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NON-LINEAR REALIZATION OF SUPERSYMMETRY IN DE SITTER SPACE

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ABSTRACT

We derive the non-linear transformation law and the non-linear Lagrangian for a Goldstone spinor corresponding to spontaneous breaking of global supersymmetry in a de Sitter space with $O(3,2)$ invariance.

With a suitable choice of the Goldstone spinor field the Lagrangian agrees with the form suggested by the coupling to supergravity. The construction is also valid for the case of extended supersymmetry.

1. INTRODUCTION

Recent developments in supergravity theory [1,2] based on Lagrangians with a cosmological term [3,6] have revived interest in the study of global supersymmetry in de Sitter space. Deser and the author have considered the interaction of the supergravity fields with a matter system with spontaneously broken supersymmetry [7]. The latter was described in terms of a Goldstone spinor in de Sitter space. Approximate formulas for its transformation properties and for its non-linear Lagrangian were given. The kinetic part of the Lagrangian had the form

$$\mathcal{L}_{kin} = -\frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda - i m \bar{\lambda} \lambda, \quad (1.1)$$

where λ is the Majorana spinor describing the Goldstone fermion and $m = r^{-1}$ is the reciprocal of the radius of de Sitter space. In the present paper we give the theory of non-linear realizations of global supersymmetry in de Sitter space^{*)} and a procedure for the construction of the complete non-linear Lagrangian. Our results generalize to de Sitter space the work of Volkov and Akulov [11] in Minkowski space. The peculiar mass term in (1.1), at first surprising for a Goldstone field, arises naturally in our construction [see (5.10) below].

The non-linear realizations in de Sitter space can be studied by the general method developed by Callan, Coleman, Wess and the author [12] and by Volkov [13]. A short review of this method, with some improvements, is given in Section 2. In Section 3 we give the Lie algebra structure of $OSp(1,4)$. In Section 4 we apply the method of Section 2 to study the non-linear realizations and in Section 5 we construct the non-linear Lagrangian. Finally, in the Appendix we show how one works with exponentials and describe some of the calculations in more detail. The method of Refs. [12,13] has already been applied by Keck [14] to the case of $OSp(1,4)$. Keck has constructed the corresponding superspace and has studied superfields and linear supermultiplets. Our Section 4 follows essentially his treatment of superspace.

The method of the present paper can be applied without any difficulty to the extended global supersymmetry $OSp(N,4)$ which has the maximal Lie superalgebra

^{*)} This is the graded Lie algebra $OSp(1,4)$. See Refs. [8-10].

$Sp(4) \oplus O(N)$. The extended results can be read off the formulas of Sections 4 and 5 simply by interpreting L to be not the Lorentz algebra but the sum of the Lorentz algebra plus that of $O(N)$.

2. NON-LINEAR REALIZATIONS AND LAGRANGIANS

Following Refs. [12,13] we consider a Lie group (or supergroup) G and a subgroup H . Let us call T_m the generators of H . We assume^{*)} that the remaining generators S_i can be chosen so that they form a representation of H . In other words the commutator $[T_m, S_i]$ should be a linear combination of the S_i alone. A group element g of G can be represented (uniquely) in the form

$$g = e^{\xi \cdot S} h \quad , \quad (2.1)$$

where h is an element of H , ξ^i are parameters and $\xi \cdot S = \xi^i S_i$ (summation convention). The ξ^i parametrize the coset space G/H ; this exponential parametrization is convenient, but any other can be used. We do not specify here the parametrization of h . We define the effect of a group element g_0 on the coset space by ($h' \in H$)

$$g_0 g = e^{\xi' \cdot S} h' \quad (2.2)$$

or

$$g_0 e^{\xi \cdot S} = e^{\xi' \cdot S} h_1 \quad (2.3)$$

$$h_1 = h' h^{-1} \quad . \quad (2.4)$$

Clearly ξ' and h_1 are functions of g_0 and ξ .

