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LARGE TRANSVERSE MOMENTA FROM STATISTICAL BOOTSTRAP WITH SPIN

R. Hagedorn

CERN - Geneva

and

U. Wambach *)

University of Dortmund, Dortmund

ABSTRACT

We show that statistical bootstrap models (SBM) with their prediction of exponentially decreasing p distributions (equivalently: constant maximal temperature $\tau_0 \approx m_\pi$) are not ruled out by the large $p_\perp$ observed above ≈ 100 GeV lab energy in hadron collisions. Usual SBM sum over all spin angular momenta of the decaying fireball; if a particular spin is fixed, SBM leads to large $p_\perp$. We discuss in this paper (after having justified a classical approach) several mechanisms yielding high-mass high-spin fireballs, select arbitrarily one of these (single fireball excitation) and calculate in SBM with fixed spin the decay structure. The cross-section $d\sigma/dM$ is taken to be an empirical function, fitted to existing $p_\perp$ data at one energy and then calculable at other energies. We can easily reproduce the $p_\perp$ distribution. Further results concern multiplicities rising with $p_\perp$, anisotropy of the single particle inclusive distribution and various correlations. Unexpectedly, but a posteriori rather naturally a jet-like structure is found in the spin-orthogonal plane.

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*) Present address: Kirschgartenstr. 11, D-6501 Essenheim.

1. INTRODUCTION

The large transverse momenta observed in hadron collisions with incident momenta above 100 GeV/c ¹⁾ seem to invalidate the conclusions drawn from statistical bootstrap models (SBM), namely that for fireballs of large mass the momentum distribution of the decay products is very nearly of the Boltzmann type and that rigorous four-momentum conservation leads, at large $p_{\perp}$, to deviations below the Boltzmann curve ²⁾⁻⁴⁾ instead of above as found experimentally. On the other hand, even at the highest available c.m.s. energies ($\sqrt{s}$), the distribution up to $p_{\perp} \approx 1.2$ GeV/c is of Boltzmann type; the corresponding temperature is independent of the collision energy and of the order of the limiting temperature $T_0 \approx 160$ MeV predicted by SBM. More than 99% of the produced particles follow this pattern and less than 1% falls into the region where the distribution deviates in an s dependent way from the Boltzmann curve. If in the small $p_{\perp}$ region ($\lesssim 1.2$ GeV/c) the distribution would be very different from Boltzmann and/or the slope $1/T$ would be s dependent and even decrease, SBM would be ruled out; as this is not the case, we only have to explain why $\lesssim 1\%$ of the decay products behave so differently.

In the terminology of SBM and thermodynamical models which combine SBM with longitudinal kinematics ^{5),6)}, several mechanisms can be invoked.

- Pre-equilibrium escape of particles. This possibility was already mentioned in the first paper on the thermodynamical model ³⁾ where it was called "superheating". It is not in contradiction with SBM but would require an extension into non-equilibrium statistical mechanics. SBM explicitly postulates equilibrium; its conclusions: exponential mass spectrum and limiting temperature, rest on the assumption that the time during which a system keeps together, is long enough to allow all available mass states of the spectrum to be produced in principle. This time should be of order 10^{-23} sec while the collision time is of order 10^{-23} sec $\cdot m_p/\sqrt{s}$, so that indeed pre-equilibrium states should start to manifest themselves at primary energies of the order of 100 GeV ⁷⁾. Parton models ⁸⁾ might be the best description in this situation.
- In SBM chain decay is predominant and one might be tempted to explain large $p_{\perp}$ as accumulated recoil momenta. Strict four-momentum conservation leads to the opposite effect for single fireballs; nevertheless large $p_{\perp}$ multiplicities above Boltzmann can result from assuming a rather large value of T_0 combined with summing over a particular type of cluster mass distribution ⁹⁾.

- Decaying clusters might have occasionally significant transverse velocities.
- Decaying clusters might have high spin.

While probably all these mechanisms are present and mix in a complicated way, we shall deal only with the last one and show that it alone would already be able to account quantitatively for the large $p_{\perp}$ yields.

Our motivation is, that while usual SBM predictions apply to spin-averaged fireballs, fixed spin fireballs decay like an "exploding fly-wheel", i.e., non-isotropically with large momenta perpendicular to the spin axis. The natural statistical spin distribution of a fireball is centered at zero and decreases so rapidly with increasing spin, that spin averaged SBM's do not show excess $p_{\perp}$; in hadron collisions, where the initial state has in general a large (orbital) angular momentum, the spin distribution of fireballs is forced (by angular momentum conservation) to center at a non-zero value which might be large in some rare events with few heavy fireballs.

This mechanism reconciles very naturally SBM with large $p_{\perp}$. Our aim is to show that it is principally possible; for this, it is sufficient to consider one particular mode of fireball production: we choose the single fireball process without claiming that others do not contribute; in fact pionization (one fireball plus two leading protons) would be even more efficient, and double fireball plus multiple fireball production might sum up to an effect as large as the one considered here (but requiring unreasonable computing time).

In Section 2 we discuss how fireballs with large spin might be produced and in Section 3 how they decay; Section 4 displays our results.

Units: $\hbar = c = 1$.

2. MECHANISMS FOR PRODUCING HIGH SPIN CLUSTERS

We use a classical picture, based on the two assumptions:

- total angular momentum = impact parameter $\times$ momentum + spin;
- excitation energy = c.m.s. kinetic energy $\times$ overlap ratio $\Delta V/V$.

Here V is the proton volume and ΔV the geometrical overlap volume; as the latter depends on the impact parameter, the two assumptions, together with angular momentum conservation, define a relation between mass and spin of clusters in the simple cases of single and double fireball excitation and of pionization. Before working this out in detail, we estimate to what approximation the classical picture might be right.

2.1 Justification of the classical approximation

We are concerned with a high-energy effect, where typical c.m.s. momenta are of order 20 GeV. The orbital angular momentum in c.m.s. of the initial state is $L = pb$ with $0 \leq b \lesssim 2/m_\pi$; so that, for $p = 20$ GeV, $0 \leq L \lesssim 280$. As long as $L \gg 1$, the correspondence principle suggests that the classical picture is a good approximation; this holds down to rather small b . A more quantitative check is obtained as follows: we consider the ℓ^{th} partial wave of a plane wave going in z direction and find out its intensity in the xy plane as a function of the distance from the origin. Thus

$$\Psi_\ell = [e^{ipz}]_\ell = i^\ell (2\ell+1) j_\ell(pr) P_\ell(\cos\theta) \quad (2.1)$$

In the xy plane $\cos\theta = 0$ and $r = b$, hence

$$W(b) := |\Psi_\ell|^2 = [(2\ell+1) j_\ell(pb) P_\ell(0)]^2 \quad (2.2)$$

$$P_\ell(0) = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ (-1)^{\ell/2} (\ell-1)! / [2^{\ell/2} (\ell/2)!] & \text{for } \ell \text{ even} \end{cases}$$

Figure 1a shows $|\Psi_\ell|^2$ for $p = 20$ GeV and $\ell = 58$ as function of b . The main peak is where it ought to be, near ℓ/p ; below $b \approx 2.5$ is nothing and above 3.3 further peaks decrease faster than $1/pb$; the width of the main peak is of order $\Delta b/b \approx 1/\ell$.

We now ask for the ℓ distribution when b is given; as b is a continuous variable we choose an interval over which we integrate with a weight function. The most natural choice is the interval under and the curve over the shaded region R of Fig. 1a. Thus

$$F(\ell) := [(2\ell+1) P_\ell(0) \cdot 2\pi \int_R db W(b) j_\ell(pb)]^2 \quad (2.3)$$

The result is shown in Fig. 1b as a function of (even) ℓ . The maximum is at $\ell_0 = 58$ where it should be, above 68 is nothing and below 54 further maxima of strongly decreasing height contribute. The width of the main peak is of order $\Delta\ell/\ell \approx 1/\ell$.

Conclusion: for large energies the classical relation $\ell \cong pb$ holds in both directions; neglecting accompanying peaks (which become unimportant like $1/p$), we can trust the formulae to be derived below within uncertainties of order $\Delta\ell/\ell \approx \Delta b/b \approx 1/\ell$.

From now on we use the classical picture to establish a relation between spin and mass of fireballs produced in collisions.

2.2 Cluster spin as consequence of angular momentum conservation

We shall consider the following processes, which we call "double fireball" (DF) with its limiting case "single fireball" (SF) and, distinct from that, pionization (PI):

We treat DF and SF together, PI separately at the end. We discuss everything in the centre of momentum (c.m.s.) frame and put the origin in the centre of gravity (CG) of the collision. Momenta and co-ordinates are denoted by p, x before and q, y after the collision:

$$p_\alpha = (E_\alpha, p, 0, 0)$$

$$p_\beta = (E_\beta, -p, 0, 0)$$

$$x_\alpha = (t, vt, b_\alpha, 0)$$

$$x_\beta = (t, -vt, -b_\beta, 0)$$

$$q_\alpha = (E_\alpha, q, 0, 0)$$

$$q_\beta = (E_\beta, -q, 0, 0)$$

$$y_\alpha = (t, v_\alpha t, c_\alpha, 0)$$

$$y_\beta = (t, -v_\beta t, -c_\beta, 0)$$

(2.4)

The choice of CG for the origin implies

$$\begin{aligned} \varepsilon_\alpha b_\alpha &= \varepsilon_\beta b_\beta \\ \varepsilon_\alpha &= \varepsilon_\beta = E/2 \quad (E = \text{C.M.S. energy}) \\ b_\alpha &= b_\beta = b/2 \end{aligned} \tag{2.5}$$

The initial angular momentum has then only one non-vanishing component: the third. After the collision the orbital angular momentum will be smaller than the initial one and the loss must be compensated by cluster spins. A lower bound on their magnitude is obtained by requiring that they both point in the direction of the initial orbital momentum (we assume scalar incoming particles). In that case the scattering plane is conserved and all angular momenta are invariant under a rotation about the z axis, the scattering angle θ of the reaction drops out and putting it equal to zero [as in Eq. (2.4)] is no loss of generality.

The orbital angular momentum tensors before (L) and after $(\bar{L})$ are

$$\begin{aligned} L^{\mu\nu} &:= (x^\mu p^\nu - x^\nu p^\mu)_\alpha + (x^\mu p^\nu - x^\nu p^\mu)_\beta \\ \bar{L}^{\mu\nu} &:= (y^\mu q^\nu - y^\nu q^\mu)_\alpha + (y^\mu q^\nu - y^\nu q^\mu)_\beta \end{aligned} \tag{2.6}$$

With Eqs. (2.4) it follows

$$\begin{aligned} L^{\mu\nu} &= \{ \text{all } L^{\mu\nu} = 0, \text{ except } L^{12} = -bp \} \\ \bar{L}^{\mu\nu} &= \{ \text{all } \bar{L}^{\mu\nu} = 0, \text{ except } \bar{L}^{12} = -cq \} \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} b &:= b_\alpha + b_\beta \\ c &:= c_\alpha + c_\beta \end{aligned}$$

Conservation of CG implies conservation of the mixed (time/space) components and yields the relations

$$\varepsilon_\alpha b_\alpha = \varepsilon_\beta b_\beta = Eb/4$$

$$C_\alpha E_\alpha = C_\beta E_\beta \quad (2.8)$$

$$C = E \frac{C_\alpha}{E_\beta} = E \frac{C_\beta}{E_\alpha}$$

Now the sum of the cluster spins (so far unknown)

$$J^{\mu\nu} = J_\alpha^{\mu\nu} + J_\beta^{\mu\nu} \quad (2.9)$$

must equal the loss in orbital momentum:

$$J^{\mu\nu} = L^{\mu\nu} - \bar{L}^{\mu\nu} \quad (2.10)$$

hence

$$J^{\mu\nu} = \{ \text{all } J^{\mu\nu} = 0, \text{ except } J^{12} = -(b_p - c_q) \} \quad (2.11)$$

