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CM-P00061704

QUARK-ANTIQUARK POTENTIAL IN QCD

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A B S T R A C T

The potential energy of colour sources as a function of their separation is computed up to two loops ; the renormalization group equations it satisfies are derived. This enables us to obtain $\beta(g)$ to $O(g^5)$ and recover its known value. The solution of this Callan-Symanzik equation to $O(g^5)$ leads to a colour-confining potential. A qualitative argument is given to reject it as an artifact of the approximation used.

Ref.TH.2321-CERN

12 May 1977

1. - INTRODUCTION

In his lecture of Les Houches 1976, Susskind ¹⁾ studied asymptotic freedom in Yang-Mills theory by looking at the force law between fictitious "infinitely massive quarks" or rather the interaction energy of two static colour sources as a function of their spatial separation. Via this approach, he obtained the well-known asymptotic freedom results ²⁾ that the strength of the force law, i.e., the coupling constant, which is taken to be small for small distances, unavoidably grows for large distances. This behaviour is governed by the one-loop term in the β function.

This work will extend Susskind's approach to include two-loop contributions. In the intermediate steps involved in this calculation one encounters infra-red and ultra-violet divergences. The former will be shown not to endanger the whole approach if one prepares the static colour sources in a singlet state. These divergences then cancel between diagrams belonging to the same gauge invariant subset of diagrams. Once the combination of renormalization constants which governs the ultra-violet divergences has been found, the Callan-Symanzik equations satisfied by the interaction energy of the sources can be derived. Feeding this equation with the two-loop value for the energy one recovers the well-known two-loop value of β ³⁾

$$\beta(g) = -\frac{11}{3} g^3 \frac{C_2(G)}{16\pi^2} - \frac{34}{3} g^5 \frac{C_2^2(G)}{(16\pi^2)^2}$$

This is not surprising since β is universal up to two-loop ⁴⁾, after which it depends on the definition of g being used.

An important fact appears when one tries to solve the renormalization group equation for the energy using the up to one or two-loop truncated β function. The energy develops a pole which renders it infinitely attractive when the sources are separated up to a critical distance. This achievement of confinement is an artifact, not so much because of the truncation of $\beta(g)$, but because of a lack of information, on the source representation, buried in the higher loop corrections.

The outline of our paper is as follows. Section 2 serves an illustrative purpose ; it contains the calculation of the energy between two static electric charges which can be carried out to all orders. Section 3 discusses the gauge invariance of the Green's function used to calculate the energy. Section 4 contains the computation of the energy

up to two loops ; it is divided into two subsections concerned with the one- and two-loop contributions to the energy, respectively. In Section 5 we derive the Callan-Symanzik equation obeyed by the interaction energy and we show how the known two-loop value for β can be recovered. We then study the implications, in this approximation, of the renormalization group equation on the behaviour of the energy as a function of the spatial separation between the colour sources and its relevance for colour confinement. The first Appendix deals with the exponentiation problem in the Abelian and non-Abelian case. The second contains a list of the various diagrams contributing to the energy up to two loops with their values obtained in momentum space using dimensional regularization ⁵⁾.

2. - THE POTENTIAL ENERGY OF TWO STATIC ELECTRIC CHARGES

The ground state energy of a system described by an action $I(\varphi) = \int d^4x \mathcal{L}(\varphi)$, in the presence of external sources can in quantum field theory be expressed ⁶⁾ as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\int \mathcal{D}\{\varphi\} e^{-[\int \mathcal{L}(\varphi) d^4x + \int J(x) \varphi(x) d^4x]}}{\int \mathcal{D}\{\varphi\} e^{-\int \mathcal{L}(\varphi) d^4x}} \quad (1)$$

where, outside the time interval $(-T/2, +T/2)$, the sources have been switched off.

An illustration of this result for a simple gauge field theory would be a system of photons interacting with two point-like static electric charges. More specifically we consider a case where the two charges have the same magnitude but opposite sign. The expression for the energy of this system then reads :

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\int \mathcal{D}\{A_\mu\} e^{-\int d^4x \left[-\frac{(F_{\mu\nu})^2}{4} + \frac{(\partial_\mu A_\mu)^2}{2\eta} + \int_\mu(x) A_\mu(x) \right]}}{\int \mathcal{D}\{A_\mu\} e^{-\int d^4x \left[-\frac{(F_{\mu\nu})^2}{4} + \frac{(\partial_\mu A_\mu)^2}{2\eta} \right]}} \quad (2)$$

where

$$\int_\mu(x) = g \int_{\mu 0} \delta^3(\vec{x} - \vec{r}) - g \int_{\mu 0} \delta^3(\vec{x} - \vec{r}')$$

Let us first show that this quantity is gauge invariant as it should be, since it is an energy. We will prove this in a manner which may be extended to the more complicated situation encountered in non-Abelian theories.