If $g_0 - 1$ is infinitesimal, (2.3) implies

$$e^{-\xi \cdot S} (g_0 - 1) e^{\xi \cdot S} - e^{-\xi \cdot S} \delta e^{\xi \cdot S} = h_1 - 1 \quad . \quad (2.5)$$

*) We proceed in a formal manner in order that our considerations be as general as possible. For instance, we do not try to establish under which conditions the above property can be derived.

The right-hand side of (2.5) is a generator of H .

Let us consider first the case in which $g_0 = h_0 \in H$. Then (2.3) gives

$$h_0 e^{\xi \cdot S} h_0^{-1} h_0 = e^{\xi' \cdot S} h_1. \quad (2.6)$$

Since the S^i form a representation of H , this implies

$$e^{\xi' \cdot S} = h_0 e^{\xi \cdot S} h_0^{-1} \quad (2.7)$$

and

$$h_1 = h_0, \quad h' = h_0 h. \quad (2.8)$$

The transformation from ξ to ξ' given by (2.7) is linear. On the other hand, consider now

$$g_0 = e^{\xi_0 \cdot S}. \quad (2.9)$$

In this case (2.3) becomes

$$e^{\xi_0 \cdot S} e^{\xi \cdot S} = e^{\xi' \cdot S} h_1. \quad (2.10)$$

This is a non-linear inhomogeneous transformation on ξ . The infinitesimal form (2.5) becomes

$$e^{-\xi \cdot S} \xi_0 \cdot S e^{\xi \cdot S} - e^{-\xi \cdot S} \delta e^{\xi \cdot S} = h_1 - 1. \quad (2.11)$$

The left-hand side of this equation can be evaluated, using the algebra of the group. Since the result must be a generator of H , one must set equal to zero the coefficient of S_i . In this way one finds an equation from which $\delta \xi^i$ can be calculated.

In many cases of physical interest the algebra admits the automorphism $S_i \rightarrow -S_i, T_m \rightarrow T_m$. Then (2.10) implies

$$e^{-\xi_0 \cdot S} e^{-\xi \cdot S} = e^{-\xi' \cdot S} h_1. \quad (2.12)$$

One can eliminate h_1 from (2.10) and (2.12) and obtain

$$e^{\mathbb{F}_0 \cdot S} e^{2\mathbb{F} \cdot S} e^{\mathbb{F}_0 \cdot S} = e^{2\mathbb{F}' \cdot S} \quad (2.13)$$

We now consider the Lie algebra valued differential form

$$\omega = e^{-\mathbb{F} \cdot S} d e^{\mathbb{F} \cdot S} = \mu \cdot S + \nu \cdot T \quad (2.14)$$

The transformation laws for the forms $\mu^i(\xi, d\xi)$ and $\nu^m(\xi, d\xi)$ are easily obtained. First observe that, if $g_0 = h_0 \in H$,

$$\omega' \equiv e^{-\mathbb{F}' \cdot S} d e^{\mathbb{F}' \cdot S} = h_0 \omega h_0^{-1} \quad (2.15)$$

The forms μ^i and ν^m transform linearly under H . On the other hand, if g_0 has the form (2.9), one finds, using (2.10),

$$\omega' = h_1 \omega (h_1)^{-1} + h_1 d(h_1)^{-1} \quad (2.16)$$

or

$$\mu' \cdot S = h_1 \mu \cdot S (h_1)^{-1} \quad (2.17)$$

$$\nu' \cdot T = h_1 \nu \cdot T (h_1)^{-1} + h_1 d(h_1)^{-1} \quad (2.18)$$

Equation (2.17) shows that the differential forms $\mu^i(\xi, d\xi)$ are transformed linearly by a group element of the form (2.9). The transformation law is the same as by an element of H , except that now this group element $h_1(\xi_0, \xi)$ is a function of the variable ξ . Therefore any expression constructed with $\mu^i(\xi, d\xi)$ which is invariant under the subgroup H will be automatically invariant under the entire group G , the elements of H operating linearly on ξ , the remaining elements non-linearly. Such an invariant expression is a candidate for a non-linear Lagrangian.

