$$

Clearly it is more convenient to avoid the conservation δ functions for p_- by removing the collective co-ordinate x_+ at the beginning. We work on the momentum (p_-) shell just as in old-fashioned perturbation theory. Taking the usual care with infinities for plane wave states, the state $|x\rangle$ is a mixed momentum ($x = q_-/p_-$) co-ordinate (x_+) state

$$|x\rangle = (\text{const.}) \times \int_0^\infty \frac{dp_-}{\sqrt{2p_-}} e^{i x p_-} |x, p_-\rangle \quad (\text{A.7})$$

evaluated at $x_+ = 0$.

It would be very useful to re-express the bound state formalism directly in configuration space. We have not done this, but expect it would clarify immensely both the partonesque ²⁾ (scaling) and string ²⁴⁾ (dual) features of the theory.

APPENDIX B - PHOTOPRODUCTION AND COMPTON SCATTERING

In this Appendix we study meson amplitudes involving external currents (photons) to study their Regge behaviour and fixed pole structures. Form factors and deep inelastic Compton scattering have already been discussed by Einhorn¹¹⁾, and we are here more concerned with photoproduction and Compton scattering of real photons.

Just as meson-meson scattering amplitudes could largely be constructed out of three-meson vertices, so also photo-amplitudes can largely be constructed out of meson form factors. Therefore we first consider an electromagnetic form factor, which Einhorn¹¹⁾ has shown to have the structure illustrated in Fig. N.1 :

$$(F_\mu)_{nm} \equiv (p_\mu + p'_\mu + \frac{m_m^2 - m_n^2}{q^2} q_\mu) F_{nm}(q^2) \equiv \langle n | J_\mu | m \rangle \quad (\text{B.1})$$

where

$$(F_-)^a_{nm} = 2e_a p_- (1-z) \left\{ \int_0^1 dz \phi_n(x+(1-x)z) \phi_m(z) \right. \\ \left. + x^2 \int_0^1 \int_0^1 du dz \frac{[\phi_n(xu) - \phi_n(x+(1-x)z)] G(u; q^2) \phi_m(z)}{[x(1-u) + (1-x)z]^2} \right\} \quad (\text{B.2})$$

is the contribution from the photon hitting the quark a , and $x \equiv q_-/p_-$ lies in the range $0 < x < 1$ for space-like q^2 . The first term in (B.2) comes from the direct photon-quark coupling of Fig. B1a, the rest goes through bound states as in Figs. B1b,c. For real q^2 we may choose a reference frame such that $q_- = 0$, i.e., $x = 0$. Then

$$(F_-)^a_{nm} = 2e_a p_- \int_0^1 dz \phi_n(z) \phi_m(z) = 2e_a p_- \delta_{nm} \quad (\text{B.3})$$

because of the orthogonality of the wave functions. This expression is not surprising. We know that in higher dimensions gauge invariance requires the vanishing of inelastic form factors at $q_\mu = 0$, but not $q^2 = 0$. In two dimensions the weaker condition is actually rather restrictive : recall from the discussion of the Adler zero condition that the $m^2 = 0$ meson decouples completely from the theory. The same argument given in that

case shows here also that the meson-dominated contribution to $(F_-)_{nm}$ vanishes at $q^2=0$ as found in deriving (B.3). In the above case of a vector photon coupling, the wave functions' orthonormality also killed the direct photon-quark coupling except for the elastic form factor, and fixed the correct bound state charge normalization.

In the case of a scalar current (scalar photon) the form factors are given by an expression similar to (B.2) but with an extra multiplicative factor

$$\frac{m}{p-} \frac{(x + 2z(1-x))}{z(1-x)(x+z(1-x))} \quad (B.4)$$

in the integrand. As before, the meson-dominated terms in (B.2) vanish when $q^2=0$ ($x=0$), but the direct scalar photon-quark diagrams no longer decouple from the inelastic states :

$$F_{nm}^q \sim \int_0^1 dz \frac{\phi_n(z) \phi_m(z)}{z} \quad (B.5)$$

Forward elastic Compton scattering of a vector photon consists of graphs which yield the squares of form factors, and a "contact" graph shown in Fig. B2. This diagram also vanishes at $q^2=0$, because it is vector-meson dominated. Thus the $q^2=0$ forward vector Compton amplitude just has a pole due to the elastic intermediate state (in agreement with current algebra). It hence has a ($J = -1$) "fixed pole" behaviour as $s \rightarrow \infty$ and no trace of Regge behaviour. For $q^2 \neq 0$ (non-leading) Regge behaviour as in purely hadronic amplitudes sets in. For scalar photons the "fixed pole", not normalized to the Born term, is non-leading for $\beta < 1$ and normal Regge behaviour dominates as in the purely hadronic amplitudes.

For photoproduction ($q^2 \leq 0$) besides "vector dominance" graphs we have those of Fig. B.3 : these include a piece which, for vector currents, reduces to the contribution of the elastic intermediate state $m=n$ for $q^2=0$ which behaves like $s^{-\beta_a - \beta_c}$, corresponding to a J plane pole at $J = -\beta_a - \beta_c$, which cannot be interpreted as Regge exchange since it carries in principle the flavour of the outgoing particle n (and that of the exchanged system only in case of an electromagnetic current). One might look at this behaviour as a $J=0$ pole modified in a similar way as the power of the electromagnetic form factor. In the case of scalar photoproduction again Regge behaviour dominates.

In conclusion we found "fixed pole" contributions in the case of vector current amplitudes not present in purely hadronic amplitudes. However a two-dimensional photon without transverse degrees of freedom surely is not a good model for real (transverse) photons in four dimensions. Hence we feel that one should not draw conclusions for the four dimensional world from this calculation.