Here arises a complication: while the only non-vanishing component of $J^{\mu\nu}$ is J^{12} , this is not true for $J_\alpha^{\mu\nu}$ and $J_\beta^{\mu\nu}$ separately. Namely, in their respective rest frames $R_{\alpha,\beta}$ they indeed have only 12 components (by assumption), but a boost to the c.m.s. will generate non-zero 02 components. We obtain the correct decomposition of $J^{\mu\nu}$ into $J_\alpha^{\mu\nu}$ and $J_\beta^{\mu\nu}$ by using the spin four-vector S^μ and its relation to $J^{\mu\nu}$ [see, e.g., Ref. 10)]

$$S^\mu = \frac{1}{m} \varepsilon^{\mu}_{\nu\rho\sigma} q^\nu J^{\rho\sigma}$$

$$J^{\mu\nu} = \frac{1}{m} \varepsilon^{\mu\nu}_{\rho\sigma} q^\rho S^\sigma \quad (2.12)$$

In the rest frame of a particle these two quantities have the same components:

$$\begin{aligned}
 (J^{\mu\nu})_R &= (J^{0k}=0, J^{23}, J^{31}, J^{12}) \\
 (S^\mu)_R &= (0, S^1, S^2, S^3) \\
 S^1 &= J^{23} \text{ etc., cycle.}
 \end{aligned} \tag{2.13}$$

Now we argue as follows: in their respective rest frames $R_{\alpha,\beta}$ only the $J_{\alpha,\beta}^{12}$ components are different from zero (by assumption) and equal to the $S_{\alpha,\beta}^3$:

$$\begin{aligned}
 (J^{\mu\nu})_{R_{\alpha,\beta}} &= \{ 0, \text{except } J_{\alpha,\beta}^{12} = S_{\alpha,\beta} \} \\
 (S^\mu)_{R_{\alpha,\beta}} &= (0, 0, 0, S_{\alpha,\beta})
 \end{aligned} \tag{2.14}$$

The vectors $S_{\alpha,\beta}^\mu$, however, are invariant under a boost in the 1 direction; using the second Eq. (2.12) in c.m.s. we find

$$\begin{aligned}
 J_\alpha^{01} &= 0 & J_\beta^{01} &= 0 \\
 J_\alpha^{02} &= q \cdot \frac{S_\alpha}{m_\beta} & J_\beta^{02} &= -q \frac{S_\beta}{m_\beta} \\
 J_\alpha^{03} &= 0 & J_\beta^{03} &= 0 \\
 J_\alpha^{23} &= 0 & J_\beta^{23} &= 0 \\
 J_\alpha^{31} &= 0 & J_\beta^{31} &= 0 \\
 J_\alpha^{12} &= \frac{E_\alpha S_\alpha}{m_\alpha} & J_\beta^{12} &= \frac{E_\beta S_\beta}{m_\beta}
 \end{aligned} \tag{2.15}$$

Now in $J^{\mu\nu} = J_\alpha^{\mu\nu} + J_\beta^{\mu\nu}$ only $J^{12} \neq 0$; hence from $J^{02} = 0$

$$\frac{S_\alpha}{m_\alpha} = \frac{S_\beta}{m_\beta} = \lambda \quad (2.16)$$

where λ is some function of kinematical variables but constant with respect to spin.

Equation (2.11) together with $E_\alpha + E_\beta = E$ and the last line of Eq. (2.15) then yield

$$\begin{aligned} S_\alpha &= \frac{m_\alpha}{E} \cdot (c q - b p) \\ S_\beta &= \frac{m_\beta}{E} \cdot (c q - b p) \\ \lambda &= (c q - b p) / E \end{aligned} \quad (2.17)$$

Before we can discuss plausible choices for the outgoing impact parameter c , we turn to the masses m_α and m_β .

2.3 Cluster masses as functions of the impact parameter

We estimate the excitation mass from the simple geometry of Fig. 2. In the overlap region matter is more or less stopped and its kinetic energy transformed into excitation mass, while the other parts of the particle move on. In reality, there is a continuous velocity distribution; we neglect these details and simply postulate

$$\text{total excitation energy} = (\text{total kinetic energy}) \cdot (\Delta V / V) .$$

The Lorentz contraction drops out; a simple geometrical calculation yields

$$\frac{\Delta V}{V} = \left(1 - \frac{b}{2R}\right)^2 \cdot \left(1 + \frac{b}{R}\right) \quad (2.18)$$

where $R = \text{hadron radius} \simeq 1/m_\pi$.

We now have to distribute this total excitation energy between two clusters: we give one of them the fraction K and the other $(1-K)$; with $\frac{1}{2} \leq K \leq 1$ all cases from symmetrical double cluster to single cluster processes are covered. Thus

$$\begin{aligned} m_\alpha &= K \cdot (E - 2m) \cdot \frac{\Delta V}{V} + m \\ m_\beta &= (1 - K) \cdot (E - 2m) \cdot \frac{\Delta V}{V} + m \end{aligned} \quad \left[\frac{1}{2} \leq K \leq 1 \right] \quad (2.19)$$

As $E \gg m$ it follows that $m_{\alpha, \beta}$ are roughly proportional to E .

For two of the possible versions we shall need the outgoing impact parameters (centre of gravity 2-components) of the excited part, c_* , and of the unexcited part, c' , as defined in Fig. 2. As we allow transfer of excitation energy from α to β , c_* will depend on K and b while c' depends only on b ; a simple calculation yields

$$\begin{aligned} c_*(K, b) &= \frac{b}{4} \cdot \frac{2R - b}{R + b} \cdot \frac{1 - K}{K} \\ c'(b) &= R - \frac{b}{4} \cdot \frac{2R - b}{3R - b} \end{aligned} \quad (2.20)$$

2.4 What is the outgoing impact parameter c ?

Apart from the freedom in choosing K , the masses m_α and m_β are definite functions of the impact parameter; they also determine q , the outgoing momentum. Thus in Eqs. (2.17) all quantities are known, apart from the outgoing impact parameter c , which cannot be fixed by any conservation law and must be guessed. In order not to be limited by one particular choice, we have considered several possibilities.

2.4.1 Geometrical impact parameter conservation

$$c = b \quad (2.21)$$

2.4.2 Separate centre of gravity conservation

The 2-components of the centres of gravity remain unchanged separately for the particle α and the particle β ; this yields

$$\left. \begin{aligned} c_\alpha E_\alpha &= b_\alpha \varepsilon_\alpha \\ c_\beta E_\beta &= b_\beta \varepsilon_\beta \end{aligned} \right\} c = b \cdot \frac{E^2}{4E_\alpha E_\beta} \quad (2.22)$$

2.4.3 Kinematical impact parameter conservation

Here we define the outgoing impact parameter c_α (and similarly for β) as the two component of the centre of gravity of α , assuming that the excited and non-excited parts go on with individually conserved impact parameters c_* and c' . Using a relativistic definition of CG [see, e.g., Ref. 10)]^{*)}

$$X_{CG} = \left(t, \frac{\sum \vec{x}_i E_i}{\sum E_i} \right) \quad (2.23)$$

we find

$$c_\alpha = \left[c' \cdot \frac{E}{2} \left(1 - \frac{\Delta V}{V} \right) + m_*(K) \cdot c_*(K) \right] / E_\alpha \quad (2.24)$$

where

$$m_*(K) = K \cdot (E - m) \frac{\Delta V}{V} \quad (2.25)$$

c_*, c' from equ. (2.20)

The resulting formulae are not as transparent as (2.21) and (2.22) and will not be given; in any case (2.24) and (2.25) fix this version.

2.4.4 Collective motion generated spin

Here we do not make a hypothesis on c , but rather try to compute the internal angular momentum of particle α due to an oversimplified collective motion¹¹⁾: we imagine that in Fig. 2 the excited part is at rest and the other part goes on with the initial velocity; as we know also the two components of their respective CG, straightforward application of Eqs. (2.12) yields

$$S_\alpha = \frac{p m_*(K)}{m_\alpha(K)} \cdot \left(1 - \frac{\Delta V}{V} \right) \cdot (c_*(K) - c') \quad (2.26)$$

In this case m_α is not determined by Eqs. (2.19), but has to be computed as the invariant mass of the "two-body system α ".

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*) The relativistic CG cannot be defined uniquely: the CG co-ordinates calculated according to (2.23) by different observers are not related by Lorentz transformations between these observers.

2.4.5 Rigorous single fireball (SF)

We require that particle β remains unexcited and does not acquire spin:

$$\frac{s_\beta}{m_\beta} = \lambda = \text{const.} \quad (2.27)$$

where const. means: independent of kinematical variables. Here appears the weakness of our classical considerations, which result in the statement (2.16) that of two initial spinless particles either both or none should become spinning (for $\lambda = 0$, $s_\alpha = s_\beta = 0$); of course the classical argument fails where the spin becomes small, such that the quantum steps Δs are of the same order as s itself. Here it would be more reasonable to replace s_β by Δs_β , which for a pp collision could be between 0 and 1 according to the degree of spin flip. Thus with a value $\lambda \approx 1 \text{ GeV}^{-1}$ we might be not far from reality.

In this case

$$\begin{aligned} s_\alpha &= \lambda m_\alpha \\ c &= \frac{\lambda E + b p}{q} \end{aligned} \quad (2.28)$$

and the model is fixed.

2.4.6 Pionization (PI)

The total excitation mass

$$M^* = (E - 2m) \cdot \frac{\Delta V}{V} \quad (2.29)$$

rests in the middle and two protons share the rest of c.m.s. energy, $E - M^*$. This fixes their momentum q and, for the plausible choice $c = b$, their orbital angular momentum bq . The spin of M^* is then

$$s^* = -b(p - q) \quad (2.30)$$

2.5 Choice of the model

In each version considered, one finds for large E that spin and mass grow very nearly proportionally to E so that for each version one single curve s_Q/E vs m_Q/E (s^*/E vs M^*/E) suffices to see what happens. These curves are shown in Figs. 3a, b; the first one displays the different versions for $K=1$ (SF) and for (PI), the second illustrates the effect of varying K from 1 over 0.75 to 0.5 (DF). We also have included in Fig. 3a the "naive guess" (non-relativistic treatment of angular momentum, $c = b$)

$$s = b \cdot (q - p) \quad (2.31)$$

which lies about in the middle of all our more elaborate curves [and has been used in exploratory calculations ¹²⁾].

We now have to choose between all these possibilities, without having any strong arguments for the one or the other. From Fig. 3b we see that if K varies from 1 (SF) to $\frac{1}{2}$ (symmetric DF) the curves are squeezed to the left without losing much height; as we do not know whether the dominance of SF over DF observed at low masses will still hold here, it might be reasonable to allow some admixture of $K < 1$ contributions. Apart from that, we do not know which of the different versions to choose -- presumably it should not be one of the extremes. The different versions need not be mutually exclusive because the impact parameter is not conserved and c is thus even less well defined than b . Looking at c and K as continuous parameters for which in the best case a probability distribution (unknown to us) could exist, we have to choose a reasonable compromise ("mean value"): we take the straight line. Apart from the "mean value" argument there is the other one that, after all, even at large masses SF (analogous to single diffraction) would probably be more important than DF (double diffraction), if it is "rigorous SF", then the straight line would be the most reasonable classical guess, which overmore agrees with some recent low-energy single diffraction data if $\lambda \cong 1/m_p$ ¹³⁾.

For the present work we use therefore the version:

$$\text{rigorous single fireball: } \begin{cases} \lambda = 1/m_p \\ s = M/m_p \end{cases} \quad (2.32)$$

We neither claim nor believe that large $p_\perp$ are produced exclusively via

single fireball excitation; we take this choice as a reasonable representative of the average over many possible mechanisms. Independently of all that follows, we can already state here that with this choice there will be a leading proton (the non-excited one) with a momentum determined by the kinematics. In PI there would be two leading protons and in DF one (if one mass is light) or none (both heavy).

3. DECAY OF HIGH-SPIN HIGH-MASS CLUSTERS

We remain in the framework of SBM where the decay structure of a fireball of mass M follows from the recipe: write the mass spectrum $\rho(M)$ as a Yellin expansion ^{14),3)} over invariant phase space integrals ^{*)}; choosing the fireball rest frame $[P = (M, 0, 0, 0)]$ we have

$$\rho(M) = \sum_{n=1}^{\infty} g_n \Omega_n(M) \quad (3.1)$$

$$\Omega_n(M) = \left[\frac{V_0}{(2\pi)^3} \right]^n \int \delta^4(P - \sum_{i=1}^n p_i) \prod_{i=1}^n d^3 p_i$$

g_n = Yellin coefficients; they follow from the recursion relation ¹⁶⁾

$$g_1 = 1$$

$$n \cdot g_n = -(n-1) \cdot g_{n-1} + 2 \sum_{k=1}^{n-1} k \cdot g_k \cdot g_{n-k} \quad (3.2)$$

Then one obtains all desired distributions and correlations by suppressing corresponding summations and/or integrations in the phase space integrals.

