REGULAR N INSTANTON FIELDS
AND SINGULAR GAUGE TRANSFORMATIONS

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A B S T R A C T

The singularities of 't Hooft's N instanton solution obscure its topological properties. It is shown that these singularities may be displaced and in fact removed to infinity by suitable singular gauge transformations. In this new regular representation, the winding number of the field can be defined following the BPST prescription. The example of two instantons is discussed in detail.

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1. - INTRODUCTION ¹⁾

Classical Euclidean gauge fields A_μ corresponding to a finite Euclidean action in $\mathbb{R}_4$ behave at infinity like the gauge transform of the vacuum $A_\mu = 0$, i.e. :

$$A_\mu(x) \underset{x \rightarrow \infty}{\simeq} g^{-1}(x) \partial_\mu g(x) + O\left(\frac{1}{|x|^{1+\epsilon}}\right)$$

It was observed by Belavin, Polyakov, Schwarz and Tyupkin ²⁾ (BPST) that, as a consequence, the gauge fields could be grouped into topologically equivalent classes characterized by a "winding number" c , where c can be any positive or negative integer. The possible physical implications of this BPST observation in conjunction with their proposal that the solutions of the field equations dominate the functional integral, have been explored in a number of subsequent papers ³⁾. The conjectured existence of an infinite number of degenerate vacua in non-Abelian gauge theories suggested by topological considerations has been verified for the explicitly solvable Schwinger model of two-dimensional QED, using operator ⁴⁾ and functional techniques ⁵⁾.

The first explicit construction of solutions with winding number $c > 1$ and cylindrical symmetry was given by Witten ⁶⁾. They correspond to an arbitrary number of "instantons" being aligned along a straight line. A more general solution corresponding to arbitrary "positions" of the instantons in an $SU(2)$ gauge theory was subsequently obtained by 't Hooft ⁷⁾ and generalized by Bitar ⁸⁾ to higher compact Lie groups containing $SU(2)$ as a subgroup. Further enlargement of this class of solutions, using the conformal group was obtained by Jackiw, Nohl and Rebbi ⁹⁾.

't Hooft solutions exhibit singularities (simple poles) at points parametrizing the position of the N instantons. This complicates their topological interpretation. In fact, in this singular representation, these solutions vanish at infinity faster than a pure gauge. The connection between the non-trivial winding number of 't Hooft solution and their behaviour at infinity, an important ingredient in the original BPST derivation is evidently lost. 't Hooft expressions represent, nevertheless, solutions of the homogeneous field equations for all x , including the points of singularities, since a simple pole in A_μ contributes an integrable singularity to the field equations in $\mathbb{R}_4$. This is, however, not true for the action, whose value may, in principle, be ambiguous and has to be considered in the frame of generalized functions (distributions).

One reasonable way of assigning a value to the action (or to the topological number, as we will consider solutions with $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$) would be to find a suitable (singular) gauge transformation which could displace the singularities and - in particular - take them to infinity ¹⁾. The winding number of the resulting regular representation will be unambiguously defined and should exhibit the expected connection with the asymptotic behaviour of A_μ^{reg} keeping the validity of the original BPST derivation.

The purpose of this paper is to contribute to the clarification of the points raised above.

After a brief review of some known facts in Section 2, we begin Section 3 by constructing a new but gauge equivalent singular solution, none of whose singularities coincide with those of 't Hooft solution. We show then that an unambiguous value can be assigned to the action by following the "patching prescription" ¹⁰⁾. We show here, further, that all singularities of the 't Hooft solution can in fact be gauged away to infinity. In Section 4 we illustrate the discussion of Section 3 with the example of two-instanton solutions and conclude in Section 5 with some general remarks. A theorem used in Sections 3 and 4 is proved in the Appendix.