The functional integral appearing in the numerator of (2) can be written as the following expectation value

$$\langle T(e^{g \int A_0(\vec{r}', t) dt} - g \int A_0(\vec{r}, t) dt) \rangle \quad (3)$$

where T stands for time-ordering. To show the gauge invariance of this Green's function, consider the gauge invariant operator $P e^{g \oint_{\Gamma} A_{\mu} dx_{\mu}}$ where Γ is a rectangular loop of spatial extent $|\vec{r} - \vec{r}'|$ and time extent T ; the symbol P means that the exponential is path-ordered. In the limit $T \rightarrow \infty$ the spatial components $A_i(\vec{r}, T/2)$, $A_i(\vec{r}', -T/2)$ reduce to pure gauge terms since $F_{\mu\nu} = 0$ at infinity and hence are gauge equivalent to $A_i = 0$. Therefore the operator $T e^{g \int A_0(\vec{r}', t) dt - g \int A_0(\vec{r}, t) dt}$ to which $P e^{g \oint_{\Gamma} A_{\mu} dx_{\mu}}$ reduces in the $T \rightarrow \infty$ limit is gauge invariant and so is its expectation value. So that the energy which is equal to

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\langle T(e^{g \int A_0(\vec{r}', t) dt} - g \int A_0(\vec{r}, t) dt) \rangle}{\langle 1 \rangle} \quad (4)$$

is gauge invariant as expected.

The evaluation of the energy may in this simple model be performed exactly since the functional integral to be computed is of Gaussian type. In momentum space, (3) becomes :

$$\int d\{A_{\mu} q\} e^{-[\frac{1}{2} \sum_q A_{\mu} q M_{\mu\nu} q A_{\nu} q + \sum_q J_{\mu} q A_{\mu} q]}$$

where

$$\begin{aligned} M_{\mu\nu} q &= -g_{\mu\nu} q^2 + (1 - \frac{1}{\eta}) q_{\mu} q_{\nu} \\ J_{\mu} q &= g(e^{-i\vec{q}\vec{r}} - e^{-i\vec{q}\vec{r}'}) \delta_{\mu 0} 2\pi \delta(q_0) \end{aligned} \quad (5)$$

Completing the squares and changing variables :

$$A_{\mu} = A'_{\mu} - \sum_{\nu} (M^{-1})_{\mu\nu} J_{\nu}$$

one obtains

$$\begin{aligned} & \int \mathcal{D}\{A_{\mu}\} e^{-[\frac{1}{2} \sum_{\mu\nu} A_{\mu} M_{\mu\nu} A_{\nu} + \sum_{\mu} J_{\mu} A_{\mu}]} \\ &= e^{\frac{1}{2} \sum_{\mu\nu} J_{\mu} (M^{-1})_{\mu\nu} J_{\nu}} 2\pi\delta(0) \int \mathcal{D}\{A'_{\mu}\} e^{-\frac{1}{2} \sum_{\mu\nu} A'_{\mu} M_{\mu\nu} A'_{\nu}} \end{aligned} \quad (6)$$

where

$$2\pi\delta(0) = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} dt = T$$

Hence the energy equation (2), using (5) and (6), is equal to

$$g^2 \sum_{\bar{q}} \frac{1}{\bar{q}^2} - g^2 \sum_{\bar{q}} e^{i\bar{q}(\bar{r}-\bar{r}')} \frac{1}{\bar{q}^2}$$

This is easily recognized as being the sum of the kinetic energies of the sources ($\sum_{\bar{q}} 1/\bar{q}^2$) which are linearly divergent due to the absence of Z graphs, and the potential energy which for this simple model reduces to the Coulomb energy as expected.

A diagrammatic derivation of this result may also be obtained. The graphical rules one encounters in expanding the Green's function (4) involve the photon propagator and Θ functions of time due to the time-ordering in (4). These Θ functions may be looked upon as propagators of the static sources. They are the amplitude for such sources located at some space point $\bar{r}$ at time t to be there at some other time. Individual diagrams have Θ functions, but when added together these Θ functions combine to exponentiate the one-photon exchange (Coulomb interaction) and the self-energy of the static electric charges (see Appendix 1).

3. - POTENTIAL ENERGY OF TWO STATIC COLOUR SOURCES

We now proceed to generalize the previous considerations to a non-Abelian gauge theory. We will evaluate the energy of two static colour sources prepared in a singlet state as a function of their spatial separation. The expression for the energy in this case takes the form

$$E = \lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\text{tr} \int D(\vec{A}_\mu) D(c_a) D(c_a^*) e^{-\int d^4x [\mathcal{L}(\vec{A}_\mu, c_a, c_a^*) + \vec{j}_\mu \cdot \vec{A}_\mu]} }{\text{tr} \int D(\vec{A}_\mu) D(c_a) D(c_a^*) e^{-\int d^4x \mathcal{L}(\vec{A}_\mu, c_a, c_a^*)}} \quad (7)$$

as follows from (1). Since we choose to work in one of the covariant gauge viz. the renormalized Feynman gauge one has to integrate over Faddeev-Popov ghost fields (c_a, c_a^*) . The Lagrangian appearing in (7) written in terms of renormalized quantities is then

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} (\partial_\mu A_\mu)^2 - Z_1 g f_{abc} \partial_\mu A_\nu A_\mu A_\nu \\ & - \frac{1}{4} g^2 \frac{Z_1^2}{Z_3} f_{abc} f_{ade} A_\mu A_\nu A_\mu A_\nu - \tilde{Z}_3 \partial_\mu c_a^* \partial_\mu c_a \\ & - \tilde{Z}_1 g f_{abc} \partial_\mu c_a^* A_\mu c_b \end{aligned}$$

The f_{abc} are the totally antisymmetric structure constants of the gauge group G which obey the relation $f_{acd} f_{bcd} = C_2(G) \delta_{ab}$ where $C_2(G)$ is the value of the quadratic Casimir operator in the regular representation of G . Since the whole system has a singlet character,

$$j_\mu(x) = g T_a j_{\mu 0} \delta^3(\vec{x} - \vec{r}) - g T_a j_{\mu 0} \delta^3(\vec{x} - \vec{r}')$$

where the matrices T_a describing the colour degree of freedom of the sources act on a finite dimensional representation space. The colour being quantized, one must sum over all the possible colour degrees of freedom, hence the appearance of the trace in (7).