3. SUPERSYMMETRY IN DE SITTER SPACE

De Sitter space can be defined as the four-dimensional manifold

$$y_A y^A = -(y^0)^2 + \vec{y}^2 - (y^4)^2 = -r^2 \quad , \quad (3.1)$$

embedded in a five-dimensional pseudo-Euclidean space with metric $\eta_{AB} = (-1, 1, 1, 1, -1)$. Here $A = (a, 4)$ and $a = 0, 1, 2, 3$. With this signature the group of motions of the space is $SO(3, 2)$ which has a spinor representation isomorphic to $Sp(4)$. The reason for choosing this signature, rather than that corresponding to $SO(4, 1)$, is that $SO(3, 2)$ admits four-component Majorana spinors and can be extended to a graded algebra, called $OSp(1, 4)$, which has the same number of generators as the usual supersymmetry algebra, which can be obtained from it by contraction.

The generators $M_{AB} = -M_{BA}$ of $SO(3, 2)$ satisfy the commutation relation

$$-i [M_{AB}, M_{CD}] = \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD} \quad (3.2)$$

We shall sometimes use the notation $m M_{a4} = P_a$, where $m = r^{-1}$. The transformations generated by P_a , which are called pseudo-translations, give rise to ordinary translations when the group $SO(3, 2)$ is contracted to the Poincaré group ($m \rightarrow 0$). They satisfy

$$[P_a, P_b] = -i m^2 M_{ab} \quad (3.3)$$

$$-i [M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a \quad , \quad (3.4)$$

while the generators M_{ab} satisfy the usual algebra of the Lorentz group.

Let γ_a be the usual gamma matrices in the Majorana representation. They are real four by four matrices which satisfy

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} \quad (3.5)$$

The five matrices $\Gamma_A \equiv (\gamma_a \gamma_5, \gamma_5)$ satisfy

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB} \quad (3.6)$$

[since $(\gamma_5)^2 = -1$, we must choose $\eta_{44} = -1$ if we want a real four-dimensional representation]. The matrices

$$\Sigma_{AB} = \frac{1}{4} [\Gamma_A, \Gamma_B] \quad (3.7)$$

or

$$\Sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b], \quad \Sigma_{4a} = \frac{1}{2} \gamma_a, \quad (3.8)$$

provide a representation $M_{AB} = -i \Sigma_{AB}$ of (3.2). The Γ_A transform like an $SO(3,2)$ vector.

The algebra (3.2) can be extended to a graded Lie algebra by adjoining spinor generators Q_α which are four-component Majorana spinors satisfying $(\bar{Q} \equiv Q^T \gamma^0)$

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= -2m (\Sigma^{AB})_{\alpha\beta} M_{AB} \\ &= -2 (\gamma^a)_{\alpha\beta} P_a - 2m (\Sigma^{ab})_{\alpha\beta} M_{ab} \end{aligned} \quad (3.9)$$

Their spinorial property is described by the commutation relation

$$-i [M_{AB}, Q_\alpha] = (\Sigma_{AB} Q)_\alpha \quad (3.10)$$

or

$$-i[M_{ab}, Q_\alpha] = (\Sigma_{ab} Q)_\alpha \quad (3.11)$$

$$-i[P_a, Q_\alpha] = -\frac{m}{2} (\gamma_a Q)_\alpha \quad (3.12)$$

As usual, (2.9) can be written as a commutation relation by using anticommuting Majorana spinor parameters α_1 and α_2 (they anticommute also with Q_α , but commute with M_{AB})

$$\begin{aligned} [\bar{\alpha}_1 Q, \bar{\alpha}_2 Q] &= -2m \bar{\alpha}_1 \Sigma^{AB} \alpha_2 M_{AB} \\ &= -2\bar{\alpha}_1 \gamma^a \alpha_2 P_a - 2m \bar{\alpha}_1 \Sigma^{ab} \alpha_2 M_{ab} \end{aligned} \quad (3.13)$$

As $m \rightarrow 0$, the graded Lie algebra given by (3.2), (3.3), (3.4), (3.9), (3.11) and (3.12) contracts to the usual supersymmetry algebra.