2. - PRELIMINARIES

We start with the ansatz ⁷⁾

$$A_\mu = \frac{1}{e} A_\mu^a \frac{\sigma_a}{2i} = i \tilde{\sigma}_{\mu\nu} a_\nu \quad \tilde{A}_\mu = i \sigma_{\mu\nu} a_\nu \quad (2.1)$$

where $\sigma^a/2$ are the generators of $SU(2)$ in the fundamental representation, $\sigma_{\mu\nu}$, $(\tilde{\sigma}_{\mu\nu})$ are the self (anti-self) dual matrices defined by

$$\begin{aligned} \sigma_{ij} &= \tilde{\sigma}_{ij} = \frac{1}{2} \epsilon_{ijk} \sigma_k \\ \sigma_{i4} &= -\tilde{\sigma}_{i4} = \frac{1}{2} \sigma_i \\ \sigma_{\mu\nu} &= \epsilon_{\mu\nu\lambda\rho} \sigma^{\lambda\rho} \quad \tilde{\sigma}_{\mu\nu} = -\epsilon_{\mu\nu\lambda\rho} \tilde{\sigma}^{\lambda\rho} \end{aligned} \quad (2.2)$$

with $\epsilon_{1234} = +1$ and where a_ν is chosen such as to satisfy the self (anti-self) duality condition

$$F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho} \quad (2.3)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (2.4)$$

As a result of the Bianchi identity $D_\mu \tilde{F}_{\mu\nu} = 0$, where D_μ is the covariant derivative, it follows that solutions of Eq. (2.3) are also solutions of the field equations

$$\partial^\mu F_{\mu\nu} + [A_\mu, F^\mu{}_\nu] = 0 \quad (2.5)$$

With the ansatz 7)

$$a_\mu = \partial_\mu \ell_\mu \rho \quad (2.6)$$

ρ is found to satisfy the equation 7)-9)

$$\frac{\square \rho}{\rho} = 0 \quad (2.7)$$

The 't Hooft solution is given by

$$\rho = 1 + \sum_{i=1}^N \frac{\lambda_i^2}{(x - a_i)^2} \quad (2.8)$$

Solution (2.8) together with (2.6) and (2.1) yields :

$$A_\mu = -2i \tilde{\sigma}_{\mu\nu} \sum_{i=1}^N f_i(x) \frac{y_{i\nu}}{y_i^2} \quad (2.9a)$$

with a corresponding expression for $\tilde{A}_\mu$, where

$$y_{i\mu} = (x - a_i)_\mu \quad (2.9b)$$

and

$$f_i(x) = \frac{\lambda_i^2}{y_i^2} \left(1 + \sum_{j=1}^N \frac{\lambda_j^2}{y_j^2} \right)^{-1} \quad (2.9c)$$

The gauge field A_μ thus exhibits singularities of the order $O((y^2)^{-\frac{1}{2}})$ at $x = a_i$ $i = 1, 2, \dots, N$. These, in turn, imply singularities of the order $O((y_i^2)^{-\frac{3}{2}})$ in the equations of motion which are, however, integrable in $\mathbb{R}_4$. Hence, in the sense of distributions, (2.9) correspond to solutions of the equations of motion (2.5) for all x . The same may not be true for the action density

$$\mathcal{F}_G(x) = -\frac{1}{2} \text{Tr} \left(F_{\mu\nu}(x) F_{\mu\nu}(x) \right)$$

and the Chern density

$$\mathcal{C}(x) = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \text{tr} (F_{\mu\nu}(x) F_{\lambda\rho}(x))$$

which may involve singularities of the order $O((y_i^2)^{-2})$ and hence need to be treated as singular generalized functions in $\mathbb{R}_4$. As is well known, the definition in such cases involves arbitrary parameters, so that the value of the integrals

$$\begin{aligned} A &= \int d^4x \mathcal{F}_G(x) \\ c &= \frac{1}{16\pi^2} \int d^4x \mathcal{C}(x) \end{aligned} \quad (2.12)$$

corresponding to the action and Chern winding number will a priori be ambiguous. However, following the method of Ref. 10) it will nevertheless be possible to assign an unambiguous value to these integrals, provided the singularities of A_μ can all be displaced by a properly chosen singular gauge transformation: that is, provided that there exists a gauge transform A'_μ of A_μ such that each of them is regular in, respectively, two overlapping regions R' and R . The situation may be even better, no such "patching up" procedure will be needed if all of these singularities can be removed to infinity.

We shall examine these questions in the following section.