To study the gauge invariance of expression (7) let us rewrite it as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\langle \text{tr} T(e^{g \int_{-T}^T A_0(\vec{r}, t) T_a dt} - g \int_{-T}^T A_0(\vec{r}', t) T_a dt) \rangle}{\langle \text{tr} 1 \rangle} \quad (8)$$

We may now follow the same method as in the Abelian case, but new aspects due to the non-Abelian character of the gauge fields will appear. Let us consider the gauge invariant operator $\text{tr} P e^{g \oint_{\Gamma} \vec{A}_{\mu} d\vec{x}_{\mu}}$ where Γ is again a rectangular loop of spatial extent $|\vec{r} - \vec{r}'|$ and time extent T . Since $\vec{F}_{\mu\nu}$ vanishes in the limit $T \rightarrow \infty$ the spatial components $\vec{A}_i(\vec{\xi}, T/2)$ and $\vec{A}_i(\vec{\eta}, -T/2)$ reduce to pure gauge terms. Here appears a property of pure Yang-Mills theories which does not have a counterpart in the pure Abelian case, namely the possibility of having gauge transformations which cannot be joined continuously with the identity⁷⁾. This is the situation encountered in the non-zero instanton sector of a non-Abelian gauge theory. We shall restrict ourselves in this paper to the zero instanton sector and hence exclude the possibility of having gauge transformations belonging to different topological classes. In this case the spatial components $\vec{A}_i(\vec{\xi}, T/2)$, $\vec{A}_i(\vec{\eta}, -T/2)$ in the limit $T \rightarrow \infty$ are gauge equivalent with $\vec{A}_i = 0$ and therefore we do not have to integrate over these when we average over Yang-Mills field configurations. Then, as for the Abelian case, the operator $\text{tr} P e^{g \oint_{\Gamma} \vec{A}_{\mu} d\vec{x}_{\mu}}$ in the limit $T \rightarrow \infty$ reduces to $\text{tr} T e^{g \int_{\vec{r}} \vec{A}_0(\vec{r}, t) T dt - g \int_{\vec{r}'} \vec{A}_0(\vec{r}', t) T dt}$ and hence one finds that the energy is gauge invariant as expected.

4. - CALCULATION OF THE ENERGY

Again the Feynman rules for calculating the energy involve Θ functions of time due to the presence of the time-ordered exponential in (8). The source terms lead to vertices proportional to $\delta_{\mu 0} T_a$, which reflects the time-like character of these source currents. In addition to these particular rules we have the usual Feynman rules for non-Abelian gauge theories.

A. - One-loop contributions

- Figure 1 -

The one-loop diagrams involved in the interaction energy calculation are shown in Fig. 1.

Because of the static character of the sources a simplification occurs in this list of diagrams ; diagram (h) involving a tri-linear self-coupling of gluons vanishes. This is a consequence of working in the Feynman gauge which preserves the gluon's polarization and spin statistics, three time-like gluons interacting at the same space-time point.

The surviving diagrams, except for the vacuum polarization loops, are present in the previous Abelian case with the sole difference that each vertex now contains a matrix, a reflection of the non-Abelian character of the gauge group. When one works out the group algebraic weight for each diagram, they all possess an "Abelian part", proportional to $C_2^{n+1}(R)$ in the n loop approximation [i.e., $C_2^2(R)$ here, where $C_2(R)I = T_a T_a$] which exponentiates as in the Abelian case a Coulomb-like potential $(g^2/4\pi|\vec{r}-\vec{r}'|)C_2(R)$.

Only diagrams (a), (b), (d) and (e) have a non-Abelian content proportional to $C_2(G)$. Two of these (d) and (e) contain infra-red divergences, but these divergences cancel because the configuration into which the sources have been put in has a singlet character. This can be qualitatively understood when one considers the following diagrams in configuration space for $|P_2 P_4| \gg |\vec{r}-\vec{r}'|$ or $|P_1 P_3| \gg |\vec{r}-\vec{r}'|$,

- Figure 2 -

$|Q_1 Q_3| \gg |\vec{r}-\vec{r}'|$ and $|R_1 R_3| \gg |\vec{r}-\vec{r}'|$ where the infra-red regime sets in. Obviously whether P_2 is on the right-hand or on the left-hand source line is equivalent and hence clearly diagram (a) with $|P_2 P_4| \gg |\vec{r}-\vec{r}'|$, and

diagram (b) with $|Q_1 Q_3| \gg |r-r'|$ have the same magnitude. Because of the singlet character of the system they have opposite signs. Since one could have taken $|P_1 P_3| \gg |\bar{r}-\bar{r}'|$, diagram (c) is needed together with (b) to cancel the infra-red divergence of (a). This can be checked from their explicit values listed in Appendix 2.