4. GOLDSTONE FERMIONS IN DE SITTER SPACE

We now apply the general theory of Section 2 to find the non-linear realization and the invariant Lagrangian for a Goldstone fermion in de Sitter space. We take as G the graded Lie algebra of Section 3 having as generators Q_α , P_a and M_{ab} . It has as subalgebra H that of the de Sitter group $SO(3,2)$ with generators P_a and M_{ab} . This in turn has as subalgebra the Lorentz algebra L with generators M_{ab} . An element of G can be represented uniquely in the form

$$g = e^{\bar{\theta} Q} h = e^{\bar{\theta} Q} e^{-ix \cdot P} \ell \quad , \quad (4.1)$$

where h is an element of H and ℓ an element of L . In analogy with (2.2), we write

$$g \circ g = e^{\bar{\theta}' Q} h' = e^{\bar{\theta}' Q} e^{-ix' \cdot P} \ell' \quad (4.2)$$

or

$$g_0 e^{\bar{\theta} Q} = e^{\bar{\theta}' Q} h_1 \quad (4.3)$$

$$h_1 e^{-ix \cdot P} = e^{-ix' \cdot P} l_1 \quad (4.4)$$

and

$$l_1 l = l' \quad (4.5)$$

Clearly $h_1(g_0, \theta)$ and $l_1(g_0, \theta, x)$ are functions of the variables indicated. The infinitesimal forms of (4.3) and (4.4) are respectively

$$e^{-\bar{\theta} Q} (g_0 - 1) e^{\bar{\theta} Q} - e^{-\bar{\theta} Q} \delta e^{\bar{\theta} Q} = h_1 - 1 \quad (4.6)$$

and

$$e^{ix \cdot P} (h_1 - 1) e^{-ix \cdot P} - e^{ix \cdot P} \delta e^{-ix \cdot P} = l_1 - 1 \quad (4.7)$$

We consider three cases. If $g_0 = l_0 \in L$ is a Lorentz transformation, (4.3) and (4.4) give

$$e^{\bar{\theta}' Q} = l_0 e^{\bar{\theta} Q} l_0^{-1} \quad (4.8)$$

$$h_1 = l_1 = l_0 \quad (4.9)$$

and

$$e^{-ix' \cdot P} = l_0 e^{-ix \cdot P} l_0^{-1} \quad (4.10)$$

Both θ and x transform linearly. If on the other hand we know only that $g_0 = h_0 \in H$, in particular if

$$g_0 = e^{-i s \cdot P} \quad (4.11)$$

is a pseudo-translation of parameters s^a , (4.3) gives

$$e^{\bar{\theta}' Q} = h_0 e^{\bar{\theta} Q} h_0^{-1} \quad (4.12)$$

$$h_1 = h_0, \quad (4.13)$$

while (4.4) gives

$$h_0 e^{-i x \cdot P} = e^{-i x' \cdot P} l_1(h_0, x) \quad (4.14)$$

In this case θ transforms linearly, but the transformation law (4.14) of x under pseudo-translations is inhomogeneous and non-linear. Infinitesimally

$$e^{i x \cdot P} (-i s \cdot P) e^{-i x \cdot P} - e^{i x \cdot P} \delta e^{-i x \cdot P} = l_1 - 1 \quad (4.15)$$

Finally, if

$$g_0 = e^{\bar{\alpha} Q} \quad (4.16)$$

is a supersymmetry transformation, one must use (4.3) and (4.4) as they stand. Observe, however, that (4.4) has the same form as (4.14) except for the fact that h_1 is a function of θ while h_0 is not. Therefore, the transformation law of x under a supersymmetry transformation has the same form as that under a de Sitter transformation but with parameters which depend in a well-defined way on θ . The algebra admits the automorphism $Q_\alpha \rightarrow -Q_\alpha$, $P_a \rightarrow P_a$, $M_{ab} \rightarrow M_{ab}$. Therefore the transformation law

$$e^{\bar{\alpha} Q} e^{\bar{\theta} Q} = e^{\bar{\theta}' Q} h_1 \quad (4.17)$$

implies

$$e^{-\bar{\alpha} Q} e^{-\bar{\theta} Q} = e^{-\bar{\theta}' Q} h_1 \quad (4.18)$$