3. - DISPLACING THE SINGULARITIES

It turns out that all singularities of A^μ in Eq. (2.9) can be removed to infinity by a suitable gauge transformation. We shall discuss this later. We begin now by addressing ourselves to the first question raised at the end of the previous section, i.e., we consider the problem of finding a new singular solution to the field equations (2.5) which is regular in a region R' overlapping with a region R in which the original solution was regular.

A - Local displacement of singularities

In order to construct the new singular solution A'_μ , we observe that one may write

$$-2i \sigma_{\mu\nu} \frac{y_{i\nu}}{y_i^2} = g_i(x) \partial_\mu g_i^{-1}(x) \quad (3.1a)$$

$$-2i \tilde{\sigma}_{\mu\nu} \frac{y_{i\nu}}{y_i^2} = g_i^{-1}(x) \partial_\mu g_i(x) \quad (3.1b)$$

where g_i is the element of the $SU(2)$ gauge group, parametrized as

$$g_i(x) = \frac{y_{i4} + i \vec{\sigma} \cdot \vec{y}_i}{\sqrt{y_i^2}} \quad (3.2)$$

The solution (2.9) thus takes the form

$$A_\mu(x) = \sum_{i=1}^N f_i(x) g_i^{-1}(x) \partial_\mu g_i(x) \quad (3.3)$$

Since

$$f_i(x) \underset{|x| \rightarrow \infty}{\simeq} \lambda_i^2 (x^2)^{-1} \quad (3.4a)$$

$$f_i(x) \underset{y_i \rightarrow 0}{\simeq} 1 - \frac{y_i^2}{\lambda_i^2} \quad (3.4b)$$

It follows that

$$A_\mu \simeq \begin{cases} O(x^{-3}) & |x| \rightarrow \infty \\ g_i^{-1} \partial_\mu g_i(x) & x \rightarrow a_i \end{cases} \quad (3.5a)$$

$$(3.5b)$$

so that A_μ behaves like a pure gauge near its singularities.

This shows that these singularities can be locally displaced by a suitable singular gauge transformation. In fact, for $N=1$ it is evident that the gauge transformation

$$A_\mu^{(N=1)} \longrightarrow A_\mu^{g_i} = g_i A_\mu^{N=1} g_i^{-1} + g_i \partial_\mu g_i^{-1}$$

actually removes the singularity to infinity. One thus finds for A_μ corresponding to one instanton :

$$A_\mu^{g_i}(x) = -2i \sigma_{\mu\nu} \frac{y_{i\nu}}{y_i^2 + \lambda_i^2} \quad (3.6)$$

where the following equality has been used :

$$g_i(x) (\tilde{\sigma}_{\mu\nu} y_{i\nu}) g_i^{-1} = -\sigma_{\mu\nu} y_{i\nu} \quad (3.7)$$

Equation (3.6) is just the regular BPST solution corresponding to winding number $c=1$. Note the interchange of the rôles of $\sigma_{\mu\nu}$ and $\tilde{\sigma}_{\mu\nu}$ in (3.7).

It is now clear from (3.5b) which properties the gauge transformation must satisfy in order to displace all singularities ¹¹⁾.

We may choose, for example :

$$g(x) = \frac{z_4(x) + i \vec{\sigma} \cdot \vec{z}(x)}{\sqrt{z^2}} \quad (3.8a)$$

with

$$z_\mu = \sum_{i=1}^N \gamma_i \frac{y_{i\mu}}{y_i^2} \quad (3.8b)$$

and γ_i some real constants. Unfortunately, such a transformation does not remove these singularities to infinity, the reason being that the corresponding $g_0 g_\mu g^{-1}$ not only has the desired singularities at the points $x=a_i$ but has additional singularities at the zeros of z_μ . Indeed, it is easy to see that

$$g(x) \partial_\mu g^{-1}(x) = -2i \sigma_{\nu\lambda} \frac{z_\lambda}{z^2} S_{\mu\nu} \quad (3.9a)$$

where $S_{\mu\nu}$ is the (symmetric) tensor

$$S_{\mu\nu} = \sum_{i=1}^N \gamma_i \left(\delta_{\mu\nu} - \frac{2 y_{i\mu} y_{i\nu}}{y_i^2} \right) \frac{1}{y_i^2} \quad (3.9b)$$

so that $g \partial_\mu g^{-1}$ will also be singular whenever $z_\mu = 0$.