Ultra-violet divergences occur in diagrams (a), (b) and (e) of Fig. 1. The divergence appearing in gluon self-energy diagrams can be handled as usual by wave function renormalization. To understand the renormalization procedure involved in removing the divergences of the non-Abelian part of diagram (e), we note that the Feynman rules involving the sources are completely mimicked by the rules emerging from the following fermion Lagrangian of Bloch-Nordsieck type $\bar{\psi}(\partial_\mu - g \vec{A}_\mu \cdot \vec{T}) N_\mu \psi$, where N_μ is a pure time-like unit vector. The fermion propagator is a θ function of time and the vertex is the same as the one coupling the sources to the gluons. It is then straightforward to see that the renormalization factor looked for is Z_1^F/Z_2^F , where Z_1^F is the fermion vertex renormalization constant and Z_2^F the fermion wave function renormalization constant. Since this fermion Lagrangian is gauge invariant, Ward-Slavnov-Taylor identities relate the Z_i^F to the renormalization constants for gluon (ghost) propagation and interaction, in particular ⁸⁾

$$\frac{Z_1^F}{Z_2^F} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3}$$

Hence the potential energy part of

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\langle \hbar T (e^{g_A \frac{Z_1}{Z_3} \int \vec{A}_0^R(\vec{r}', t) \vec{T} dt - g_A \frac{Z_1}{Z_3} \int \vec{A}_0^R(\vec{r}, t) \vec{T} dt}) \rangle}{\langle \hbar \rangle}$$

is ultra-violet convergent, which bears as an immediate consequence the canonical dimensionality of the interaction energy between the colour sources.

Taking these remarks into account, one may compute the energy which, in the one-loop approximation, is :

$$E = \frac{-u^2 C_2(R)}{\bar{q}^2} \left[1 + \left(\frac{11}{3} \log \frac{\mu^2}{\bar{q}^2} - \frac{11}{3} \gamma + \frac{31}{9} \right) u^2 \frac{C_2(G)}{16\pi^2} \right] \quad (9)$$

where γ is the Euler number. The explicit values of the various contributions computed in momentum space by dimensional regularization are listed in Appendix 2.

B. - Two-loop contributions

- Figure 3 -

The two-loop diagrams contributing to the potential energy are shown in Fig. 3 (a blob on a vertex and a blob with index (1)/(2) on a gluon propagator denotes that it is one/two loop renormalized). Again because of the static character of the sources and the use of the Feynman gauge, some of the diagrams vanish ; the quadrilinear vertex involved in diagram (u) vanishes, and, in diagrams (v) and (w), the antisymmetric structure constant present in the trilinear vertex multiplies a symmetric tensor and hence they also disappear.

As in the one-loop calculation, individual diagrams have infra-red and ultra-violet divergences. The subsets of diagrams which are infra-red finite are easily constructed on the pattern briefly described in the previous section. One considers all possible ways of bringing vertices from one source propagator to the other without crossing any vertex, i.e., generating commutators. These subsets of diagrams are shown in Appendix 2 with their individual values ; it is easily checked that the infra-red divergences indeed cancel.

The renormalized diagrams are obtained as in the one-loop case by expanding

$$\frac{\langle k T(e^{g_R \frac{Z_1}{Z_2} \int \vec{A}_0(\vec{r}, t) \vec{T} dt - g_R \frac{Z_1}{Z_2} \int \vec{A}_0(\vec{r}, t) \vec{T} dt}) \rangle}{\langle k 1 \rangle}$$

Again the sum of diagrams lead to exponentiation through the combinations of pieces from various diagrams which have the same group algebraic structure (see Appendix 1). Collecting the various contributions listed in Appendix 2 results in the following value for the energy in the two-loop approximation

$$E(1\bar{q}, \mu, u) = - \frac{u^2 C_2(R)}{\bar{q}^2} \left[1 + \left(\frac{11}{3} \log \frac{\mu^2}{\bar{q}^2} - \frac{44}{3} + \frac{31}{9} \right) \frac{C_2(G) u^4}{16 \pi^2} \right. \\ \left. + \left(\frac{121}{9} \log^2 \frac{\mu^2}{\bar{q}^2} - \frac{242}{9} \log \frac{\mu^2}{\bar{q}^2} + \frac{388}{27} \log \frac{\mu^2}{\bar{q}^2} + cte \right) \frac{C_2^2(G) u^6}{(16 \pi^2)^2} \right] \quad (10)$$

This result clearly shows the attractive character of the interaction between the sources. Unfortunately the logarithmic corrections to the Coulomb-like potential, although they have the appropriate sign, do not change radically the possibility of separating the colour sources up to infinity. The renormalization group with its property of iterating contributions from diagrams to all orders may contain some more information and, as with the Coleman-Weinberg⁹⁾ mechanism, may even produce phenomena not seen in the straightforward perturbation expansion.

5. - RENORMALIZATION GROUP EQUATION FOR THE ENERGY

As was shown in Section 4A, there is no anomalous dimension for the energy. Since there are no masses present in this problem, the renormalization group equation takes the simple form

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(u) \frac{\partial}{\partial u} \right) E(\bar{q}, \mu, u) = 0 \quad (11)$$

which states that a change in subtraction point may be re-absorbed by an appropriate change in coupling constant.