One can eliminate h_1 between these two equations and obtain

$$e^{2Q} e^{2\bar{\theta}Q} e^{2Q} = e^{2\bar{\theta}'Q} \quad (4.19)$$

The transformation law of x under a pseudo-translation

$$e^{-is.P} e^{-ix.P} = e^{-ix'.P} \ell_1 \quad (4.20)$$

refers purely to the algebra of H , which admits the automorphism $P_a \rightarrow -P_a$, $M_{ab} \rightarrow M_{ab}$. Therefore (4.20) implies

$$e^{-is.P} e^{-2ix.P} e^{-is.P} = e^{-2ix'.P} \quad (4.21)$$

The parameters x^a can be taken as curvilinear co-ordinates in de Sitter space. Their connection with the co-ordinates y^a can be easily found if one observes that the point $x^a = 0$ goes over into the point $x^a = s^a$ under a pseudo-translation, while a simple geometrical argument shows that the effect of the same pseudo-translation is to transform the point $y^a = 0, y^4 = r$ into the point $y^a = r(s^a/s) \sinh s/r, y^4 = r \cosh s/r$, where $s = (s_a s^a)^{1/2}$. Therefore

$$\begin{aligned} y^a &= r (x^a/x) \sinh x/r \\ y^4 &= r \cosh x/r \end{aligned} \quad (4.22)$$

Observe that the functions on the right-hand side of (4.22) are even in the argument x/r . Therefore, only $x^2 = x_a x^a$ enters and not its square root. If one wishes an explicit formula for the transformation of x^a under a pseudo-translation, one can make use of (4.22), and its inverse, since the transformation formulas for y^a are immediate. Alternatively, one can make use of (4.15) (see Appendix). The result is

$$\delta x^a = s^a + \left[\frac{(m^2 x^2)^{1/2} \cosh(m^2 x^2)^{1/2}}{\sinh(m^2 x^2)^{1/2}} - 1 \right] \left(s^a - \frac{x \cdot s x^a}{x^2} \right). \quad (4.23)$$

The transformation of x^a under an infinitesimal Lorentz transformation $\ell_0 = 1 + \frac{i}{2} t^{ab} M_{ab}$ is of course simply

$$\delta x^a = t^{ab} x_b \quad . \quad (4.24)$$

Using (4.6) with $g_{0-1} = \bar{\alpha}Q$ one finds ^{*}) (see Appendix)

$$\delta \theta = \alpha - \frac{im}{6} (5 \bar{\theta} \theta + \bar{\theta} \Gamma_A \theta \Gamma^A) \alpha + \frac{m^2}{9} (\bar{\theta} \theta)^2 \alpha \quad (4.25)$$

and

$$\begin{aligned} h_{1-1} &= -m \left(1 + \frac{im}{6} \bar{\theta} \theta \right) \bar{\alpha} \Sigma^{AB} \theta M_{AB} \\ &= \left(1 + \frac{im}{6} \bar{\theta} \theta \right) (\bar{\alpha} \gamma^a \theta P_a + m \bar{\alpha} \Sigma^{ab} \theta M_{ab}) \quad . \end{aligned} \quad (4.26)$$

The corresponding transformation of x^a is given by (4.7). As explained above, it is a transformation of H but with θ dependent parameters, given in (4.26). So we must use (4.23) plus (4.24) with

$$S^a = -i \left(1 + \frac{im}{6} \bar{\theta} \theta \right) \bar{\alpha} \gamma^a \theta \quad (4.27)$$

and

$$t^{ab} = -2im \left(1 + \frac{im}{6} \bar{\theta} \theta \right) \bar{\alpha} \Sigma^{ab} \theta \quad . \quad (4.28)$$

If one interprets θ as a field $\theta(x)$, its transformation law is

$$\Delta \theta = \delta \theta(x) - \delta x^a \partial_a \theta(x) \quad . \quad (4.29)$$

^{*}) Formulas (4.25) and (4.26) appear to agree with those given previously by Keck [14].