Clearly $\rho(M)$ is summed over all spins, because the momentum space integrals correspond to counting plane wave states in V_0 , containing all angular momenta.

We now consider fixed spin along the third axis:

$$J = J_z = S \quad (3.3)$$

Instead of (3.1) we then have

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*) For the difference between invariant phase space (IPS) and invariant momentum space (IMS) see Ref. 15); here IPS is the correct one.

$$\rho(M, s) = \sum_{n=1}^{\infty} f_n(M, s) \cdot \Omega_n(M, s) \quad (3.4)$$

The $\Omega_n(M, s)$ can be computed following Cerulus ¹⁷⁾ (see Appendix A), but we have to find the new coefficients $f_n(M, s)$. We proceed as follows: write

$$\begin{aligned} \rho(M, s) &= \rho(M) \cdot G(M, s) \\ \Omega_n(M, s) &= \Omega_n(M) \cdot F_n(M, s) \end{aligned} \quad (3.5)$$

From comparison of (3.1), (3.4) and (3.5) follows

$$f_n(M, s) = g_n \cdot \frac{G(M, s)}{F_n(M, s)} \quad (3.6)$$

The function $F_n(M, s)$ can be taken from old papers ¹⁸⁾

$$F_n(M, s) \cong C \cdot \frac{n^{3/2} \cdot \exp\left[-\frac{9ms^2}{4R^2M^2}\right]}{M^3 R^3} \quad (3.7)$$

$$R = \text{Radius of } V_0; \text{ i.e., } R = m_\pi^{-1}$$

The other unknown function, $G(M, s)$ follows from a simple consideration: for a classical spinning body with rest mass M_0 we have

$$\begin{aligned} M &= M_0 + \frac{s^2}{2\theta_0} + \dots \\ \theta_0 &= \text{rest moment of inertia} = \alpha \cdot M_0 R^2 \\ \alpha &= \text{constant of order 1; } \alpha(\text{sphere}) = 2/5 \end{aligned} \quad (3.8)$$

Writing $M = M_0(1 + s^2/(2\alpha M_0^2 R^2) + \dots)$ we see that with our assumption $s = M/m_p$ and $R = 1/m_\pi$ the correction term is $s^2/(2\alpha M_0^2 R^2) \approx 1/50$. Thus it is irrelevant whether in the correction term we write M_0 or M , the difference giving a correction of order $(1/50)^2$. Hence, to first order

$$M_0 = M \cdot \left(1 - \frac{s^2}{2\alpha M^2 R^2}\right) = M \cdot \left(1 - \frac{s^2}{2M\Theta}\right) \quad (3.9)$$

$$\Theta = \alpha M R^2$$

Turning to the density of states we argue that it does not change under uniform rotation; therefore

$$\rho(M, s) = \rho(M_0, s=0) \quad (3.10)$$

Furthermore, the total density $\rho(M)$ is

$$\rho(M) = \int_{-\infty}^{\infty} \rho(M, s) ds = 2 \int_0^{\infty} \rho(M_0, s=0) ds \quad (3.11)$$

There is no factor $2s+1$ since our $s = s_z$; the s dependence of $\rho(M_0, s=0)$ is in M_0 . We know that ³⁾ $\rho(M) \sim \text{const. } M^{-3} \exp(M/T_0)$ and that therefore $\rho(M_0, s=0)$ must be exponential in M_0 . We make the ansatz

$$\rho(M, s) = \rho(M_0, 0) = u(M_0) e^{M_0/T_0} \quad (3.12)$$

where $u(M_0)$ is bounded by a polynomial. Using (3.9) we have

$$\rho(M, s) = u(M) e^{\frac{M}{T_0}} e^{-\frac{s^2}{2\Theta T_0}} \cdot \left(1 - \frac{u'}{u} \cdot \frac{s^2}{2\Theta} + \dots\right) \quad (3.13)$$

and integrate this over s from $-\infty$ to ∞ to obtain

$$\rho(M) = u(M) e^{\frac{M}{T_0}} \sqrt{2\pi\Theta T_0} \cdot \left(1 - \frac{T_0}{2} \frac{u'}{u} + \dots\right) \quad (3.14)$$

Knowing that $\rho(M) \sim \text{const. } M^{-3} \exp(M/T_0)$, we see that the leading power of u is $M^{-7/2}$ so that $u'/u = -7/(2M)$. Using this and Eq. (3.13) we find

$$p(M) = p(M, s) e^{\frac{s^2}{2\theta T_0}} \sqrt{2\pi\theta T_0} \cdot \left(1 - \frac{7}{2M} \left(\frac{s^2}{2\theta} - \frac{T_0}{2}\right) + \dots\right)$$

(3.15)

and finally, to first order in $s^2/(2M\theta)$:

$$p(M, s) = p(M) e^{-\frac{s^2}{2\theta T_0}} \cdot \frac{1}{\sqrt{2\pi\theta T_0}} \cdot \left(1 + \frac{7}{2M} \left(\frac{s^2}{2\theta} - \frac{T_0}{2}\right) + \dots\right)$$

$$\theta = \alpha M R^2$$

(3.16)

This result agrees with that of Chiu and Heimann ¹⁹⁾ up to a numerical factor in the parenthesis, which depends on the choice of α ; for $\alpha = 1/7$ both formulae agree. Their constant d has the physical significance $d = 2\alpha R$ (if $R = 1/T_0$). In contrast to the complicated derivation of Ref. 19) our classical derivation shows the physical origin of the Gaussian: since part of the actual mass M is due to rotational energy, the level density (which is independent of the rotation) has to be evaluated not at M but at $M_0 = M(1 - s^2/2M\theta)$. The further factor $M^{-1/2}$ contained in the square root is due to the integration. Note that in our derivation the rotating body was taken as a whole; its internal structure is hidden in the constant α . In Ref. 19) linear chain decay was assumed, resulting in $\alpha = 1/7$. Full SBM will presumably not alter the result of Ref. 19), except for the value of α .

Our above approximations are good as long as

$$\frac{T_0}{M} \ll 1 \quad (\text{here } \approx 1/200)$$

(3.17)

$$\frac{s^2}{2M\theta} \ll 1 \quad (\text{here } \approx 1/50)$$

We shall therefore even drop the terms in the parenthesis of Eq. (3.16) and identify, according to Eq. (3.5)

$$G(M, s) = \frac{\exp\left[-\frac{s^2}{2\alpha M R^2 T_0}\right]}{\sqrt{2\pi\alpha M R^2 T_0}}$$

(3.18)

Noting that $T_0 \approx m_\pi = 1/R$ we obtain with (3.18), (3.7) and (3.6)

$$f_m(M, s) = g_m \frac{\exp\left[\frac{9ms^2}{4M^2R^2}\right]}{n^{3/2}} \cdot \frac{\exp\left[-\frac{s^2}{2\alpha MR}\right]}{C \cdot \sqrt{2\pi\alpha MR}} \quad (3.19)$$

We take $\alpha = \frac{1}{2}$. This defines the new coefficients for Eq. (3.4); note that the last factor is eliminated by normalization if we work at fixed M and s .

We now have the old recipe: compute (3.4) with the above $f_n(M, s)$ and obtain all desired distributions and correlations by suppressing corresponding integrations and/or summations. In doing so, one cannot, of course, use again Eq. (3.5) and simply suppress integrations in the ordinary phase space integrals: Eq. (3.5) is only valid for the complete integral, while the integrand has now acquired a complicated structure due to spin. Therefore, the integrals $\Omega_n(M, s)$ must be computed with the help of the usual Monte-Carlo method, in which suppression of integrations is automatically incorporated. As the application of standard methods leads to some difficulties, we discuss our procedure more explicitly in Appendix A.

Before presenting the results, a comment on average spin might be in order. From Eq. (3.16) it follows that

$$\sqrt{\langle s^2 \rangle} \cong \sqrt{\alpha MR^2 T_0} \cong \sqrt{\frac{M}{2m_\pi}} \quad (3.20)$$

while earlier in (2.32) we stated for the single fireball case that

$$s \cong M/m_p \quad (3.21)$$

There is no contradiction, because (3.21) describes the spin mass relation resulting from a particular spin exciting mechanism, while (3.20) is the average spin in a statistical ensemble of masses M ; namely

$$\begin{aligned} \langle E_{\text{rot}} \rangle &= \frac{1}{2} T_0 = \frac{\langle s^2 \rangle}{2\theta} + \dots \\ \langle s^2 \rangle &= \theta T_0 = \alpha MR^2 T_0 \end{aligned} \quad (3.22)$$

4. RESULTS

We have chosen "rigorous single fireball" (SF) as a typical mechanism to produce large spin fireballs. We stress again that we did this in order to keep computation in reasonable limits, not because we believe that other processes would not contribute; in particular, pionization and multiple cluster contributions would be probably as important as SF. This has to be kept in mind during the following considerations.

The model is now fixed in three respects:

- i) rigorous single fireball excitation; $s = M/m_p$;
- ii) phase space with spin;
- iii) spin-adjusted SBM (Yellin solution) superposition of phase space with spin.

There is one ingredient missing: the single fireball differential cross-section $d\sigma_{SF}/dM$. For masses of the order 10-50 GeV no experimental data are available and theoretical models -- mostly based on Regge models -- are valid only for small masses. We therefore are forced to turn the argument and ask: supposing our model assumptions i), ii), iii) were qualitatively correct, what must $d\sigma_{SF}/dM$ be in order to produce the observed large transverse momentum distribution?

Thus our procedure: take a given c.m.s. energy E , select a number of masses M_i on the straight line of Fig. 3a with their corresponding spins s_i , compute for each of them, according to Section 3, the superposition over all important multiplicities (3 to 40) of all interesting distributions, select out of these the transverse momentum (p_x) distributions ^{*)} $(d\sigma/dp_x)_{M_i}$ and sum them up:

$$\frac{d\sigma}{dp_x} = \frac{1}{\sigma_{ind.}} \int \frac{d\sigma}{dM} \cdot \frac{d\sigma}{dp_x}(M) dM \cong \sum_{i=1}^6 W_i \left(\frac{d\sigma}{dp_x} \right)_i \Delta M_i$$

$$W_i \cong \frac{1}{\sigma_{ind.}} \cdot \frac{d\sigma}{dM} \quad ;$$

(4.1)

take the experimental $(d\sigma/dp_x)_{exp}$ and fix the weights W_i to fit the above $d\sigma/dp_x$ to the experimental one. Then compute the superposition of all other distributions with the same weights. While this is in principle

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*) See our definition of co-ordinates in subsection 4.4.1.

easy, computing time and storage capacity limitations forced us to select only six different masses and two different c.m.s. energies. One energy and one mass would have sufficed for showing that large p_x can be produced, but we wanted to find out possible regularities to be discussed below. We selected

$$\begin{array}{c|cccccc} M_{SF} & 45 & 31 & 28 & 22 & 18 & 14 & [\text{GeV}] \\ \hline \text{spin} & 48 & 32 & 29 & 23 & 19 & 15 & [\text{rounded}] \end{array} \quad (4.2)$$

and

$$E = E_{\text{c.m.s.}} = 52 \text{ and } 32 \text{ GeV} \quad (4.3)$$

The resulting weights W_i are not to be interpreted as a prediction of $(1/\sigma_{\text{inel}})d\sigma/dM$ from our model, because they have been determined as if only SF would contribute, while we know that even inside our model other mechanisms contribute (not to speak of mechanisms foreign to our present model); the W_i may be at best correct as order of magnitude estimates.

4.1 Qualitative summary of results

Following the procedure just described, we find the following main features, which will be discussed in the following subsections:

- reduced multiplicities and increased mean energies as compared to SBM without spin; multiplicities increase with p_x (subsection 4.3);
- anisotropic momentum distribution: more particles with larger momenta in the plane orthogonal to spin than in other directions (subsection 4.4.2);
- strong two-body correlations of particles with large p_x on both, same and opposite side (fireball rest frame) of a given large p_x particle (in the plane orthogonal to spin). One might call this a "jet", although we do not know more about its structure. We did not include resonances as produced particles; their decay would make the jet-like appearance more pronounced (subsection 4.4.3);
- very reasonable weights W_i of order 10^{-3} to 10^{-6} produce the experimental p_x distribution at 52 GeV c.m.s. energy; only the six weights for 52 GeV have been treated as free parameters, those for 32 GeV are then computed using the notion of "energy independent velocity distribution $F_0(\lambda)$ " borrowed from the old thermodynamical model ⁵⁾. With these computed weights a satisfactory p_x distribution for 32 GeV is predicted (subsection 4.2).