As we show in the Appendix, there exists in general no judicious choice of the real constants γ_i such that z_μ in Eq. (3.8b) does not vanish for finite x_μ , unless the poles at $x = a_i$ all happen to lie on a straight line. Thus, in general, when this will not be the case, we may as well choose

$$\gamma_i = 1 \quad i = 1, \dots, N \quad (3.10)$$

Then, using

$$g(x) g_i^{-1}(x) = \frac{(z \cdot y_i) + 2i \sigma_{\mu\nu} z_\mu y_{i\nu}}{\sqrt{z^2 y_i^2}} \quad (3.11)$$

and noting that

$$4 \sigma_{\mu\nu} \sigma_{\lambda\rho} = \epsilon_{\mu\nu\lambda\rho} + (\delta_{\mu\lambda} \delta_{\nu\rho} - \delta_{\nu\lambda} \delta_{\mu\rho}) + 2i (\delta_{\mu\lambda} \sigma_{\nu\rho} - \delta_{\nu\lambda} \sigma_{\mu\rho} - \delta_{\mu\rho} \sigma_{\nu\lambda} + \delta_{\nu\rho} \sigma_{\mu\lambda}) \quad (3.12)$$

$$4 \tilde{\sigma}_{\mu\nu} \tilde{\sigma}_{\lambda\rho} = -\epsilon_{\mu\nu\lambda\rho} + (\delta_{\mu\lambda} \delta_{\nu\rho} - \delta_{\nu\lambda} \delta_{\mu\rho}) + 2i (\delta_{\mu\lambda} \sigma_{\nu\rho} - \delta_{\nu\lambda} \sigma_{\mu\rho} - \delta_{\mu\rho} \sigma_{\nu\lambda} + \delta_{\nu\rho} \sigma_{\mu\lambda})$$

one computes

$$g (g_i^{-1} \partial_\mu g_i) g = 2i \left[\frac{2(y_i \cdot z)}{y_i^2} \sigma_{\mu\nu} \frac{z_\nu}{z^2} - \sigma_{\mu\nu} \frac{y_{i\nu}}{y_i^2} - 2 z_\mu \sigma_{\lambda\rho} \frac{y_{i\lambda}}{y_i^2} \frac{z_\rho}{z^2} \right] \quad (3.13)$$

Observe that expression (3.13) is singular only at the points $x = a_i$. It is convenient to write the corresponding gauge transformed field

$$A'_\mu(x) = g(x) A_\mu(x) g^{-1}(x) + g(x) \partial_\mu g^{-1}(x) \quad (3.14)$$

as the sum of a regular (R_μ) and singular (S_μ) part by introducing $h_i(x) = f_i(x) - 1$. As

$$\sum_{i=1}^N g(g_i^{-1} \partial_\mu g_i) g^{-1} = 2i \sigma_{\mu\nu} z_\nu = - \sum_{i=1}^N g_i \partial_\mu g_i^{-1} \quad (3.16)$$

we thus obtain for the gauge transformed field (3.14) $A'_\mu = S_\mu + R_\mu$ where

$$S_\mu = g \partial_\mu g^{-1} - \sum_{i=1}^N g_i \partial_\mu g_i^{-1}$$

$$R_\mu = \sum_{i=1}^N h_i(x) g (g_i^{-1} \partial_\mu g_i) g$$

With the relevant quantities given by Eqs. (3.1), (3.13) and using Eq. (3.9) with the choice (3.10) it is easy to prove that S_μ and R_μ have the following properties :

R_μ is regular everywhere ;

S_μ is regular at the points $x = a_i$ and singular at the points corresponding to $z_\mu = 0$. For $|x| \rightarrow \infty$,

$$A'_\mu \simeq g \partial_\mu g^{-1} \simeq -2i \sigma_{\mu\nu} \frac{x_\nu}{x^2}$$

We observe that if A'_μ were regular, one would obtain from its asymptotic behaviour the winding number $c=1$. Actually its winding number is N , as shown below. Hence, the equation $z_\mu = 0$ necessarily possesses non-trivial solutions.