5. NON-LINEAR LAGRANGIAN

According to the general theory of Section 2, we introduce the Lie algebra valued differential form

$$\begin{aligned}\omega &= e^{ix \cdot P} e^{-\bar{\theta} Q} d(e^{\bar{\theta} Q} e^{-ix \cdot P}) \\ &= \bar{\tau} Q - i\mu \cdot P + \frac{i}{2} v^{ab} M_{ab}\end{aligned}\tag{5.1}$$

where τ_α , μ^a and v^{ab} are forms which are functions of x^a , θ_α , dx^a and $d\theta_\alpha$. Using (4.3) and (4.4) we find

$$\omega' = l_1 \omega (l_1)^{-1} + l_1 d(l_1)^{-1}\tag{5.2}$$

so that, in particular

$$\mu' \cdot P = l_1 \mu \cdot P (l_1)^{-1}.\tag{5.3}$$

Under a supersymmetry transformation of parameter α (or under a pseudo-translation of parameters s^a) the forms $\mu^a(x, dx, \theta, d\theta)$ transform like a Lorentz vector, but the Lorentz transformation l_1 is a function of α , θ and x (of s^a , θ and x) determined by the group structure.

At any rate the product

$$\mu^0 \mu^1 \mu^2 \mu^3 = -2a^2 \mathcal{L} dx^0 dx^1 dx^2 dx^3\tag{5.4}$$

is Lorentz invariant and therefore invariant under the entire group. Apart from a scaling factor it gives the non-linear Lagrangian for the Goldstone fermion. The evaluation of the forms μ^a is not difficult.

By the methods described in the Appendix one can easily extract the P_a component of (5.1). In this way one obtains

$$\begin{aligned} \mu^a = & dx^a + \left(\frac{\sinh(m^2 x^2)^{1/2}}{(m^2 x^2)^{1/2}} - 1 \right) \left(dx^a - \frac{x \cdot dx}{x^2} x^a \right) \\ & + i \left(1 - \frac{im}{6} \bar{\theta} \theta \right) \bar{\theta} \gamma^b d\theta \left[\delta_b^a + (\cosh(m^2 x^2)^{1/2} - 1) \left(\delta_b^a - \frac{x_b x^a}{x^2} \right) \right] \\ & + 2im \left(1 - \frac{im}{6} \bar{\theta} \theta \right) \bar{\theta} \Sigma^{ba} d\theta x_b \frac{\sinh(m^2 x^2)^{1/2}}{(m^2 x^2)^{1/2}} \end{aligned} \quad (5.5)$$

which, together with (5.4), gives the complete Lagrangian. To exhibit the first few terms we write

$$\begin{aligned} \mu^a = & dx^a + i \bar{\theta} \gamma^a d\theta + 2im \bar{\theta} \Sigma^{ba} d\theta x_b + \dots \\ = & dx^c \left(\delta_c^a + i \bar{\theta} \gamma^a \partial_c \theta + 2im \bar{\theta} \Sigma^{ba} x_b \partial_c \theta + \dots \right), \end{aligned} \quad (5.6)$$

where the dots denote terms with higher powers of θ or x . Therefore

$$\mathcal{L} = -\frac{1}{2a^2} \det \left(\delta_c^a + i \bar{\theta} \gamma^a \partial_c \theta + 2im \bar{\theta} \Sigma^{ba} x_b \partial_c \theta + \dots \right) \quad (5.7)$$

Introducing the correctly normalized Goldstone fermion field $\psi = a^{-1} \theta$, we have finally

$$\mathcal{L} = -\frac{1}{2a^2} - \frac{i}{2} \bar{\psi} \gamma \cdot \partial \psi - im \bar{\psi} \Sigma^{ab} x_a \partial_b \psi + \dots \quad (5.8)$$