Remark:

All numerical results from Monte-Carlo calculations show statistical fluctuations which decrease like $1/\sqrt{t}$ when t is the computing time. We had to keep t reasonable and to accept fluctuations which made a direct presentation of our results impossible and forced us to use a smoothing procedure described in Appendix C. Even so, our curves show still some remnants of the original fluctuations.

4.2 The weight function $W(M,E)$; p_x distribution

We could just determine six weights at 52 GeV and another six at 32 GeV, making thus a 2×6 parameter fit to the large p_x data. We prefer to take a more educated approach by fitting only six weights at 52 GeV and predict the other ones (at any energy) by borrowing the notion of "energy independent velocity distribution" from the old thermodynamical model ^{5),6)}. There it was found that the continuous velocity distribution of the collective motion of hadron matter along the collisions axis, averaged over space, time and impact parameter, could be written as an empirical function

$$\begin{aligned}
 F(\lambda) & \quad \text{for newly created matter} \\
 F_o(\lambda) & \quad \text{for throughgoing initial protons} \\
 & \quad \text{becoming excited} \\
 \lambda: & = \text{sign}(v)(\gamma - 1)/(\gamma_o - 1) \\
 \gamma_o & = \text{initial c.m.s. Lorentz factor} \\
 \gamma & = \text{actual Lorentz factor of outgoing fireball}
 \end{aligned} \tag{4.4}$$

and that this pair of functions, when fitted to experimental data, was practically independent of

c.m.s. energy (γ_o)
the nature of secondary produced
the nature of the colliding particles .

This universality of $F(\lambda)$ and $F_o(\lambda)$ has been confirmed in the following years at all available energies up to ISR ²⁰⁾. It seems that the universality of these functions represents some universal property of excited hadron matter, probably just geometry. With the data available it was not possible to detect any fine structure in these functions; in particular, $F_o(\lambda)$ was assumed to go smoothly to zero at $\lambda \rightarrow 0$ while having a peak

for $\lambda \rightarrow 1$ (diffraction dissociation of initial proton). In our case we have to consider the function $F_0(\lambda)$ for λ very near zero, since we excite very large masses with low velocity; so far data did not allow us to determine $F_0(\lambda)$ in that region. We take the point of view that in determining our weights $W_i = W(M_i, E)$ we actually fix $F_0(\lambda)$ at very small λ , where it then will not be zero but have some tiny peak. If that is so, and if $F_0(\lambda)$ is indeed nearly energy independent, it suffices to fix it at one energy and compute $W(M, E)$ from it at other energies. This is what we did.

Let us call the velocity distribution $F_{SF}(\lambda)$, according to the special situation considered. From SF kinematics the fireball mass for given γ is at c.m.s. energy E

$$M(\gamma) = \gamma E - \sqrt{m^2 + E^2(\gamma^2 - 1)} \quad (4.5)$$

Thus, with $\gamma_0 = E/(2m)$ and Eq. (4.4), $\lambda = \lambda(M, E)$ is calculable from M and E . For fixed E we have $|F_{SF}(\lambda)d\lambda| = |W(M, E)dM|$, hence

$$F_{SF}(\lambda) = F_{SF}(\lambda(M, E)) = W(M, E) \cdot \left| \frac{dM}{d\lambda} \right| \quad (4.6)$$

and, as $F_{SF}(\lambda)$ is by hypothesis independent of E ,

$$W(M, E_1) \cdot \left| \frac{dM}{d\lambda}(E_1) \right| = W(M, E_2) \cdot \left| \frac{dM}{d\lambda}(E_2) \right| \quad (4.7)$$

From (4.4) and (4.5)

$$\left| \frac{dM}{d\lambda} \right| = \frac{EM^2(E - 2m)}{m(E^2 - M^2 - m^2)} \quad (4.8)$$

Finally,

$$W(M, E) = W(M, E_0) \cdot \frac{E_0}{E} \cdot \frac{E_0 - 2m}{E - 2m} \cdot \frac{E^2 - M^2 - m^2}{E_0^2 - M^2 - m^2} \quad (4.9)$$

where E_0 is the energy where the weights $W(M_i, E)$ are fitted to the experiment. We took $E_0 = 52$ GeV and fitted the weights of the six masses given in (4.2). The six values W_i were furthermore constrained to lie on a smooth curve; we succeeded with the ansatz

$$W(M, E_0) = P_2(M) \exp\left(-\frac{M}{M_0}\right) / \left|\frac{dM}{d\lambda}(E_0)\right| \quad (4.10)$$

We found

$$M_0 = 5 \text{ GeV}$$

$$P_2(M) = 61.6 - 5.68 M + 0.189 M^2 \quad (4.11)$$

This fixes $W(M, E_0)$ and permits us to calculate $W(M, E)$ from Eq. (4.9) at other c.m.s. energies.

Figure 4 shows $\log_{10} W(M, E)$ as a function of M for $E = 52$ GeV (fitted to data) and $E = 32$ GeV [computed from Eq. (4.9)]. The comparison with actual data ¹⁾ is satisfactory (Fig. 5):

- the p_x distribution at 52 GeV is well fitted by a superposition of the usual Boltzmann curve $\sim e^{-6p_x}$ plus the contribution from SF with high spin;
- the fitted weights $W(M_i, 52)$ lie on a smooth curve (Fig. 4), dropping roughly exponentially;
- the fitted weights are very small, ranging roughly from 10^{-3} to 10^{-6} [the lowest mass (14 GeV) was found to have no influence]; the largest p_x (near 6 GeV/c) are exclusively due to the largest mass with weight $\sim 10^{-6}$. In other words: an effect concerning $\lesssim 1\%$ of the produced particles can be accounted for by a fraction of 10^{-3} to 10^{-6} of collisions, and at $p_x = 6$ GeV/c 10^{-6} of the collisions produce an increase of $d\sigma/dp_x$ by a factor $\sim 10^7$ with respect to the exponential cut-off $\sim \exp(-6p_x)$ predicted by the ordinary spin-averaged SBM.
- the weights for $E = 32$ GeV as computed from Eq. (4.9) yield a predicted p_x distribution which passes through the corresponding data.

The irregularities in the computed curves in Fig. 5 are due to the small number of (effectively only 5) contributing masses; with more computing effort the curves would become smooth. From now on we consider the weights to be fixed. We stress again that we attach little physical meaning to their actual numerical values, because these are the weights which a fictitious single fireball process should have in order to simulate the whole of all processes contributing to large p_x (double fireball, pionization, multifireball, pre-equilibrium emission, fireballs with transverse momentum and what not).

4.3 Multiplicities

It has been known for long ^{17),18)} that angular momentum conservation leads to a significant reduction of multiplicities in phase-space models; this alone implies a larger average energy per particle and is one main reason for large momenta (the others being anisotropy and jet structure; see below).

In the following table we give our values of mass, spin and corresponding multiplicity; for comparison we give also the multiplicity value predicted by the ordinary (spin averaged) SBM ³⁾ as well as the ratio of the latter to the former, which is the factor by which the mean (total) energy per particle is increased by giving the decaying fireball our SF spin value. Note that the total experimental multiplicity is of order 17 at 52 GeV and 14 at 32 GeV ²¹⁾.

Multiplicities

Mass	14	18	22	28	31	45
Spin	15	19	23	29	32	46
Multiplicity in SBM with spin	16.6	17.2	17.8	18.8	19.3	21.8
Multiplicity in ordinary SBM	28	36	44	56	62	90
Factor of increment of $\langle E \rangle$	1.69	2.09	2.47	2.98	3.21	4.13

From the third and last rows of this table we learn that the mean multiplicity per event increases with the mean energy per particle $\langle E \rangle$; in other words: large p_x events have larger multiplicities than small p_x events and their multiplicities are larger than average multiplicities (17 at 52 GeV). The results displayed in the table illustrate a compromise between compensating effects: the relation between mean multiplicity $\langle N \rangle$, mean energy per particle $\langle E \rangle$ and fireball mass is $\langle N \rangle \cdot \langle E \rangle \approx M$; in ordinary SBM, $\langle E \rangle$ is fixed and $\langle N \rangle$ increases proportionally to M ; in SBM with spin $\langle E \rangle$ increases with spin (which increases with M) but not enough to reduce $\langle N \rangle$ to a constant; thus, although spin reduces multiplicity for given M , it increases sufficiently slowly with M to allow still some (reduced) growth of $\langle N \rangle$ (and $\langle E \rangle$) with M . We remember that in all our spin generating mechanisms (and particularly in SF) spin grows over large intervals, roughly linearly with M (Fig. 3a); if it would grow like M^2 (as for linear leading Regge trajectories), multiplicities would presumably decrease with increasing M and large p_x events would have lower multiplicities than small p_x events in contradiction with the experiment.

4.4 Momentum distributions and correlations

All figures refer to $E_{\text{cms}} = 52$ GeV, weighed sum over the six masses.

4.4.1 Co-ordinates

We fix our coordinates as follows: p_x, p_y, p_z refer to the c.m.s. frame, such that (Fig. 6)

- $p_y \equiv p_{\parallel}$ is along the beam direction;
- $p_z \equiv p_s$ is along the spin direction ($\perp$ beam);
- p_x forms with p_y and p_z a right-handed frame.

Thus with respect to the collision axis, p_x and p_z are transverse momenta; the "large $p_{\perp}$ " will be produced along p_x , not along p_z . As p_x is parallel to the impact parameter, the direction of p_x will in reality change randomly from event to event.

In computing p_x distributions and correlations we proceeded as follows: the Monte-Carlo method yields a distribution in the fireball rest frame. In each event we pick two arbitrary particles calling them No. 1 and No. 2. As in the fireball rest frame no direction in the plane $\perp$ spin is preferred, we can rotate the event by an arbitrary angle φ about the spin axis; we chose this angle such that a subsequent boost ^{*)} from

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*) In single fireball kinematics this boost is completely fixed by the fireball mass, provided transverse motion of the fireball can be neglected.

the fireball rest frame to c.m.s. will make $p_y = 0$ for particle No. 1. Furthermore, we keep the spin direction always the same (unlike reality). Thus p_x is the momentum of a trigger particle hitting a counter placed at 90° to the beam axis and having a very small square opening (Fig. 6). The fact that in reality the spin direction is not fixed but randomly distributed about the collision axis, is automatically taken into account by leaving the p_z component of particle No. 1 unconstrained: if $p_z = p \cos \theta$, then this particle would have hit the small square opening of the counter if it had come from an actual event in which the angle between the spin direction and the line collision-centre-to-counter was θ . Particle No. 2 will, of course, in general miss the trigger counter and come out in any direction; we give results only for its p_x distribution.

In addition to the above co-ordinates, we also use the momentum transverse to the spin direction in the fireball rest frame:

$$p_{\perp} = \left(\sqrt{p_x^2 + p_y^2} \right)_{\text{fireball rest frame}}$$

As in the average over many decays the $p_{\perp}$ distribution of a given fireball must be symmetric about the spin axis, the $p_{\perp}$ distribution can be measured in any direction in the plane $_{\perp}$ s; in particular for instance, in the x direction; hence, as far as the fireball mass is large and thus the boost to c.m.s. small, the $p_{\perp}$ distribution in the fireball rest frame will be very similar to the p_x distribution in c.m.s. (apart from the geometrical factor $2\pi p_{\perp}$).

4.4.2 Anisotropy in single-particle distributions

The picture of the "exploding fly wheel" suggests that along any direction in the spin-orthogonal plane, more and faster debris are emitted than along the spin direction. Our results confirm this: Fig. 7 shows a comparison between $\log_{10} W(p_x)$ and $\log_{10} W(p_z)$ in arbitrary units; $W(p_z)$ drops much faster than $W(p_x)$. Figure 8 shows the ratio $W(p_x)/W(p_z)$, Fig. 9 the double differential cross-section $W(p_x, p_z)$ in arbitrary units, and Fig. 10 the $p_x p_z$ correlation (all W 's are normalized to 1)

$$C(p_x, p_z) := \frac{W(p_x, p_z)}{W(p_x) W(p_z)} \quad (4.14)$$

which is varying between 0.23 and 2.2 (rather a weak correlation); it is largest (~ 2) for p_x and p_z either both large or both small and smallest for p_x small and p_z large.