B - Instantons aligned on a line

If the poles all happen to lie on a straight line - which may be conveniently chosen to be the z_4 axis, then there exists a choice for the constants γ_i in Eq. (3.8) for which all the singularities are removed to infinity. As our discussion in the Appendix shows, this choice is given by

$$\gamma_i = \frac{1}{\prod_{j=1, j \neq i}^N (a_i - a_j)} \quad (3.17)$$

corresponding to the solution of the equations

$$\sum_{i=1}^N \gamma_i a_i^l = 0 \quad l=0, 1, 2 \dots N-2$$

which is unique up to normalization.

In particular, for $N=2$, one may choose $\gamma_1 = -\gamma_2 = 1$. We shall use this fact in Section 4.

C - Removal of singularities to infinity [see also Ref. 1]

We shall now show that the gauge transformation

$$A_\mu \xrightarrow{G} A_\mu^{\text{reg}} = G A_\mu G^{-1} + G \partial_\mu G^{-1}$$

$$G(x) = G_N(x) G_{N-1}(x) \dots G_1(x) \quad (3.18a)$$

with $G_1(x) = g_1(x)$ and the $g_i(x)$ defined recursively by

$$G_i(x) = g_i(x) \prod_{j=1}^{i-1} (a_j)$$

$$\Pi_R(x) = G_1^{-1}(x) G_2^{-1}(x) \dots G_N^{-1}(x) \quad (3.18b)$$

removes all singularities to infinity.

Proof

The transformation G will not introduce any new singularities in A_μ^{reg} . This result is evident from the fact that

$$G(x) \partial_\mu G^{-1}(x) = G_N(x) \partial_\mu G_N^{-1}(x) + \sum_{j=1}^{N-1} G_N(x) \dots G_{j+1}^{(x)} \left(G_j(x) \partial_\mu G_j^{-1}(x) \right) G_{j+1}^{-1}(x) \dots G_N^{-1}(x)$$

Moreover, from

$$G(x) \partial_\mu G^{-1}(x) \underset{\substack{x \rightarrow a_i \\ i \neq N}}{\sim} G_N(a_i) \cdot G_{i+1}(a_i) (G_i(x) \partial_\mu G_i^{-1}(x)) G_{i+1}^{-1}(a_i) \dots G_N^{-1}(a_i) \\ \underset{x \rightarrow a_N}{\sim} G_N(x) \partial_\mu G_N^{-1}(x) \quad (3.19)$$

and the property

$$G_i(x) \partial_\mu G_i^{-1}(x) \underset{x \rightarrow a_i}{\simeq} g_i(x) \partial_\mu g_i^{-1}(x) \quad (3.20)$$

we deduce that

$$A_\mu^{\text{reg}} \underset{x \rightarrow a_i}{\sim} G_N(a_i) \dots G_{i+1}(a_i) B_\mu(x) G_{i+1}^{-1}(a_i) \dots G_N^{-1}(a_i) + \text{finite terms}$$

where

$$B_\mu(x) = G_i(x) \prod_{i=1}^{-1} (g_i^{-1}(a_i) (g_i^{-1}(x) \partial_\mu g_i(x)) \prod_{i=1}^{-1} G_i^{-1}(a_i) + g_i(x) \partial_\mu g_i^{-1}(x)) = 0$$

which proves the assertion.

D - Winding number of A_μ^{reg}

For $|x| \rightarrow \infty$ A_μ^{reg} tends to a gauge transformation of the vacuum

$$A_\mu^{\text{reg}} \simeq G \partial_\mu G^{-1} \quad (3.21)$$

Since A_μ^{reg} is regular everywhere, its associated winding number (2.12) is entirely determined by its topology on a three-dimensional sphere S_R^3 of radius $R \rightarrow \infty$ 2)

$$C = \frac{1}{16\pi^2} \int d^4x \text{Tr}(\bar{F}_{\mu\nu}^{\text{reg}} \tilde{F}_{\mu\nu}^{\text{reg}}) = \frac{i}{16\pi^2} \int_{S_R^3} d\sigma_\mu I_\mu \quad (3.22a)$$

where

$$\begin{aligned} d\sigma_\mu &= x^2 \chi_\mu d\Omega & d\Omega &= \sin^2 \theta \sin \nu d\theta d\nu d\phi \\ 0 \leq \phi &\leq 2\pi & 0 \leq \theta, \nu &\leq \pi \end{aligned} \quad (3.22b)$$

and

$$I_\mu = \epsilon_{\mu\nu\lambda\rho} \text{tr} \left[A_\nu F_{\lambda\rho} - \frac{2}{3} A_\nu A_\lambda A_\rho \right] \quad (3.22c)$$