Except for the (negative) constant $-1/2a^2$, this kinetic term has a structure similar to that suggested by Dirac [15] for a Lagrangian in de Sitter space. It can

be given a more familiar form by the transformation^{*)}

$$\lambda = \left(1 - \frac{m}{2} \gamma \cdot x\right) \psi \quad . \quad (5.9)$$

One finds

$$\mathcal{L} = -\frac{1}{2\alpha^2} - \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda - im \bar{\lambda} \lambda + \dots \quad (5.10)$$

which is the form used in Ref. [7]. The origin of the peculiar mass term is thereby explained.

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*) Transformations similar to (5.9) have been used by Gürsey and Lee [16] in their study of the Dirac equation in de Sitter space.

APPENDIX

For any two quantities X and Y we define

$$\begin{aligned} X \wedge Y &= [X, Y] \\ X^2 \wedge Y &= [X [X, Y]] \end{aligned} \quad (\text{A.1})$$

etc.

The expression $f(X) \wedge Y$ is defined as a series of multiple commutators, obtained by expanding the function $f(X)$ as a power series in X . One can easily see that

$$g(X) \wedge [f(X) \wedge Y] = [g(X)f(X)] \wedge Y. \quad (\text{A.2})$$

As a consequence, the equation

$$f(X) \wedge Y = Z \quad (\text{A.3})$$

can be solved for Y in the form

$$Y = [f(X)]^{-1} \wedge Z. \quad (\text{A.4})$$

In this notation the formulas

$$e^X Y e^{-X} = e^X \wedge Y \quad (\text{A.5})$$

and

$$e^X \delta e^{-X} = \frac{1 - e^X}{X} \wedge \delta X \quad (\text{A.6})$$

apply, where δ is any variation.

We now use these results to perform the explicit evaluation of some expressions given in the text. For instance, (4.15) can be written

$$e^{ix.P} \wedge (-i s.P) - \frac{1 - e^{ix.P}}{ix.P} \wedge (i \delta x.P) = t_1 - 1. \quad (\text{A.7})$$

Since this is a Lorentz generator we must evaluate the pseudo-translation component of the left-hand side and set it equal to zero. Only commutators of even order contribute to it. Therefore we must take the even part (even powers of $ix \cdot P$) of the functions occurring in (A.7). This gives

$$\cosh ix \cdot P \wedge (-is \cdot P) + \frac{\sinh ix \cdot P}{ix \cdot P} \wedge (i\delta x \cdot P) = 0 \quad (A.8)$$

which can be solved as

$$\begin{aligned} \delta x \cdot P &= \frac{ix \cdot P \cosh ix \cdot P}{\sinh ix \cdot P} \wedge s \cdot P \\ &= s \cdot P + \left(\frac{ix \cdot P \cosh ix \cdot P}{\sinh ix \cdot P} - 1 \right) \wedge s \cdot P \end{aligned} \quad (A.9)$$

The commutation relations (3.3) and (3.4) give immediately

$$(ix \cdot P)^{2n} \wedge s \cdot P = (m^2 x^2)^n \left(s \cdot P - \frac{x \cdot s \cdot x \cdot P}{x^2} \right), \quad n > 0. \quad (A.10)$$

Therefore (A.9) becomes

$$\delta x \cdot P = s \cdot P + \left(\frac{(m^2 x^2)^{1/2} \cosh (m^2 x^2)^{1/2}}{\sinh (m^2 x^2)^{1/2}} - 1 \right) \left(s \cdot P - \frac{x \cdot s \cdot x \cdot P}{x^2} \right) \quad (A.11)$$

which is identical with (4.23).