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FIGURE CAPTIONS

- Figure 1.1   The three quark-antiquark exchange diagrams contributing to the meson-meson scattering amplitude in order $1/N_c$.
- Figure 2.1   The Feynman rules for two-dimensional QCD after all traces over γ matrices have been done. Colour indices are indicated $(i,j,...)$, flavour indices $(a,b,...)$.
- Figure 2.2   (a) The quark-antiquark scattering amplitude, with
(b) a typical planar diagram contributing in leading order in $1/N_c$.
- Figure 2.3   (a) One gluon contribution to the quark self-energy, and
(b) a multi-gluon graph which does not contribute to the quark self-energy.
- Figure 2.4   Integral equation for the quark-antiquark scattering matrix $T^{a\bar{b}}(q,q';p)$.
- Figure 2.5   Integral equation for the proper vertex function $\Gamma_n^{a\bar{b}}(q,q';p)$.
- Figure 2.6   (a) Absorbing the $q\bar{q}$ denominator into a proper vertex Γ_n (2.33) for $x \in [0,1]$, and
(b) turning round a quark with $x < 0$ using a gluon $\hat{V}$ (2.34).
- Figure 2.7   Three forms for the $q\bar{q}$ scattering matrix $T^{a\bar{b}}(x,x';p_-)$ convenient for
(a) $0 < x, x' < 1$,
(b) $x < 0, 0 < x' < 1$,
(c) $x, x' < 0$.
- Figure 2.8   (a) The three-meson vertex G_{123} expressed
(b) as an integral d^2k with internal quark propagators, and decomposed (c) and (d) into two old-fashioned perturbation theory diagrams after performing the k_+ integral.
- Figure 2.9   The bound state vertex G_{123} with $\phi_i(x)$ wave functions at the vertices, and no quark propagators.

- Figure 3.1   The three different contributions to A^{st} .
- Figure 3.2   The absorptive part of A^{st} , the dashed line indicating a discontinuity.
- Figure 3.3   Possible orderings for the meson $q-\bar{q}$ vertices in Fig. 3.1a.
- Figure 3.4   Possible orderings for the meson $q-\bar{q}$ vertices in Fig. 3.1b.
- Figure 3.5   Possible orderings for the meson $q-\bar{q}$ vertices in Fig. 3.1c.
- Figure 3.6   Diagrams contributing to A^{st} .
- Figure 4.1   An expression for the $q\bar{q}$ T matrix when there is a large momentum transfer between a and a' .
- Figure 4.2   A multiple scattering contribution ²³⁾ to meson-meson scattering.
- Figure B.1   Contributions to the meson form factor.
- Figure B.2   "Contact" graph for forward Compton scattering.
- Figure B.3   Contributions to photoproduction amplitudes.

FIG.1.1

FIG. 2.1

FIG. 2.2

(a)

(b)

FIG. 2.3

FIG. 2.4

FIG. 2.5

(a)

(b)

FIG. 2.6

Diagram (a) illustrates the decomposition of a box diagram $T^{a\bar{b}}$ into a sum of a tree-level exchange and a sum over n loops. The left side shows the box diagram $T^{a\bar{b}}$ with external momenta a, x , a, x' , $\bar{b}, 1-x$, and $\bar{b}, 1-x'$. The right side shows the decomposition:

$$T^{a\bar{b}} = \text{Tree-level exchange} - \sum_n \text{Loop diagrams}$$

The tree-level exchange is represented by a vertical dashed line connecting two horizontal lines. The loop diagrams are represented by two vertices $\phi_n^{a\bar{b}}$ connected by a horizontal line, with external momenta a, x , a, x' , $\bar{b}, 1-x$, and $\bar{b}, 1-x'$.

(a)

Diagram (b) illustrates the decomposition of a box diagram $T^{a\bar{b}}$ with a negative external momentum $a, x < 0$ into a sum over n loops. The left side shows the box diagram $T^{a\bar{b}}$ with external momenta $a, x < 0$, a, x' , $\bar{b}, 1-x$, and $\bar{b}, 1-x'$. The right side shows the decomposition:

$$T^{a\bar{b}} = \sum_n \text{Loop diagrams}$$

The loop diagrams are represented by two vertices $\phi_n^{a\bar{b}}$ connected by a horizontal line, with external momenta $a, x < 0$, a, x' , $\bar{b}, 1-x$, and $\bar{b}, 1-x'$.

(b)

Diagram (c) illustrates the decomposition of a box diagram $T^{a\bar{b}}$ with two negative external momenta $a, x' < 0$ and $a, x < 0$ into a sum over n loops. The left side shows the box diagram $T^{a\bar{b}}$ with external momenta $a, x' < 0$, $a, x < 0$, $\bar{b}, 1-x$, and $\bar{b}, 1-x'$. The right side shows the decomposition:

$$T^{a\bar{b}} = \text{Tree-level exchange} - \sum_n \text{Loop diagrams}$$

The tree-level exchange is represented by a vertical wavy line connecting two horizontal lines. The loop diagrams are represented by two vertices $\phi_n^{a\bar{b}}$ connected by a horizontal line, with external momenta $a, x < 0$, $a, x' < 0$, $\bar{b}, 1-x$, and $\bar{b}, 1-x'$.

(c)

FIG.2.7

FIG. 2.8

FIG. 2.9

FIG.3.1

FIG.3.2

FIG.3.3

FIG.3.4

FIG.3.5

(a)

(b.1)

(b.2)

(b.3)

(b.4)

(c.1)

(c.2)

(c.3)

(c.4)

FIG.4.1

FIG. 4.2

FIG. B. 1

FIG. B. 2

FIG. B. 3