4.4.3 Two-particle distributions; jets?

The "exploding fly wheel" picture suggests further that we should expect several large $p_{\perp}$ in each event and not just one large $p_{\perp}$ compensated by many small ones. This is confirmed by our results: Fig. 11 shows in arbitrary units the two-particle distribution $\log_{10} W(p_{\perp 1}, p_{\perp 2})$; it is seen that $W(p_{\perp 1}, p_{\perp 2})$, as a function of $p_{\perp 2}$ for fixed $p_{\perp 1}$, drops faster when $p_{\perp 1}$ is small than when it is large; Fig. 12 dramatizes this in the form of $p_{\perp 1} p_{\perp 2}$ correlation

$$C(p_{\perp 1}, p_{\perp 2}) := \frac{W(p_{\perp 1}, p_{\perp 2})}{W(p_{\perp 1})W(p_{\perp 2})} \quad (4.15)$$

which varies from 0.6 to 320 and thus makes it obvious that there is a strong correlation such that if there is one large $p_{\perp}$, there is with overwhelming probability (at least) a second one.

As $p_{\perp} \cong |p_x|$ it could be that the second large $p_{\perp}$ is mostly in the opposite direction to the first and is just a convenient means to accomplish momentum conservation in the plane orthogonal to spin. We therefore looked also into the p_{x1}, p_{x2} distribution and found to our surprise something indicating a jet structure: namely if there is one particle with large p_x , there is (at least) another one with comparable $|p_x|$, and it may as well be on the same as on the opposite side (with a slight preference of the opposite side because of momentum conservation). Figure 13 shows $\log_{10} W(p_{x1}, p_{x2})$ in arbitrary units; the distribution is almost symmetric with respect to the p_{x1} axis (a little higher for $p_{x2} < 0$). Again the correlation function is spectacular:

$$C(p_{x1}, p_{x2}) := \frac{W(p_{x1}, p_{x2})}{W(p_{x1})W(p_{x2})} \quad (4.16)$$

is plotted logarithmically in Fig. 14, and linearly in Fig. 15 (in Fig. 14 the asymmetry, due to momentum conservation, is clearly visible but unimportant compared to the variations inside each half of the plot). The correlation varies from 0.2 to 4600; this is much more than the variation of $C(p_{\perp 1}, p_{\perp 2})$ which varied from 0.6 to 320 only.

That this indicates a jet structure is seen from a simple example: consider a momentum distribution on a circle, such that all momenta are equal (p)

$$\begin{aligned}
 p_x &= p_{\perp} \cos \varphi \\
 p_y &= p_{\perp} \sin \varphi \\
 W(p_{\perp}) &= \delta(p_{\perp} - p) \\
 W(p_{\perp 1}, p_{\perp 2}) &= \delta(p_{\perp 1} - p) \delta(p_{\perp 2} - p)
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} p_x &= p_{\perp} \cos \varphi \\ p_y &= p_{\perp} \sin \varphi \\ W(p_{\perp}) &= \delta(p_{\perp} - p) \\ W(p_{\perp 1}, p_{\perp 2}) &= \delta(p_{\perp 1} - p) \delta(p_{\perp 2} - p) \end{aligned}} \right\} \quad (4.17)$$

Then $C(p_{\perp 1}, p_{\perp 2}) = 1$, no matter what the angular distribution is. From simple geometry:

$$W(p_x) = W(\varphi) \cdot W(p_{\perp}) \cdot 2 \sqrt{p_{\perp}^2 - p_x^2} \quad (4.18)$$

where $W(\varphi)$ is the angular distribution. Now consider

$$W(p_{x1}, p_{x2}) = W(\varphi_1, \varphi_2) \cdot W(p_{\perp 1}, p_{\perp 2}) \cdot 2 \sqrt{p_{\perp 1}^2 - p_{x1}^2} \cdot 2 \sqrt{p_{\perp 2}^2 - p_{x2}^2} \quad (4.19)$$

Hence

$$C(p_{x1}, p_{x2}) = \frac{W(\varphi_1, \varphi_2)}{W(\varphi_1) W(\varphi_2)} \quad (4.20)$$

Now put

$$W(\varphi_1, \varphi_2) = \sqrt{\frac{A}{4\pi^3}} e^{-A(\varphi_1 - \varphi_2)^2}; \quad A \gg 4\pi^2 \quad (4.21)$$

Then, because of the large A , integrations can be extended to $\pm \infty$; we obtain

$$W(\varphi_2) = \sqrt{\frac{A}{4\pi^3}} \int_{-\infty}^{\infty} e^{-A(\varphi_1 - \varphi_2)^2} d\varphi_1 = \frac{1}{2\pi} \quad (4.22)$$

so that the single-particle distribution is isotropic. With this choice

$$C(p_{x1}, p_{x2}) = \sqrt{4\pi A} e^{-A(\varphi_1 - \varphi_2)^2} \quad (4.23)$$

reaches a peak value $\sqrt{4\pi A}$. Thus $C(p_{x1}, p_{x2})$ varies considerably between very small and very large values in contradistinction to $C(p_{\perp 1}, p_{\perp 2})$ which is identically 1. We thus are entitled to take the fact that our $C(p_{x1}, p_{x2})$ varies much more and reaches much higher values than $C(p_{\perp 1}, p_{\perp 2})$, as an indication of a jet structure.

This example shows also that the angular symmetry about the z axis, expressed by $W(\varphi) = 1/2\pi$, does in no way exclude jets; the jet directions are then randomly distributed about the symmetry axis, which in our case is the spin axis.

Nevertheless, we did not really expect jets from phase space with spin, because phase space normally aims at equidistribution. However, closer inspection of the complicated weight function, which spin forces onto phase space, reveals that indeed this weight function favours a distribution in the form of two jets in opposite directions ^{*)}; the jet axis lies preferentially in the plane perpendicular to the spin and varies randomly from event to event. It is remarkable that even for zero spin this structure survives (then the jet axis is arbitrary). As the details are somewhat technical, we defer them to Appendix B. We have not been able to find an intuitive explanation for this jet structure; however, the "exploding fly wheel" analogue does not forbid it.

5. CONCLUSIONS, ACKNOWLEDGEMENTS

We have shown that SBM and large transverse momenta do not exclude each other. Without claiming to have presented here the mechanism of large $p_{\perp}$ generation, it has become clear that high spin does play an important role in a mixture of different mechanisms which might be most difficult to disentangle. In any case we have demonstrated that among others, spin must be considered and that neglecting it (as one has done so far) charges other mechanisms with too heavy a burden.

Due to the technical complications in working out our intuitively simple model, we were not able to present results directly comparable to experiments; we started this work more than two years ago and could not readapt

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*) The two jets are opposite in the fireball rest frame, not in c.m.s.

it each time new experimental results became available; besides, having chosen a particular mechanism (single fireball) amongst those offered by the model and only a few mass values, we should not expect more than a qualitative agreement between our results and experiments. What unfortunately is missing, is a clear qualitative signature of our mechanism which could unmistakably distinguish it from other models and thereby lead to a prediction for some experimental crucis: all the qualitative features of other models, as well as those of experiments, reappear in our model.

The present work should be pushed further in several respects:

- i) incorporate transverse momentum of fireballs;
- ii) consider other spin generating processes, e.g., pionization;
- iii) incorporate resonances in phase space; we conject that not only higher momenta, but also the production of heavier masses, might be favoured in the decay of high spin high mass fireballs. This might not only be interesting in itself (corrections to the SBM prediction of exponential decrease with mass of the production rate of heavy pairs), it might substantially increase jet-like correlations;
- iv) improve computational methods for phase space with spin (most difficult!);
- v) find an exact solution (or a better approximation) for the Yellin coefficients with spin [Eqs. (3.6) and (3.19)];
- vi) do more computations (after the above improvements) at more masses, and aim at results directly related to presently measured quantities.

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APPENDIX A

INVARIANT PHASE SPACE WITH SPIN

1. Definition

In the rest frame of the normalization volume V_0 , the invariant phase space integral [IPS; see Ref. 15)] for n particles of equal mass m and total energy M is written

$$\Omega_m(M) = \left[\frac{V_0}{(2\pi)^3} \right]^n \int \delta(M - \sum_{i=1}^n \sqrt{p_i^2 + m^2}) \delta^3(\sum_{i=1}^n \vec{p}_i) \prod_{i=1}^n d^3 p_i \quad (\text{A.1})$$

In order to apply the procedure of Cerulus¹⁷⁾, we rewrite this in a more compact form which allows immediate generalization. Let n be kept fixed. We define

$$\begin{aligned} |\vec{p}\rangle &:= |\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n\rangle \\ |\vec{x}\rangle &:= |\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\rangle \end{aligned} \quad (\text{A.2})$$

with

$$\begin{aligned} \langle \vec{p}' | \vec{p} \rangle &= \delta^{3n}(\vec{p}' - \vec{p}) \\ \langle \vec{x}' | \vec{x} \rangle &= \delta^{3n}(\vec{x}' - \vec{x}) \\ \langle \vec{p} | \vec{x} \rangle &= (2\pi)^{-3n/2} e^{-i \vec{p} \vec{x}} \\ \vec{p} \vec{x} &:= \sum_{i=1}^n \vec{p}_i \vec{x}_i \end{aligned} \quad (\text{A.3})$$

Then

$$\Omega_m(M) = \text{Tr}(\delta(\hat{E} - M) \cdot \delta^3(\hat{\vec{P}}) \cdot \mathcal{P}_{V_0}) \quad (\text{A.4})$$

where $\hat{E}$ and $\hat{\vec{P}}$ are energy and three-momentum operators, and $\mathcal{P}_{V_0}$ the projection operator onto the states confined to V_0 . The normalization expressed by Eqs. (A.3) corresponds to IPS¹⁵⁾.

The generalization to fixed spin is now obvious: we introduce the projection operator onto total spin s, s_3 into (A.4). This operator reads ²³⁾

$$\mathbb{P}_{s, s_3} = \frac{2s+1}{8\pi^2} \int d\alpha d(\omega\beta) d\gamma D_{s_3 s_3}^{*(s)}(\alpha, \beta, \gamma) \cdot \hat{R}_a(\alpha, \beta, \gamma) \quad (\text{A.5})$$

$$\{0 \leq \alpha, \gamma < 2\pi; 0 \leq \beta < \pi\} = \text{Euler angles}$$

$$D_{mm}^{(s)}(\alpha, \beta, \gamma) = \text{irred. repr. of rotation group}$$

$$\hat{R}_a(\alpha, \beta, \gamma) = \text{operator of active rotation of states}$$

We abbreviate

$$\{\alpha, \beta, \gamma\} =: \omega \quad (\text{A.6})$$

$$d\alpha d(\omega\beta) d\gamma = d\omega$$

Now we define

$$\Omega_m(M, s, s_3) = \text{Tr}(\delta(\hat{E}-M) \delta^3(\vec{p}) \mathbb{P}_{V_0} \mathbb{P}_{ss_3}) \quad (\text{A.7})$$

$$= \left(\frac{1}{2\pi}\right)^{3n} \int_{V_0} d\vec{x} d\vec{p}' d\vec{p} \delta(M - \sum \sqrt{\vec{p}_i^2 + m^2}) \delta^3(\sum \vec{p}_i) e^{i(\vec{p}' - \vec{p}) \cdot \vec{x}} \langle \vec{p}' | \mathbb{P}_{ss_3} | \vec{p} \rangle \quad (\text{A.8})$$

where (A.3) has been used.