In a similar manner we can make use of (4.6), with $g_0 - 1 = \bar{\alpha} Q$. It becomes

$$e^{-\bar{\theta} Q} \wedge \bar{\alpha} Q - \frac{1 - e^{-\bar{\theta} Q}}{\bar{\theta} Q} \wedge \delta \bar{\theta} Q = h_1 - 1 \quad (A.12)$$

Here there is a simplification: any power of θ_α higher than the fourth vanishes identically due to the anticommuting property of the spinor components. Therefore we need only

$$\bar{\theta}Q \wedge \bar{\alpha}Q = -2m \bar{\theta} \Sigma^{AB} \alpha M_{AB} \quad , \quad (A.13)$$

$$(\bar{\theta}Q)^2 \wedge \bar{\alpha}Q = -\frac{5}{2}im \bar{\theta}\theta \bar{\alpha}Q - \frac{im}{2} \bar{\theta} \Gamma_A \theta \bar{\alpha} \Gamma^A Q \quad (A.14)$$

$$(\bar{\theta}Q)^3 \wedge \bar{\alpha}Q = 4im^2 \bar{\theta}\theta \bar{\theta} \Sigma^{AB} \alpha M_{AB} \quad (A.15)$$

and

$$(\bar{\theta}Q)^4 \wedge \bar{\alpha}Q = -5m^2 (\bar{\theta}\theta)^2 \bar{\alpha}Q \quad . \quad (A.16)$$

If one sets equal zero in the left-hand side of (A.12) the part with the even powers of $\bar{\theta}Q$, one finds

$$\begin{aligned} \delta \bar{\theta}Q &= \bar{\theta}Q \frac{\cosh \bar{\theta}Q}{\sinh \bar{\theta}Q} \wedge \bar{\alpha}Q \\ &= \left[1 + \frac{1}{3} (\bar{\theta}Q)^2 - \frac{1}{45} (\bar{\theta}Q)^4 \right] \wedge \bar{\alpha}Q \quad . \end{aligned} \quad (A.17)$$

On the other hand, using (A.17), the part with the odd powers gives

$$\begin{aligned} h_1 - 1 &= \frac{1 - \cosh \bar{\theta}Q}{\sinh \bar{\theta}Q} \wedge \bar{\alpha}Q \\ &= \left[-\frac{1}{2} \bar{\theta}Q + \frac{1}{24} (\bar{\theta}Q)^3 \right] \wedge \bar{\alpha}Q \quad . \end{aligned} \quad (A.18)$$

If one now makes use of (A.13) to (A.15) one obtains the results given in (4.25) and (4.26).

The differential forms (5.5), which enter in the construction of the Lagrangian, are evaluated by exactly the same methods.

REFERENCES

- [1] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D13 (1976) 3214.
- [2] S. Deser and B. Zumino, Phys. Letters 62B (1976) 335.
- [3] D.Z. Freedman and A. Das, Stony Brook preprint ITB-SB-76-64.
- [4] S.W. MacDowell and F. Mansouri, Phys. Rev. Letters 38 (1977) 739.
- [5] P.K. Townsend, Stony Brook preprint ITP-SB-77-2.
- [6] P.K. Townsend and P. van Nieuwenhuizen, Stony Brook preprint ITP-SB-77-18.
- [7] S. Deser and B. Zumino, Brandeis University preprint (1977), submitted to Phys. Rev. Letters.
- [8] P.G. Freund and I. Kaplansky, Journ. Math. Phys. 17 (1976) 228.
- [9] W. Nahm, M. Scheunert and V. Rittenberg, Phys. Letters B61 (1976) 383; Journ. Math. Phys. 17 (1976) 1626, 1640; 18 (1977) 146, 155.
- [10] V.G. Kac, Functional Anal. Appl. 9 (1975) 91 (Russian).
- [11] D.V. Volkov and V.P. Akulov, Phys. Letters 46B (1973) 109.
- [12] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239; C.G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.
- [13] D.V. Volkov, Soviet Journal of Particles and Nuclei 4 (1973) 3 (Russian).
- [14] B.W. Keck, Journ. Phys. A8 (1975), 1819.
- [15] P.A.M. Dirac, Annals of Math. 36 (1935) 657.
- [16] F. Gürsey and T.D. Lee, Proc. Nat. Acad. 49 (1963) 179.