On the large sphere S_R^3 we have, according to the asymptotic behaviour (3.21)

$$I_\mu \simeq I_\mu^G = -\frac{2}{3} \epsilon_{\mu\nu\lambda\rho} \text{tr} [A_\nu^G A_\lambda^G A_\rho^G] \quad (3.23)$$

where

$$A_\mu^G \equiv G \partial_\mu G^{-1} \quad (3.24)$$

The integral is most conveniently evaluated by observing that the tensor $F_{\mu\nu}^G$ which corresponds to the pure gauge (3.24) vanishes wherever G is regular. Let ΔV be the volume excluding all singularities of A^G at $x = a_i$.

Gauss law then yields

$$0 = \int_{\Delta V} d^4x \text{tr} (F_{\mu\nu}^G \tilde{F}_{\mu\nu}^G) = \int_{S_R^3} d\sigma_\mu I_\mu^G - \sum_{i=1}^N \int_{S_{\epsilon_i}^3} d\sigma_\mu I_\mu^G \quad (3.25)$$

where $\int_{S_{\epsilon_i}^3}$ denotes a small sphere of radius $\epsilon_i \rightarrow 0$ enclosing the singular point a_i . But near the singularities, the behaviour of $G \partial_\mu G^{-1}$ is given by Eq. (3.19) which, when combined with Eq. (3.20) in (3.23) shows that near the singularity one can effectively replace I_μ^G by $I_\mu^{g_i}$. A straightforward computation using Eqs. (3.1) and (3.12) yields, for each singularity

$$\left(I_\mu^{g_i} \right)_{S_{\epsilon_i}^3} = 8 \frac{(\epsilon_i)_\mu}{(\epsilon_i^2)^2} \quad (3.26)$$

From (3.22), (3.24) and (3.25), we have

$$C = \sum_{i=1}^N \int \frac{d\Omega}{2\pi^2} = N \quad (3.27)$$

We observe that the successful removal of all singularities to infinity by the gauge transformation (3.18) justifies implicitly the Jackiw-Nohl-Rebbi prescription ⁹⁾, which states

$$C = \frac{1}{16\pi^2} \int_{\Delta V} d^4x \operatorname{tr} (F_{\mu\nu}^{\text{sing}}(x) \tilde{F}_{\mu\nu}^{\text{sing}})$$

for the problem in question, ΔV denoting the volume excluding the N singular points $x = a_i$ $i = 1, 2, \dots, N$

4. - THE $N = 2$ INSTANTON

It is instructive to illustrate the concepts discussed in the preceding section for the particular case $N = 2$.

A - Patching

Consider the element of the $SU(2)$ gauge group

$$g(z) = \frac{z_4 + i \vec{\sigma} \cdot \vec{z}}{\sqrt{z^2}} \quad (4.1)$$

with

$$z_\mu = \frac{y_{1\mu}}{y_1^2} + \frac{y_{2\mu}}{y_2^2} \quad (4.2)$$

The vector z_μ vanishes at the point

$$\bar{x}_\mu = \frac{1}{2} (a_1 + a_2)_\mu \quad (4.3)$$

Hence, for $N=2$ the singular gauge transformation (3.8) with the constants γ_i given by (3.10) takes the 't Hooft field (2.9) with singularities at $x=a_i$ $i=1,2$ into a new field A'_μ with just one singularity at x_μ given by (4.3). We may thus define two overlapping regions R and R' in which A_μ and A'_μ are regular respectively, excluding the pertinent singular points. Following the Wu and Yang prescription¹⁰⁾ we perform the integration in Eq. (2.12) by performing the gauge transformation (4.1) with z given by (4.2) taking us from one plane to another, whenever a particular integration path approaches a singularity of the corresponding field. The prescription is seen to be equivalent to the Jackiw-Nohl-Rebbi prescription (3.28) of excluding all singularities, since the integrand is invariant under a gauge transformation in the region of overlap.