Denote the rotated state

$$\hat{R}_a(\alpha, \beta, \gamma) |\vec{p}\rangle =: |\vec{p}_\omega\rangle \quad (\text{A.9})$$

then the matrix element of the projector $\mathbb{P}_{ss_3}$ becomes

$$\langle \vec{p}' | P_{ss_3} | \vec{p} \rangle = \frac{2s+1}{8\pi^2} \int d\omega D_{s_3 s_3}^{*(s)}(\omega) \delta^{3n}(\vec{p}' - \vec{p}_\omega) \quad (\text{A.10})$$

and $\Omega_n(M, s, s_3)$ can be written

$$\Omega_n(M, s, s_3) = \left[\frac{V_0}{(2\pi)^3} \right]^n \int \delta(M - \sum_{i=1}^n \sqrt{p_i^2 + m^2}) \delta^3(\sum_{i=1}^n \vec{p}_i) F_{nss_3}(\vec{p}) \prod_{i=1}^n d^3 p_i$$

where

$$V_0^n F_{nss_3}(\vec{p}) := \frac{2s+1}{8\pi^2} \int d\omega D_{s_3 s_3}^{*(s)}(\omega) \int \prod_{k=1}^n d^3 k_k e^{i \vec{k}_k \cdot \vec{K}_k} \quad (\text{A.11})$$

$$\vec{K}_k := [\hat{R}_a(\omega) \cdot \vec{p}_k] - \vec{p}_k$$

In what follows, we shall put $|s_3| = s$, according to our physical situation.

2. Volume integration

We replace the volume integration by an integration over a properly normalized Gaussian density ¹⁷⁾:

$$\int_{V_0} d^3 k e^{i \vec{k} \cdot \vec{r}} \Rightarrow \int_{\infty} d^3 k e^{i \vec{k} \cdot \vec{r} - \frac{k^2}{4a^2}} = (a\sqrt{4\pi})^3 e^{-\frac{r^2}{4a^2}} \quad (\text{A.12})$$

The cut-off radius a is defined by requiring that for $K=0$ the integral has the value V_0 :

$$V_0 = \frac{4\pi}{3} m_\pi^{-3} = (a\sqrt{4\pi})^3 \quad (\text{A.13})$$

$$a^2 = 10.5 \text{ [GeV}^{-2}\text{]}$$

3. The weight function

The whole effect of spin is contained in the weight function $F_{n,s}$; if it is replaced by 1, ordinary phase space is retained. From (A.11) and (A.13):

$$F_{ns}(\vec{p}_1 \dots \vec{p}_n) = \frac{2s+1}{8\pi^2} \int d\omega D_{ss}^{*(s)}(\omega) e^{-a^2 \sum_{i=1}^n K_i^2}$$

$$K_i^2 = 2[p_i^2 - (\vec{p}_i, \hat{R}_a(\omega) \cdot \vec{p}_i)] \quad (A.14)$$

$$\begin{aligned} (\vec{p}, \hat{R}_a(\omega) \cdot \vec{p}) &= (p_x^2 + p_y^2) \cdot \cos(\alpha + \gamma) \cdot \frac{1 + \cos\beta}{2} \\ &\quad - (p_x^2 - p_y^2) \cdot \cos(\alpha - \gamma) \cdot \frac{1 - \cos\beta}{2} \\ &\quad - (p_x p_y) \cdot \sin(\alpha - \gamma) \cdot (1 - \cos\beta) \\ &\quad + (p_y p_z) \cdot (\sin\alpha + \sin\gamma) \cdot \sin\beta \\ &\quad + (p_z p_x) \cdot (\cos\alpha - \cos\gamma) \cdot \sin\beta \\ &\quad + p_z^2 \cdot \cos\beta \end{aligned} \quad (A.15)$$

$$D_{ss}^{*(s)}(\omega) = e^{is_3(\alpha + \gamma)} \left(\frac{1 + \cos\beta}{2} \right)^s;$$

$$|s_3| = s \quad (A.16)$$

Equation (A.15) suggests the introduction of the following invariants:

$$\begin{aligned}
 P &= 2a^2 \sum_{i=1}^m \vec{p}_i^2 \\
 T &= 2a^2 \sum_{i=1}^m (p_x^2 + p_y^2)_i \\
 Z &= 2a^2 \sum_{i=1}^m (p_z^2)_i \\
 D &= 2a^2 \sum_{i=1}^m (p_x^2 - p_y^2)_i \\
 C &= 2a^2 \sum_{i=1}^m (p_x p_y)_i \\
 Y &= 2a^2 \sum_{i=1}^m (p_y p_z)_i \\
 X &= 2a^2 \sum_{i=1}^m (p_x p_z)_i
 \end{aligned}
 \quad (T+Z=P)$$

(A.17)

With these, and inserting (A.16) and (A.15) into (A.14) we obtain

$$\begin{aligned}
 F_{ms} &= \frac{2s+1}{8\pi^2} e^{-P} \int d\alpha d\gamma d(\omega\beta) \cdot \left(\frac{1+\omega\beta}{2}\right)^s \cdot e^{iS_3(\alpha+\gamma)} \times \\
 &\times \exp \left\{ Z\omega\beta + \frac{T}{2}\cos(\alpha+\gamma)(1+\omega\beta) - \frac{D}{2}\cos(\alpha-\gamma)(1-\omega\beta) \right. \\
 &\quad - C\sin(\alpha-\gamma)(1-\omega\beta) + Y(\sin\alpha + \sin\gamma)\sin\beta \\
 &\quad \left. + X(\cos\alpha - \cos\gamma)\sin\beta \right\}
 \end{aligned}
 \quad (A.18)$$

Thus $F_{n,s}(\vec{p}_1 \dots \vec{p}_n)$ is a function of s, s_3 and the invariants defined in (A.17).

We show that the integral does not depend on the sign of s_3 and therefore must be real: as α and γ are integration variables and the integrand has period 2π in each of them, the integrand must be invariant under the three substitutions (among others)

$$\left\{ \begin{array}{l} \alpha \rightarrow \gamma \\ \gamma \rightarrow \alpha \end{array} \right\} \quad \left\{ \begin{array}{l} \alpha \rightarrow -\alpha \\ \gamma \rightarrow -\gamma \end{array} \right\} \quad \left\{ \begin{array}{l} \alpha \rightarrow \alpha + \pi \\ \gamma \rightarrow \gamma - \pi \end{array} \right\}$$

Carrying out these substitutions, one sees that they imply that the integral is invariant under

$$\left\{ \begin{array}{l} C \rightarrow -C \\ X \rightarrow -X \\ Y \rightarrow Y \\ S_3 \rightarrow S_3 \end{array} \right\} \quad \left\{ \begin{array}{l} C \rightarrow -C \\ X \rightarrow X \\ Y \rightarrow -Y \\ S_3 \rightarrow -S_3 \end{array} \right\} \quad \left\{ \begin{array}{l} C \rightarrow C \\ X \rightarrow -X \\ Y \rightarrow -Y \\ S_3 \rightarrow S_3 \end{array} \right\}$$

The product of these three invariance operations changes $s_3 \rightarrow -s_3$, leaving everything else as it is; hence $F_{n,s} = F_{n,s}^*$. We would have expected this result because we have nowhere built in parity violating factors, and must have a momentum distribution invariant under changing $s_3 = s \rightarrow s_3 = -s$. From now on we replace $\exp [is_3(\gamma + \alpha)]$ by $\cos s(\gamma + \alpha)$.

4. Computing phase space with the weight function $F_{n,s}$

Even without a weight function, phase space integrals for $n > 3$ cannot be solved analytically. The usual Monte-Carlo methods ²⁴⁾ allow for the presence of weight functions such as our $F_{n,s}$ and there are in principle no difficulties. However, for a reasonable accuracy of Monte-Carlo phase space we need a large number of samples $\{\vec{p}_1 \dots \vec{p}_n\}$ and for each of these samples the invariants P, T, Z, D, C, Y, X have to be computed, fed into (A.18), and $F_{n,s}$ has to be found. It turns out that for given M, s , we must altogether draw 10^5 to 10^6 such samples and therefore computing $F_{n,s}$ must be achieved in a fraction of a second, otherwise we run into prohibitive computing times. We are therefore forced to approximations, since ordinary three-dimensional integration routines take much too long. Computing $F_{n,s}$ fast and within a few per cent of accuracy is the main problem; the difficulty lies in the rapid oscillatory changes of the integrand,

typical values of the invariants T, Z, D, C, Y, X ranging from 10 to 1000; they appear in exponential functions!

5. Computing the weight function

We substitute

$$t^2 = \frac{1 + \cos \beta}{2}$$

$$4t dt = d(\cos \beta)$$

$$2t \sqrt{1-t^2} = \sin \beta$$

$$\xi = \alpha + \gamma - 2\pi$$

$$\eta = \gamma - \alpha$$

(A.19)

and obtain

$$F_{ms} = e^{-P} \frac{2s+1}{2\pi^2} \int_0^1 dt t^{2s+1} e^{Z(2t^2-1)} \times$$

$$\times \int_0^{2\pi} d\xi \cos(s\xi) e^{Tt^2 \cos \xi} H(\xi, t)$$

(A.20)

$$H(\xi, t) = \int_0^{2\pi-\xi} d\eta \times$$

$$\left\{ e^{(1-t^2)(2C \sin \eta - D \cos \eta)} \cdot \cosh \left[4t \sqrt{1-t^2} \left(Y \cos \frac{\eta}{2} + X \sin \frac{\eta}{2} \right) \sin \frac{\xi}{2} \right] \right.$$

$$\left. + e^{(1-t^2)(-2C \sin \eta - D \cos \eta)} \cdot \cosh \left[4t \sqrt{1-t^2} \left(Y \cos \frac{\eta}{2} - X \sin \frac{\eta}{2} \right) \sin \frac{\xi}{2} \right] \right\}$$

(A.21)

Now the magnitudes of the invariants are important. In an average over 50 samples we find for $M = 38 \text{ GeV}$

$n =$	10	20	25	30	40
$\langle Z \rangle$	1150	575	435	361	273
$\langle T \rangle$	2200	1100	900	746	551
$\langle D \rangle$	561	183	143	98	83
$\langle C \rangle$	309	117	61	55	41
$\langle Y \rangle$	260	107	72	54	31
$\langle X \rangle$	327	103	78	56	39

(A.22)

As our average multiplicity is about 20, we take the values for $n = 20$ as representative. Obviously, $F_{n,s}$ cannot be calculated analytically; what then, is the numerical behaviour of the integrand? Clearly T and Z govern its variation which goes from 0 (at $t = 0$) to $2\pi \exp(Z + T) = 4\pi \exp(1700)$ (at $t = 1, \xi = 0$); moreover it contains the rapidly oscillating factor $\cos(s\xi)$. Obviously ordinary numerical integration will meet with great difficulties. However, the strong variation of the exponential functions produce a very sharp maximum at $t = 1, \xi = 0$, so that contributions come only from the vicinity of that point.

The ξ integrand has two factors, $\cos(s\xi) \cdot \exp(Tt^2 \cos \xi)$, which together decrease (near $t = 1, \xi = 0$) rapidly like a Gaussian and $H(\xi, t)$, which increases very slowly compared to the fast decrease of the first factors. Extensive numerical studies have taught us that the function $\cos(s\xi) \cdot \exp(Tt^2 \cos \xi) \cdot H(\xi, t)$ has one absolute maximum at $t = 1, \xi = 0$ and, as a function of ξ , drops very rapidly. As $H(\xi, t)$ varies slowly compared to $\cos(s\xi) \exp(Tt^2 \cos \xi)$, it is reasonable to replace $H(\xi, t)$ by $H(0, t)$, which can be computed analytically, and then integrate over ξ , which then also can be done analytically so that we end up with a single numerical integration over t .

As our numerical studies are much too long to be reproduced here, we show by a rough estimate that the error induced by replacing $H(\xi, t)$ by $H(0, t)$ is indeed of the order of a few per cent only. We limit ourselves to the immediate neighbourhood of $t=1, \xi=0$; knowing that all contributions come from there, it does not matter if our error estimate becomes wrong farther away from that point.