One may use this equation to derive the value of $\beta(u)$ by applying it on the perturbation expansion for E and then identify the constant and different powers of logarithm so generated. In our case, using Eq. (10) for E , we recover the known two-loop value for β ,

$$\beta(u) = -\frac{11}{3} u^3 \frac{C_2(G)}{16\pi^2} - \frac{34}{3} u^5 \frac{C_2^2(G)}{(16\pi^2)^2}$$

Let us now use Eq. (11) to gather some information on how the energy varies when one rescales the distance separating the sources. Since $\bar{q}^2 E(\mu, |\bar{q}|, u)$ is a function of μ through $\mu^2/\bar{q}^2$ one has $\mu(\partial/\partial \mu) \bar{q}^2 E = -\bar{q}(\partial/\partial \bar{q}) \bar{q}^2 E$. Equation (11) takes then the form $\bar{q}(\partial/\partial \bar{q}) E - \beta(u)(\partial/\partial u) E = -2E$ which in configuration space becomes

$$\bar{x} \frac{\partial}{\partial \bar{x}} - \beta(u) \frac{\partial}{\partial u} E = -2E \quad (12)$$

Let us introduce a parameter λ which scales the distance between the colour sources. Since the energy has dimension $(\text{length})^{-1}$ it follows that

$$\left(\lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu} \right) E(\mu, |\bar{x}|, u) = - E(\mu, |\bar{x}|, u)$$

Combining this result with (12), one finally obtains

$$\left(\lambda \frac{\partial}{\partial \lambda} - \beta(u) \frac{\partial}{\partial u} \right) E = - E \quad (13)$$

for which the general solution is

$$E(\lambda |\bar{x}|, u, \mu) = \lambda^{-1} E(|\bar{x}|, \bar{u}, \mu) \quad (14)$$

where the function $\bar{u} = \bar{u}(u, \lambda)$ is defined by

$$\int_u^{\bar{u}} \frac{d\xi}{\beta(\xi)} = \ln \lambda, \quad \bar{u}(u, 1) = u$$

For simplicity, let us consider (14) where we restrict ourselves to the one-loop value for $\beta(u)$. One obtains in this approximation for the energy :

$$E(\mu, |\bar{x}|, u) = \frac{-u^2 C_2(R) / 4\pi |\bar{x}|}{1 - \frac{22}{3} \frac{C_2(G) u^2}{16\pi^2} \log \mu |\bar{x}|} \quad (15)$$

which corresponds to the summation of all leading logarithms.

The energy develops a pole at

$$|\bar{x}|_{\text{pole}} = \mu^{-1} e^{\frac{24\pi^2}{11 C_2(G)} \frac{1}{u^2}}$$

which is reminiscent of the pole in $\bar{u}$ due to the fact that the truncated β function decreases sufficiently fast¹⁰⁾. It would be tempting to take the pole in the energy seriously because in such a case, confinement would have been realized, the cost in energy to separate the quarks more than $|\bar{x}|_{\text{pole}}$ being infinite. Unfortunately this behaviour is an artifact of the approximation we are restricted to, two-loop included. In fact, the two-loop solution of the renormalization group equation for the energy (which we were unable to write down explicitly due to the complicated implicit equation for $\bar{u}$) has the same qualitative behaviour as the one-loop

solution ; only the location of the pole is shifted. One could think that it is the unknown behaviour of $\beta(u)$ up to all orders which causes the trouble, but this is not the case. Consider a situation where the β function decreases to $-\infty$ sufficiently fast such that the effective coupling $\bar{u}$ develops a pole. We will give a qualitative argument which shows that the interaction energy between the two quarks is extremely sensitive to the representation in which the quarks lie ¹¹⁾, despite the fact that the β function for pure Yang-Mills does not depend on the nature of the colour sources. Let us consider the class of diagrams shown in Fig. 4.

- Figure 4 -

Suppose first that the sources are in the octet representation [we consider the case of $SU(3)$]. The gluon propagating along a quark line can make a singlet state with this quark, so that the physical situation would be : each quark has his gluon partner propagating around him and as such being screened from the other quark. Such a situation could never arise if the sources are in triplet representation where they cannot build a singlet state with the complicity of a gluon. Unfortunately the first sign of such a behaviour appears at the three-loop level. Here the diagram shown in Fig. 5 and its gauge partners contain some additional information about the

- Figure 5 -

influence of the quark representation, due to their quadratic and hence non-linear dependence on $C_2(R)$. We were unable to render our previous argument more quantitative due to the complexity of the calculations involved.

The conclusion which can be drawn from these considerations is that although the effective interaction between the sources increases with spatial separation it is far from sufficient to confine the colour sources. It might perhaps be possible to improve this situation by considering the topological structure of the gauge group as advocated by Polyakov¹²⁾ and take into account the non-zero instanton sector of the theory to the interaction between the colour sources.

ACKNOWLEDGEMENTS

I wish to thank L. Susskind for very stimulating discussions and for his hospitality at the Belfer Graduate School where this work started. I would like also to thank R. Brout, R. Crewther, F. Englert and C. Sachrajda for helpful discussions, and N. Phinney for a careful reading of the manuscript.