B - Regular representation

We remark that an additional gauge transformation

$$g' = \frac{\rho_4 - i \vec{\sigma} \cdot \vec{\rho}}{\sqrt{\rho^2}} \quad (4.4)$$

with

$$\rho_\mu = \frac{8}{a^2} \left[y_\mu - \frac{(y \cdot a)}{a^2} a_\mu \right] \quad (4.5)$$

$$y_\mu = x_\mu - \frac{1}{2} (a_1 + a_2)_\mu \quad (4.6)$$

also removes the remaining singularity at $x = \frac{1}{2}(a_1 + a_2)$ of the above new solution A'_μ to infinity. The resulting regular representative now has the asymptotic behaviour

$$A_\mu^{\text{reg}} \simeq (g'g) \partial_\mu (g'g)^{-1} \quad \text{for } |x| \rightarrow \infty$$

which exhibits the winding property $c=2$ of A_μ^{reg} as the product of two gauge transformations, each "carrying" winding number equal to 1.

Alternatively we may gauge away both singularities of the 't Hooft $N=2$ solution in "one go" by noting that two points always lie on a line. For the case of "instantons" aligned on an axis, a general solution was already given in Subsection 3B. For $N=2$, it corresponds to choosing $\gamma_1 = -\gamma_2 = 1$ in Eq. (3.8b). Indeed, the vector

$$z_\mu = \frac{y_{1\mu}}{y_1^2} - \frac{y_{2\mu}}{y_2^2} \quad (4.7)$$

vanishes nowhere, except at infinity, so that the gauge transformation (3.8a) with the choice (4.7) removes both singularities to infinity, we find for the corresponding regular representation after having done the gauge transformation

$$A_\mu^{\text{reg}} \underset{|x| \rightarrow \infty}{\simeq} g(x) \partial_\mu g^{-1}(x) \quad (4.8)$$

with

$$\begin{aligned} g(x) \partial_\mu g^{-1}(x) &= -\frac{2i}{a^2} \sigma_{\mu\lambda} \left(\frac{y_{1\lambda}}{y_1^2} - \frac{y_{2\lambda}}{y_2^2} \right) (y_2^2 - y_1^2) + \frac{4i}{a^2} \left(\frac{y_{1\mu}}{y_1^2} - \frac{y_{2\mu}}{y_2^2} \right) \sigma_{\nu\lambda} y_{2\nu} y_{1\lambda} \\ &\simeq -\frac{4i}{a^2 x^2} \left[(x \cdot a) \sigma_{\mu\lambda} \eta_\lambda - \eta_\mu \sigma_{\nu\lambda} x_\nu \eta_\lambda \right] \end{aligned} \quad (4.9a)$$

with

$$a_\mu = (a_1 - a_2)_\mu \quad (4.9b)$$

and

$$\eta_\mu = 2 x_\mu \frac{(x \cdot a)}{x^2} - a_\mu \quad (4.9c)$$

From Eqs. (4.8) and (4.9) we obtain for the current, e.g., (3.22b)

$$I_\mu^{\text{reg}} \simeq \frac{64}{a^2} \frac{(x \cdot a)^2}{(x^3)} x_\mu \quad N=2 \quad (4.10)$$

corresponding to an angle dependent, radial "hedgehog" field in contrast to the BPST $N=1$, solution whose value was

$$I_{\mu}^{BPST} \sim 8 \frac{x_{\mu}}{(x^2)^2} \quad (4.11)$$

which is uniform over the S_R^3 sphere. The winding number corresponding to (4.10) is easily computed from Eq. (3.22a) to be equal to two.