We put

$$\begin{aligned} t &= 1-\varepsilon \\ 1-t^2 &= 2\varepsilon \end{aligned} \tag{A.23}$$

Next we majorize the exponents in $H(\xi, t)$ by replacing them with

$$\begin{aligned} R &:= \text{Max} (\pm 2C \sin \eta - D \cos \eta) = \sqrt{4C^2 + D^2} \\ Q &:= \text{Max} (Y \cos \frac{\eta}{2} \pm X \sin \frac{\eta}{2}) = \sqrt{X^2 + Y^2} \end{aligned} \tag{A.24}$$

Numerically

n	10	20	25	30	40
Q	418	149	106	78	50
R	805	297	188	147	117

(A.25)

We further write

$$\begin{aligned} \cosh(Q \cdot 4t \sqrt{1-t^2} \sin \frac{\xi}{2}) &\approx 1 + 2t^2(1-t^2)\xi^2 Q^2 \approx e^{\varepsilon \xi^2 4Q^2} \\ \cos(S\xi) &\approx 1 - \frac{1}{2}S^2\xi^2 \approx e^{-\frac{1}{2}S^2\xi^2} \\ t^{2S+1} &= e^{(2S+1)\ln t} \approx e^{-\varepsilon(2S+1)} \end{aligned} \tag{A.26}$$

With these approximations the whole integrand of (A.20) can be written near $t = 1$, $\xi = 0$

$$\text{Integrand} \approx \exp(-\alpha \epsilon) \exp(-\xi^2(\beta_0 - \epsilon \beta_1))$$

with

$$\alpha := 2s+1 + 4Z - 2T - 2R$$

$$\beta_0 := \frac{1}{2}(s^2 + T)$$

$$\beta_1 := T + 4Q^2 \approx 4Q^2 \quad (\text{A.27})$$

Numerically:

	s = 0	s = 40
α	≈ 4000	≈ 4000
β_0	550	1350
β_1	$\approx 9 \cdot 10^4$	$\approx 9 \cdot 10^4$

(n = 20)

(A.28)

The ξ dependence of $H(\xi, t)$ is contained in the term $-\epsilon \beta_1 \xi^2$ in the exponent of the second factor of (A.27). Replacing $H(\xi, t)$ by $H(0, t)$ is essentially equivalent to dropping that term. Now the first factor, $\exp(-\alpha \epsilon)$, decreases so rapidly that only very small ϵ values will contribute to the integral; if for these small ϵ values we have $\epsilon \beta_1 \ll \beta_0$, we may drop the term $\beta_1 \epsilon$. Indeed, by the time when $\epsilon \beta_1$ is of order β_0 , $\exp(-\alpha \epsilon)$ is of order $\exp(-50)$. We therefore obtain a reasonable error estimate by integrating (A.27) first over ξ from 0 to ∞ while considering $\epsilon \beta_1 \ll \beta_0$, and integrating then over ϵ from 0 to ∞ :

$$\begin{aligned} \int_0^\infty \exp(-\xi^2(\beta_0 - \epsilon \beta_1)) d\xi &\approx \sqrt{\frac{\pi}{\beta_0}} \left(1 + \frac{\epsilon \beta_1}{2\beta_0}\right) \\ \int_0^\infty \exp(-\alpha \epsilon) \left(1 + \frac{\epsilon \beta_1}{2\beta_0}\right) d\epsilon &= \text{const.} \cdot \left(1 + \frac{\beta_1}{2\alpha \beta_0}\right) \end{aligned} \quad (\text{A.29})$$

Replacing $H(\xi, t)$ by $H(0, t)$ is equivalent to neglect the correction term, which represents the relative error:

$$\frac{\beta_1}{2\alpha\beta_0} \approx \begin{cases} 2\% & \text{for } S=0 \\ 0.8\% & \text{for } S=40 \end{cases} \quad (\text{A.30})$$

We have supported this handwaving argument by a large number of numerical studies ranging over $0 \leq t \leq 1$ and $0 \leq \xi \leq 2\pi$.

We can in fact evaluate $H(\xi, t)$ up to linear terms in ξ : expanding $\cosh(\dots \sin \xi/2 \dots)$ will yield quadratic terms and thus only the ξ dependence in the upper limit of the integral remains. We write [see (A.21)]

$$\int_0^{2\pi-\xi} d\eta [\dots] = \int_0^{2\pi} d\eta [\dots] - \int_{2\pi-\xi}^{2\pi} d\eta [\dots] =: I_1 - I_2$$

the first integral is

$$I_1 = 4\pi I_0(R(1-t^2)) \quad (\text{A.31})$$

The second is written (after a Fourier expansion of $\cosh$)

$$\begin{aligned} I_2 &= 2 \int_0^{\xi} d\eta \cosh[R(1-t^2)\omega\eta] = \\ &= 2\xi + \sum_{n=1}^{\infty} a_n \sin n\xi \end{aligned} \quad (\text{A.32})$$

If the sum over $\sin n\xi$ is introduced into the integral over ξ [see (A.20)] it vanishes, while the term 2ξ does not. I_1 does not depend on ξ , hence the ξ integral in (A.20) gives up to terms linear in ξ

$$\begin{aligned} &\int_0^{2\pi} d\xi \omega(s\xi) e^{Tt^2\omega\xi} \cdot (4\pi I_0(R(1-t^2)) - 2\xi) \\ &= 4\pi^2 I_s(Tt^2) \cdot (2I_0(R(1-t^2)) - 1) \end{aligned} \quad (\text{A.33})$$

Equation (A.33) into (A.20) yields, substituting $t^2 \rightarrow x$

$$F_{ns}(\vec{p}_1 \dots \vec{p}_n) =$$

$$= (2s+1) e^{-P} \int_0^1 dx x^s e^{Z(2x-1)} I_s(Tx) \cdot [2I_0(R(1-x)) - 1] \quad (\text{A.34})$$

It may seem inconsistent to expand $H(\xi, t)$ to terms linear in ξ and then integrate over ξ ; in fact it is better than taking simply $H(0, t)$, which is already a good approximation, and it makes our approximation exact in the case $Z = P, R = T = Q = 0$ where the integral is different from zero only for $s = 0$ (for $s \neq 0$ there must be non-vanishing $p_{x,y}$) and also for $R = Q = 0$, but $T \neq 0$ [the case considered by Cerulus, Ref. 17)]. We take it for granted now that Eq. (A.34) represents our weight function within an error of a few per cent for reasonable values of the invariants (A.17); for values far off those given in the table (A.22), we do not know the error.

Quite apart from the remaining uncertainties in the computation of the weight function at non-typical values of the invariants, the Monte-Carlo method adds another inherent difficulty: the generation of events produces the ordinary phase space distribution with weight function 1, our weight function is then computed and attached to the event a posteriori. If our weight function favours a region in phase space which is relatively very thinly populated, then a correct result is obtained only if the total number of samples is so large that even this thinly populated region gets a large absolute number of events; if the number of samples is too small, it may happen that no event falls into the region favoured by $F_{n,s}$ and the resulting distributions are grossly wrong. Importance sampling would have been the obvious answer, but it seemed to us too complicated. We thus took always such a large number of events that in each phase space integral the effect of the weight function was clearly visible.

APPENDIX B

JET-LIKE STRUCTURE

We consider the weight function $F_{n,s}(\vec{p}_1 \dots \vec{p}_n)$ as given in Appendix A, Eqs. (A.17) and (A.18). As we saw that numerically jets seem favoured (Figs. 13, 14, subsection 4.4.3) and since our numerical results were obtained using the approximate weight function Eq. (A.34), this approximation with the invariants entering there must be sufficient. That leaves us with P, Z, T, R where

$$P = 2a^2 \sum_{i=1}^n \vec{p}_i^2$$

$$Z = 2a^2 \sum_{i=1}^n (p_z^2)_i$$

$$T = 2a^2 \sum_{i=1}^n (p_x^2 + p_y^2)_i = 2a^2 n \langle p_\perp^2 \rangle$$

$$R = \sqrt{4C^2 + D^2}$$

$$C = 2a^2 \sum_{i=1}^n (p_x p_y)_i$$

$$D = 2a^2 \sum_{i=1}^n (p_x^2 - p_y^2)_i$$

(B.1)

All momenta refer to the fireball rest frame.

The invariants X and Y ,

$$\begin{aligned} X &= 2a^2 \sum_{i=1}^n (p_x p_z)_i \\ Y &= 2a^2 \sum_{i=1}^n (p_y p_z)_i \end{aligned} \quad (\text{B.2})$$

have not survived the approximations leading to Eq. (A.34) and cannot be the origin of the jet effect.

Thus, the jet structure must come via the invariant R , the only one sensitive to the angular distribution in the xy plane.

We introduce polar co-ordinates

$$\begin{aligned} p_x &= p_{\perp} \cos \varphi \\ p_y &= p_{\perp} \sin \varphi \end{aligned} \quad (\text{B.3})$$

then

$$R = 2a^2 \left[\left(\sum_{i=1}^n (p_{\perp}^2 \sin 2\varphi)_i \right)^2 + \left(\sum_{i=1}^n (p_{\perp}^2 \cos 2\varphi)_i \right)^2 \right]^{1/2} \quad (\text{B.4})$$

As the $p_{\perp}$ distribution has nothing to do with the angular distribution, we assume, for the sake of argument only, that

$$(p_{\perp}^2)_i = \langle p_{\perp}^2 \rangle = \text{const. for all } i = 1 \dots n .$$

With that simplification

$$R = 2a^2 \langle p_{\perp}^2 \rangle \left[\left(\sum_{i=1}^n \sin 2\varphi_i \right)^2 + \left(\sum_{i=1}^n \cos 2\varphi_i \right)^2 \right]^{1/2} \quad (\text{B.5})$$

Now consider three limiting cases:

- i) equidistribution: $\varphi_i = 2\pi/n \cdot i$; $R = 0$;
- ii) statistical distribution: $R \cong 2a^2 \langle p_{\perp}^2 \rangle \cdot \sqrt{n}$;

iii) perfect jet:

$$\varphi_{2k} = \pi ; \varphi_{2k+1} = 0 ; R = 2a^2 \langle p_{\perp}^2 \rangle \cdot n$$

Thus, with $2a^2 \langle p_{\perp}^2 \rangle = T/n$

$$R_{\text{equidistr.}} : R_{\text{stat. distr.}} : R_{\text{perfect jet}} \approx 0 : \frac{T}{\sqrt{n}} : T \quad (\text{B.6})$$

Numerically from the table (A.25),

$$\left. \begin{array}{l} R_{\text{equidistr.}} = 0 \\ R_{\text{stat. distr.}} \approx 250 \\ R_{\text{perfect jet}} \approx 1100 \end{array} \right\} \text{typical values } (n \approx 20) \quad (\text{B.7})$$

While the influence of the angular distribution on R is spectacular, not much remains if the influence of R on the weight function (A.34) is studied. Obviously, F increases if R increases, but R enters in the combination $R \cdot (1-x)$ and noticeable contributions to the integral come only from $x \approx 1$. We get an idea of what happens numerically by replacing in Eq. (A.34)

$$I_s(Tx) \Rightarrow \frac{1}{\sqrt{2\pi Tx}} e^{Tx} ; 1-x \Rightarrow t \quad (\text{B.8})$$

and using $Z = P - T$ to obtain

$$\begin{aligned} F_{ns}(\vec{p}_1, \dots, \vec{p}_n) &\approx \\ &\approx \frac{2s+1}{\sqrt{2\pi T}} \int_0^{t_0} dt (1-t)^{s-1/2} e^{-t(2P-T)} [2I_0(Rt) - 1] \end{aligned} \quad (\text{B.9})$$

Our numerical studies of the original integral (A.34) show that all essential contributions come from a small neighbourhood of $x=1$ ($t=0$) where the approximation just made is very good; in this region $(1-t)^{s-1/2} \approx 1$, so we drop this factor and can now extend the integration to ∞ (unless $T \cong R \cong P$; we assume that $P-T$ may be $\ll P$ but not zero). Hence

$$F_{ns}(p_1 \dots p_n) \approx \frac{2s+1}{\sqrt{2\pi T}} \int_0^\infty dt e^{-t(2P-T)} [2I_0(Rt) - 1]$$

$$= \frac{2s+1}{\sqrt{2\pi T}} \left[\frac{2}{\sqrt{(2P-T)^2 - R^2}} - \frac{1}{2P-T} \right] \quad (B.10)$$

We see that $F_{n,s}$ increases when R increases and the more so the larger T is.

In order to consider the tendency for jet formation at fixed fireball mass M , we try to estimate P as a function of M and of the multiplicity n . In the decay we have

$$\sum_{i=1}^n (p_i^2 + m^2) = nm^2 + \sum p_i^2 = n \cdot \overline{\epsilon^2}$$

$$M^2 = n^2 \cdot \overline{\epsilon^2} \quad (B.11)$$

If roughly $\overline{\epsilon^2} \approx \bar{\epsilon}^2$, then

$$\sum_{i=1}^n p_i^2 \approx \frac{M^2 - (nm)^2}{n} \approx \frac{M^2}{n} \quad (B.12)$$

so that by order of magnitude

$$P \approx \frac{2a^2 M^2}{n} \approx \frac{21}{n} \cdot M^2 \quad (B.13)$$

With $M = 38$ GeV, $n = 20$ we find $P \approx 1500$ in reasonable agreement with the table (A.22) [$\langle P \rangle = \langle T \rangle + \langle Z \rangle \simeq 1700$ there].