APPENDIX 1

A. - Exponentiation in the Abelian case

Let us first look at the pure exchange diagrams shown in Fig. 6 :

- Figure 6 -

For the sake of simplicity we consider the sum of the one-loop diagrams, since one is obtained from the other by permuting two vertices on the same quark line ; adding these diagrams together destroys one time ordering, $\Theta(t) + \Theta(-t) = 1$. As there is no privileged source line, one repeats the same operation on the other line, the net result of both permutations is then $2!(6(b)+6(c)) = (6(a))^2$. The general argument for the order $(g^2)^n$ goes as follows : there are n vertices on each source line ; label them from 1 to n ; identical numbers should be joined by a photon line. All the diagrams of $O(g^{2n})$ are generated by permuting the n vertices on one of the source lines keeping the configuration on the other line fixed ; this operation results in $n!$ diagrams. The way to destroy the ordering and get rid of the sources' propagation is to allow the vertices which have been kept fixed in the construction of the diagrams to permute ; in this way we generate each diagram $n!$ times. Hence the result

$$n! \left(\underbrace{\text{diagram (c)} + \dots}_{O(g^{2n}) \text{ exchange diagrams}} \right) = (\text{diagram (a)})^n$$

$O(g^{2n})$ exchange diagrams

which implies $\text{diagram (a)} + \text{diagram (b)} + \text{diagram (c)} + \dots = \exp(\text{diagram (a)})$

For diagrams where additional photons are exchanged on the same line, an identical procedure leads to the exponentiation of the self-energies and one recovers the result obtained for

$$\frac{\langle T(e^{g \int A_0(F;t) dt} - g \int A_0(F;t) dt) \rangle}{\langle 1 \rangle}$$

in Section 2 by functional integration.

B. - Exponentiation in the non-Abelian case

The complication here is the presence of non-commuting generators at each vertex which satisfy commutation relations $[\mathbb{T}_a, \mathbb{T}_b] = i f_{abc} \mathbb{T}_c$. This leads to various group theoretical factors appearing in the diagrams. Let us look at the one-loop pure exchange diagrams

- Figure 7 -

The group theoretical factor involved in (a) is

$$\text{tr } T_a T_b T_b T_a / \text{tr } 1 = C_2^2(R)$$

for (b) we have

$$\text{tr } T_a T_b T_a T_b / \text{tr } 1 = C_2^2(R) - C_2(R) C_2(G) / 2$$

It is easy to see that the $C_2^2(R)$ terms combine as in the Abelian case to start exponentiating the one-gluon exchange. Indeed, these terms arise from commuting the T 's as if they were commuting objects. As an immediate consequence one has that the highest power in $C_2(R)$ for each order conspire to exponentiate the one-gluon exchange. To see what happens to the surviving piece of diagram (b) proportional to $C_2(R)C_2(G)$ we consider the two-loop pure exchange diagrams containing among others the group theoretical factor $C_2^2(R)C_2(G)$.

(a)

(b)

(c)

(d)

$$C_2^3(R) - C_2^2(R) C_2(G)/2$$

$$C_2^3(R) - C_2^2(R) C_2(G) + C_2(R) C_2^2(G)/4$$

(e)

$$C_2^3(R) - \frac{3}{2} C_2^2(R) C_2(G) + C_2^2(G) C_2(R)/2$$

- Figure 8 -

These diagrams are listed in Fig. 8 with their group theoretical weight. We treat them as if they were Abelian diagrams, i.e., as if the generators are replaced by the identity. Adding diagrams (a), (d) and (e) together destroys one source correlation and one obtains

Adding diagrams (b), (c) and (e) together gives

repeating the same operation with (c), (d) and (e) gives

Hence the total sum reduces to the product of the surviving diagram Fig. 7(b) by the one-gluon exchange. The last summation has destroyed the correlation on the left-hand source involving the gluon which is already uncorrelated on the right-hand source. The important point is that the number of times each diagram has contributed to this sum is exactly equal to the factor multiplying the $C_2^2(R)(C_2(G)/2)$ coefficient for each diagram ; this constitutes the start of the exponentiation in the non-Abelian case. For diagrams which are not of the pure exchange type, a similar conspiracy leads to a starting exponentiation. We do not know a general proof for the exponentiation in the non-Abelian case.

APPENDIX 2 - NUMERICAL VALUE OF THE DIFFERENT CONTRIBUTIONS TO THE ENERGY UP TO TWO-LOOP

The various diagrams have been calculated in momentum space by using dimensional regularization. The symbols we use are : $\epsilon_i = d-4$, i stands for infra-red ; $\epsilon = 4-d$, γ = Euler number, the coupling constant $g = u(\mu)^{2-d/2}$, $\bar{q}$ corresponds to the Fourier transform of the distance $\bar{x}$ between the two sources,

$$(1) \quad \bullet \text{---} \bullet = \text{one-loop renormalized vertex} = \left\{ \text{diagram} \right\} + \times \text{---} g_R \left(\frac{Z_1}{Z_3} - 1 \right)^{(1)}$$

$$\begin{aligned} (1) \quad \text{---} \bullet \text{---} &= \text{one-loop renormalized propagator} \\ &= \text{diagram} + \text{diagram} + \text{diagram} \delta Z_3^{-1} \end{aligned}$$