5. - CONCLUSION

The motivation for the study undertaken in this paper was two-fold. Although 't Hooft's singular N instanton solution satisfies the equations of motion everywhere, the associated action could a priori be ambiguous. Moreover, for these singular representatives, one of the attractive features of the original BPST argument seems to be lost, as there no longer exists a simple connection between the topological properties of the vacuum at infinity and the topology of the Euclidean gauge field on a large three-dimensional sphere. We observed, however, that one would expect to recover such a simple connection, if one could find a non-singular gauge transform of the 't Hooft solution. This would, at the same time eliminate possible ambiguities associated with the original action integral. Although it became clear that all of the singularities of the 't Hooft solutions could be locally displaced by a suitable singular gauge transformation once the solution had been written in the form of Eq. (3.3), the existence of a transformation removing all the singularities to infinity was not evident a priori.

By our systematic construction of such a transformation leading to a solution regular everywhere in $\mathbb{R}_4$ (except for the north pole of the stereographic sphere) the simple connection between the associated Chern number and the topological properties of the vacuum at infinity was found to be restored.

Although the explicit expression for this regular representative is too complicated to be written in a compact way useful for practical calculations involving Feynman path integrals, the existence of a gauge

transformation exhibiting such a simple connection is not without physical interest. Taken together with the BPST proposal that solutions of the classical field equations dominate the Feynman path integral, it suggests that there indeed exists, as expected, an infinite number of topologically inequivalent vacua characterized by a winding number. The existence of such an infinite set of degenerate vacua has been the basis of several papers ³⁾ exploring its consequences.

ACKNOWLEDGEMENTS

The authors are indebted to Professors J.A. Swieca and B. Schroer for many clarifying discussions. One of us (K.D.R.) would also like to thank M. Capdequi and M. Talon for a very useful discussion. Discussions with D. Amati, J. Crewther and S. Sciuto are also acknowledged.

APPENDIX

Theorem

There exists in general no set of real numbers $\{\gamma_i\}$ such that

$$\sum_{i=1}^N \gamma_i \frac{(x-a_i)_-}{(x-a_i)^2} \neq 0 \quad \forall \text{ finite } x, a_i \text{ all distinct} \quad (\text{A.1})$$

unless all points $a_{i\mu}$ lie on a straight line. In the latter case there always exists a unique set of real constants $\{\gamma_i\}$ given by

$$\gamma_i = \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^N (a_i - a_j)} \quad (\text{A.2})$$

where a_i are the positions of the poles on one of the axes.

Proof

It is sufficient to show that there exists in general no solution if all singularities lie in a plane, say the (x,y) plane. Let (x_i, y_i) denote the position of the singularities in the x,y plane. Associate with them the complex vector $a_i = x_i + iy_i$. Setting $z = x + iy$, it is easy to see that the condition (A.1) is equivalent to the condition

$$f(z) \neq 0 \quad \forall z \text{ finite}$$

where $f(z)$ is the meromorphic function

$$f(z) = \sum_{i=1}^N \frac{1}{z - a_i}$$

Now

$$\frac{1}{2\pi i} \oint dz \frac{f'(z)}{f(z)} = n - N$$

where n is the number of zeros of $f(z)$ and the closed integration contour is conveniently taken to be a circle of radius $R \rightarrow \infty$. Assuming

$$f(z) \sim C z^{-p} \quad |z| \rightarrow \infty$$

one obtains $n - N = \rho$. Hence, for $f(z)$ to have no zeros, one must have $\rho = N$ which leads to the conditions

$$\sum_{i=1}^N \gamma_i a_i^l = 0 \quad l=0, 1, 2, \dots, N-2$$

Since this is a homogeneous set of equations, we may choose one of the γ_i at will, which is equivalent to supplementing the above equations with the additional equation

$$\sum_{i=1}^N \gamma_i a_i^{N-1} = 1$$

The existence of a non-trivial solution requires that the corresponding (Vandermonde) determinant is non-zero

$$\mathcal{D} = \prod_{j=1}^{N-1} \prod_{i=j+1}^N (a_i - a_j) \neq 0$$

This is evidently the case if $a_i \neq a_j$, $i \neq j$. The corresponding solution for the coefficients γ_i is given by (A.2) with a_i now complex numbers. Hence the γ_i 's will in general be real only provided all poles lie on a straight line, conveniently chosen to be the x or y axis.

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