We have studied the numerical behaviour of Eq. (B.10) for $M = 40$ GeV and various values of n and T . For R we considered the three cases

$$\begin{aligned} \text{equidistribution} & : R = 0 \\ \text{statistical distribution} & : R = T/\sqrt{n} \\ \text{jet distribution} & : R = T \end{aligned} \quad (B.14)$$

It turns out that for $n \gtrsim 5$ the difference in $F_{n,s}$ between equidistribution and statistical distribution is negligible; we thus consider from now on only the difference between statistical distribution and jet. We found that if one plots the ratio

$$J: = \frac{F_{ns}(R=T(\text{jet}))}{F_{ns}(R=\frac{T}{\sqrt{n}}(\text{statist.}))} \quad (\text{B.15})$$

as a function of T/P ($T/P = 2/3$ for isotropy, $= 1$ for a distribution in the spin orthogonal plane: all $p_z = 0$), all curves for different $n = 5, 6, \dots, 35$ fall nearly on top of each other. We show the average of all these curves in Fig. 16. The effect on the weight function is surprisingly small in view of the large $p_{x1}p_{x2}$ correlations it produces; we have no explanation for this puzzle.

SMOOTHING MONTE-CARLO RESULTS

Our Monte-Carlo output is too strongly fluctuating for immediate use. Figure 17 shows the original Monte-Carlo scattergram of $\log_{10} W(p_{\perp 1}, p_{\perp 2})$. We apply the following smoothing procedure to all such scattergrams (not to their logarithm): let W_{ik} be the (30×30) matrix representing the values, then

$$\begin{aligned} \overline{W}_{ik} := \frac{1}{9} [& W_{i-1, k-1} + W_{i-1, k} + W_{i-1, k+1} \\ & + W_{i, k-1} + W_{i, k} + W_{i, k+1} \\ & + W_{i+1, k-1} + W_{i+1, k} + W_{i+1, k+1}] \end{aligned} \quad (C.1)$$

for all $2 \leq i, k \leq 29$ (i.e., first and last rows and columns are left untouched). This smoothing is applied four times, then the logarithm is taken and the first and last rows and columns are found by linear extrapolation of the logarithms. Figure 18 shows the result of this procedure applied to Fig. 17. Next we reduce the number of bins from 30×30 to 15×15 and smooth once more; Fig. 19 shows the final result.

Comparing Fig. 19 with Fig. 17 one feels perhaps that this is a very bad procedure; in fact it is a good one; but by plotting logarithmically the result of an arithmetic smoothing, it looks bad, while the result of a geometric smoothing [procedure (C.1) applied to logarithms of scattergrams] would look much better. Figure 20 shows the difference between geometric and arithmetic smoothing on a momentum spectrum: the arithmetic one is correct, because we smooth out histograms and scattergrams representing counting rates.

REFERENCES

- 1) F.W. Büsser et al. (CERN-Columbia-Rockefeller Collaboration),
Phys. Letters 46B, 471 (1973);
and a wealth of other papers which may be found from conference
proceedings and review articles.
- 2) S.C. Frautschi and C.J. Hamer, Phys. Rev. D4, 2125 (1971).
- 3) R. Hagedorn and I. Montvay, Nuclear Phys. B59, 45 (1973).
- 4) E.M. Ilgenfritz and J. Kripfganz, Nuclear Phys. B56, 241 (1973).
- 5) R. Hagedorn and J. Ranft, Suppl. Nuovo Cimento 6, 169 (1968).
- 6) R. Hagedorn, "Thermodynamics of Strong Interactions", Lectures given
in the Academic Training Programme of CERN 1970-1971, CERN 71-12.
- 7) K. Imaeda, Nuovo Cimento Letters 1, 290 (1971).
- 8) See, for instance,
S.D. Ellis, M. Jacob and P.V. Landshoff, Nuclear Phys. B108, 93 (1976),
and the earlier literature quoted there.
- 9) A. Bouquet, J. Letessier and A. Tounsi, Phys. Letters 51B, 235 (1974).
- 10) R. Hagedorn, "Relativistic Kinematics", Section 9.6, W.A. Benjamin Inc.,
Reading, Mass. (3rd printing, 1973).
- 11) P. Grassberger, private communication (1975).
- 12) R. Hagedorn and U. Wambach, "Structure of Large $p_{\perp}$ Events in a
Statistical Model with A.M. Conservation", Talk delivered by
U. Wambach, "Workshop on Theoretical Physics", Center for
Scientific Culture "Ettore Majorana", Erice, Sicily, April/May
1975 (unpublished); and
"Large Transverse Momenta from High Spin Clusters", Talk delivered
by U. Wambach at the CERN ISR Meeting June 12, 1975 (unpublished;
contained in "ISR Discussion Meetings between Experimentalists and
Theorists", No. 14, edited by M. Jacob, CERN).
- 13) P. Bosetti et al., Aachen-Berlin-Bonn-CERN-Heidelberg-London-Vienna
Collaboration, Nuclear Phys. B101, 304 (1975).
- 14) J. Yellin, Nuclear Phys. B52, 583 (1973).
- 15) M. Chaichian, R. Hagedorn and M. Hayashi, Nuclear Phys. B92, 445 (1975).
- 16) K. Fabricius and U. Wambach, Nuclear Phys. B62, 212 (1973).
- 17) F. Cerulus, Nuovo Cimento 22, 958 (1961).
- 18) T. Ericson, Nuovo Cimento 21, 605 (1961); see also Ref. 17);
H. Satz, Fortschr. d. Phys. 11, 445 (1963);
H. Joos and H. Satz, Nuovo Cimento 34, 619 (1964);
K. Zalewski, Acta Phys. Pol. 28, 933 (1965);
A. Kotanski and K. Zalewski, Nuovo Cimento 40, 134 (1965).

- 19) C.B. Chiu and R.L. Heimann, Phys. Rev. D4, 3184 (1971).
- 20) See Ref. 5), furthermore
H. Grote, R. Hagedorn and J. Ranft, "Particle Spectra", Atlas of
Particle Production Spectra, Black Report CERN 1970;
R. Hagedorn and J. Ranft, Nuclear Phys. B48, 157 (1972);
J. Ranft, Phys. Letters 41B, 613 (1972);
E. Matthäus and J. Ranft, Phys. Letters 40B, 230 (1972);
H.R. Gerhold and B. Buschbeck, Nuovo Cimento 22A, 39 (1974);
K. Hänssgen, Leipzig Preprint KMU-HEP 7605 (March 1976).
- 21) M. Antinucci et al., Lett. Nuovo Cimento 6, 121 (1973).
- 22) R. Hagedorn, J. Reinfelds, C. Vandoni and L. Van Hove, "SIGMA, a
New Language for Array-Oriented Computing", Yellow Report
CERN 73-5 (1973).
- 23) E.P. Wigner, "Group Theory and ... ", Academic Press (1959).
- 24) For an introduction and bibliography, see:
E. Byckling and K. Kajantie, "Particle Kinematics" Section IX,
J. Wiley and Sons Ltd. (1973).

Fig. 1a Impact parameter distribution for fixed l , p . Shaded part R serves as weight function for Fig. 1b.

FIG. 1a

Fig. 1b Angular momentum distribution for fixed p and for b taken from R (Fig. 1a) with weight $W(b)$.

FIG. 1b

FIG. 2

Fig. 2 Geometry of the collision. Shaded region is "geometrical overlap ΔV " of particle α .

FIG. 3a

Fig. 3a Different model assumptions and their corresponding spin-mass relations.

- A Pionization (PI)
- B Collective motion generated spin
- C Naive guess [$s = b(q - p)$]
- D Kinematical impact parameter conservation
- E Geometrical impact parameter conservation
- F Rigorous single fireball (SF)
- G Separate centre-of-gravity conservation (α and β)
- CROSSES: experimental values with errors [Ref. 13)].

FIG. 3 b

Fig. 3b The effect of varying the amount of energy transfer K between the clusters; $K=1$ is a single fireball, $K=0.5$ a symmetrical double fireball. The letters B, D, E, G have the same meaning as in Fig. 3a.

FIG. 4

Fig. 4 Weights representing roughly $d\sigma/dM$ for single fireball excitation. Black circles: fitted to 52 GeV data; black triangles: computed for 32 GeV from Eq. (4.9).

FIG.5

Fig. 5 The transverse momentum distribution $d\sigma/dp_x$ obtained by superposing the low p_x exponential curve and the single fireball contribution with weights $W(M,E)$. Six weights to produce the 52 GeV curve were fitted to the data, those for the 32 GeV curve were computed from Eq. (4.9).

FIG.6

Fig. 6 Our c.m.s. co-ordinates for single particle distributions and correlations.

FIG. 7

Fig. 7 Spin-parallel $[W(p_z)]$ and spin-orthogonal $[W(p_x)]$ single particle distribution, showing a much faster decrease of $W(p_z)$. See also Fig. 8.

FIG. 8

Fig. 8 Ratio $W(p_x)/W(p_z)$ of spin-orthogonal to spin-parallel single particle distribution. The spin-parallel component was computed only up to 3 GeV.

FIG.9

Fig. 9 The double differential single particle distribution $W(p_x, p_z)$ up to 3 GeV/c, showing the angular anisotropy suggested by the "exploding fly wheel" picture: large momenta mainly in the spin-orthogonal plane.

FIG.10

Fig. 10 The single particle $p_x p_z$ correlation

$$C(p_x, p_z) = W(p_x, p_z) / [W(p_x)W(p_z)]$$

FIG. 11

Fig. 11 The logarithmic two-particle transverse momentum distribution (fireball rest frame) $W(p_1, p_2)$.

FIG.12

Fig. 12 The two-particle transverse momentum correlation (fireball rest frame)

$$C(p_{\perp 1}, p_{\perp 2}) = W(p_{\perp 1}, p_{\perp 2}) / [W(p_{\perp 1})W(p_{\perp 2})],$$

showing a strong preference for events with several (at least two) large $p_{\perp}$. Compare, however, Figs. 14 and 15.

FIG.13

Fig. 13 The logarithmic two-particle p_x distribution

$$(0 \leq p_{x1} \leq 6, -6 \leq p_{x2} \leq 6 \text{ GeV}) .$$

The drop in $|p_{x2}|$ is more pronounced for small p_{x1} than for large ones.

FIG.14

Fig. 14 The two-particle p_x correlation (logarithm)

$$C(p_{x1}, p_{x2}) = W(p_{x1}, p_{x2}) / [W(p_{x1})W(p_{x2})] \quad ,$$

showing a strong preference for a second large $|p_x|$ on the same as well as on the opposite side. This logarithmic representation shows that the correlation increases smoothly to the maxima and is not due to remnants of Monte-Carlo fluctuations.

FIG.15

Fig. 15 The two-particle $p_{x1}p_{x2}$ correlation

$$C(p_{x1}, p_{x2}) = W(p_{x1}, p_{x2}) / [W(p_{x1})W(p_{x2})]$$

showing very strong peaks for two large p_x , same or opposite sign. This correlation is by an order of magnitude larger than the $p_{\perp 1}, p_{\perp 2}$ correlation (Fig. 12), and thus suggests a jet-like structure (see subsection 4.4.3).

FIG. 16

Fig. 16 The ratio of the Cerulus weight functions ¹⁷⁾ for perfect jet and for random distribution in the spin orthogonal plane, as function of the ratio $T = \text{sum of transverse momenta squared}$ to $P = \text{sum of total momenta squared}$. The effect is surprisingly small in view of the large correlation it produces (Fig. 15).

FIG.17

Fig. 17 Our smoothing procedure: original Monte-Carlo output of $W(p_{11}, p_{12})$, unsuitable for direct use.

FIG.18

Fig. 18 Our smoothing procedure: Eq. (C.1) applied four times to Fig. 17.

FIG.19

Fig. 19 Our smoothing procedure: reducing the number of bins to 15×15 and smoothing once more with linear logarithmic extrapolation to obtain the boundaries: final results used in all figures.

FIG.20

Fig. 20 The difference between arithmetic and geometric smoothing: in a log representation the geometric average looks better than the arithmetic one, but is in fact wrong because counting rates must be averaged arithmetically.