One-loop contributions

As we have seen in the previous Appendix, the contributions proportional to $C_2^2(R)$ correspond to the one-gluon (Coulomb-like) exchange so that we only retain terms proportional to $C_2(R)C_2(G)$:

$$\begin{aligned} \left\{ \text{diagram} \right\} &= \frac{u^4}{\bar{q}^2} \frac{C_2(R)C_2(G)/2}{16\pi^2} \left(\frac{4}{\epsilon_i} + \frac{4}{\epsilon} \right) \\ \left\{ \text{diagram} \right\} &= - \frac{u^4}{\bar{q}^2} \frac{C_2(R)C_2(G)/2}{16\pi^2} \left(\frac{8}{\epsilon_i} - 4\gamma + 4 \log \frac{\mu^2}{\bar{q}^2} \right) \end{aligned}$$

Hence the sum

$$\left\{ \text{diagram} \right\} + \left\{ \text{diagram} \right\} + \left\{ \text{diagram} \right\}$$

is infra-red finite as the pole in $1/\epsilon_i$ has disappeared. To obtain all the renormalized diagrams one has to add the vacuum polarization contributions and the vertex counter term, i.e. :

$$\begin{aligned}
 & \text{Diagram 1} + \text{Diagram 2} + \delta Z_3^{-1} \text{Diagram 3} = -\frac{u^4}{\bar{q}^2} \frac{C_2(R)C_2(G)}{16\pi^2} \left(\frac{5}{3} \log \frac{\mu^2}{\bar{q}^2} - \frac{5}{3} \gamma + \frac{31}{9} \right) \\
 & \delta(Z_1/Z_3) \text{Diagram 3} = \frac{u^4}{\bar{q}^2} \frac{C_2(R)C_2(G)}{16\pi^2} \left(-\frac{2}{\epsilon} \right)
 \end{aligned}$$

One then obtains for the energy in the one-loop approximation

$$E = -\frac{u^2}{\bar{q}^2} C_2(R) \left[1 + \left(\frac{11}{3} \log \frac{\mu^2}{\bar{q}^2} - \frac{11}{3} \gamma + \frac{31}{9} \right) u^2 \frac{C_2(G)}{16\pi^2} \right]$$

Two-loop contributions

We only keep the $C_2(R)C_2^2(G)$ part of the diagrams since the other contributions involve the Coulomb-like interaction (one-gluon exchange) in some way (see Appendix 1). We will group the various diagrams in subsets which are infra-red finite. These subsets are constructed following the recipe given in Section 4B.

Subset A

$$\text{Diagram 1} = -\frac{u^6}{\bar{q}^2} A \left(\frac{2}{\epsilon_i^2} - \frac{2}{\epsilon^2} \right) \quad A \equiv C_2(R) C_2^2(G) / 16\pi^2$$

$$\text{Diagram 2} = -\frac{u^6}{\bar{q}^2} A \left(\frac{2}{\epsilon_i^2} - \frac{2}{\epsilon_i} - \frac{2}{\epsilon^2} - \frac{2}{\epsilon} \right)$$

$$\text{Diagram 3} = -\frac{u^6}{\bar{q}^2} A \left(-\frac{4}{\epsilon_i^2} + \log^2 \frac{\mu^2}{\bar{q}^2} - 2\gamma \log \frac{\mu^2}{\bar{q}^2} + cte \right)$$

$$\begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} = -\frac{u^6}{\bar{q}^2} A\left(\frac{4}{\epsilon_i^2}\right)$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} = -\frac{u^6}{\bar{q}^2} A\left(\frac{4}{\epsilon_i^2} + \frac{4\gamma}{\epsilon_i} - \frac{4}{\epsilon_i} \log \frac{\mu^2}{\bar{q}^2} - 4\gamma \log \frac{\mu^2}{\bar{q}^2} + 2 \log^2 \frac{\mu^2}{\bar{q}^2} + \text{cte}\right)$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} = -\frac{u^6}{\bar{q}^2} A\left(-\frac{4}{\epsilon_i^2} + \frac{2}{\epsilon_i} - \frac{4\gamma}{\epsilon_i} + \frac{4}{\epsilon_i} \log \frac{\mu^2}{\bar{q}^2} - 2 \log^2 \frac{\mu^2}{\bar{q}^2} + 4\gamma \log \frac{\mu^2}{\bar{q}^2} + \text{cte}\right)$$

Hence the sum

$$\begin{array}{c} \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} \\ + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} \\ + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} + \begin{array}{c} (1) \\ \bullet \end{array} \text{---} \text{---} \text{---} \begin{array}{c} (1) \\ \bullet \end{array} \end{array}$$

is infra-red finite as no poles in $1/\epsilon_i, 1/\epsilon_i^2$ survive.

Subset B

$$\text{Diagram 1} = -\frac{\mu^6}{q^2} A \left(\frac{4}{\epsilon_i} + \frac{4}{\epsilon} \right)$$

$$\text{Diagram 2} = -\frac{\mu^6}{q^2} A \left(-\frac{8}{\epsilon_i^2} - \frac{8\gamma}{\epsilon_i} + \frac{8}{\epsilon_i} \log \frac{\mu^2}{q^2} + 8\gamma \log \frac{\mu^2}{q^2} - 4 \log^2 \frac{\mu^2}{q^2} \right)$$

$$\text{Diagram 3} = -\frac{\mu^6}{q^2} A \left(\frac{16}{\epsilon_i^2} - \frac{8}{\epsilon_i} + \frac{16\gamma}{\epsilon_i} - \frac{16}{\epsilon_i} \log \frac{\mu^2}{q^2} + 8 \log^2 \frac{\mu^2}{q^2} + 8 \log \frac{\mu^2}{q^2} - 16\gamma \log \frac{\mu^2}{q^2} \right)$$

The sum

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}$$

is infra-red finite.

Subset C

$$\text{Diagram 1} = -\frac{\mu^6}{q^2} A \left(\frac{62}{9\epsilon_i} - \frac{10}{3} \frac{\gamma}{\epsilon_i} + \frac{10}{3\epsilon_i} \log \frac{\mu^2}{q^2} - \frac{5}{6} \log^2 \frac{\mu^2}{q^2} + \frac{5}{3} \gamma \log \frac{\mu^2}{q^2} - \frac{31}{9} \log \frac{\mu^2}{q^2} \right)$$

$$\text{Diagram 2} = -\frac{\mu^6}{q^2} A \left(-\frac{5}{\epsilon_i^2} - \frac{101}{9\epsilon_i} + \frac{15}{4} \log^2 \frac{\mu^2}{q^2} + \frac{101}{9} \log \frac{\mu^2}{q^2} - \frac{15\gamma}{2} \log \frac{\mu^2}{q^2} + \frac{10}{3} \frac{\gamma}{\epsilon_i} - \frac{10}{3\epsilon_i} \log \frac{\mu^2}{q^2} + \text{etc} \right)$$

$$\text{Diagram 3} = -\frac{\mu^6}{q^2} A \left(\frac{5}{\epsilon_i^2} + \frac{13}{3\epsilon_i} - \frac{5}{\epsilon^2} + \frac{13}{3\epsilon} \right)$$

Again the sum

is infra-red finite.

Subset of infra-red finite diagrams

$$\text{Diagram 1} = -\frac{u^6}{\bar{q}^2} A \left(\frac{5}{4\epsilon^2} - \frac{37}{48\epsilon} - \frac{5}{16} \log^2 \frac{\mu^2}{\bar{q}^2} - \frac{17}{48} \log \frac{\mu^2}{\bar{q}^2} + \frac{58}{8} \log \frac{\mu^2}{\bar{q}^2} + cte \right)$$

$$\text{Diagram 2} = -\frac{u^6}{\bar{q}^2} A \left(-\frac{7}{16\epsilon} - \frac{7}{16} \log \frac{\mu^2}{\bar{q}^2} + cte \right)$$

$$\text{Diagram 3} = -\frac{u^6}{\bar{q}^2} A \left(-\frac{1}{32\epsilon} - \frac{1}{32} \log \frac{\mu^2}{\bar{q}^2} + cte \right)$$

$$\text{Diagram 4} = -\frac{u^6}{\bar{q}^2} A \left(-\frac{3}{4\epsilon} - \frac{3}{4} \log \frac{\mu^2}{\bar{q}^2} + cte \right)$$

and

are equal to a constant times $\frac{u^6}{\bar{q}^2} A$

$$\text{---}\overset{(1)}{\bullet}\text{---}\overset{(1)}{\bullet}\text{---} = -\frac{u^6}{\bar{q}^2} A \left(\frac{25}{9} \log^2 \frac{\mu^2}{\bar{q}^2} + \frac{310}{27} \log \frac{\mu^2}{\bar{q}^2} - \frac{508}{9} \log \frac{\mu^2}{\bar{q}^2} + \text{cte} \right)$$

$$\text{---}\overset{(2)}{\bullet}\text{---} = -\frac{u^6}{\bar{q}^2} A \left(\frac{25}{12} \log^2 \frac{\mu^2}{\bar{q}^2} - \frac{258}{6} \log \frac{\mu^2}{\bar{q}^2} + \frac{395}{36} \log \frac{\mu^2}{\bar{q}^2} + \text{cte} \right)$$

where $\text{---}\overset{(2)}{\bullet}\text{---}$ is the renormalized two-loop gluon propagator.

The over-all sum of diagrams still contains ultra-violet divergences (poles in $1/\epsilon, 1/\epsilon^2$) which are handled by taking into account diagrams involving the vertex counter terms

$$g_R \left(\frac{Z_1}{Z_3} - 1 \right)^{(2)} \times \text{---} + \text{---} \times \left(\frac{Z_1}{Z_3} - 1 \right)^{(2)} g_R$$

Summing all these contributions leads to the two-loop value for the energy :

$$E = -\frac{u^2}{\bar{q}^2} C_2(R) \left[1 + \left(\frac{11}{3} \log \frac{\mu^2}{\bar{q}^2} - \frac{11}{3} \delta + \frac{31}{9} \right) u^2 \frac{C_2(G)}{16\pi^2} \right. \\ \left. + \left(\frac{121}{9} \log^2 \frac{\mu^2}{\bar{q}^2} - \frac{242}{9} \delta \log \frac{\mu^2}{\bar{q}^2} + \frac{388}{27} \log \frac{\mu^2}{\bar{q}^2} + \text{cte} \right) \frac{u^4 C_2^2(G)}{(16\pi^2)^2} \right]$$

The appearance of the Euler number is due to the particular regularization scheme used and can be reabsorbed in the coupling constant.

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