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A QUANTUM FIELD THEORY FOR THE SCATTERING OF HADRONS WITH
AN INFINITE MASS SPECTRUM

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A B S T R A C T

We examine the possibility of constructing a quantum field theory of the scattering of a system of hadrons, which lie on Regge trajectories, by associating the spectrum with a mass operator involving an internal coordinate. We show that, provided the construction ensures no tachyon states, the latter can be used to derive a causal bare theory, which leads to a generalized Feynman graph expansion of the S-matrix. In an explicit example involving an oscillator mass spectrum and a generalized ϕ^3 interaction, the tree graphs are shown to be multi-Regge behaved. However the latter is linked to unphysical infra-red singularities coming from the structure of the mass operator. We try to resolve the latter problem in the renormalization program, where we indicate that the theory is renormalizable and that loop insertions in the generalized propagator leave its structure basically unchanged. As far as the physical interpretability of the theory is concerned we propose a solution, which merges the bootstrap idea with a quark model type mass spectrum.

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1. PREAMBLE

Hadrons are known to lie on Regge trajectories and can be thought of as excited states of some basic system in analogy to the excited states of an atomic system. Furthermore if one rationalizes this implicit internal structure of hadrons in terms of quark degrees of freedom one obtains a number of striking phenomenological successes, the most notable recently is the charmonium picture¹⁾. This strongly suggests that quarks are indeed the essential ingredients of hadronic structure. However, in hadron-hadron collisions, even at the highest available energies, one never sees the elusive quark constituents, instead an ever increasing abundance of new hadrons is produced. This should be contrasted with high-energy atom-atom collisions, in which the opposite situation is true, namely the production of new intact atomic systems will be very rare compared with the production of electrons, protons and photons. This means that the binding mechanism in the case of hadrons must in some way be fundamentally different and it is by no means clear a priori that one should treat the quarks as conventional field quanta^{2,3,4)}. On the other hand, the hadrons themselves appear very much like observable quanta of a Q.F.T. The bootstrap hypothesis was hoped to be a way of rationalizing this state of affairs, since it suggests that the forces, which bind the hadrons, are mediated by the hadrons themselves. However, this idea has never been realized within a Q.F.T. and it is difficult to reconcile it with the quark model. With regard to this problem, perhaps the most promising lines of development is the dual model⁵⁾, which while being a realization of the bootstrap hypothesis, allows a peculiar quark interpretation associated with the boundaries of planar diagrams⁶⁾. The current hope is to derive the latter model within a non-Abelian Q.F.T. of quarks and gluons⁷⁾. However, the formidable task of solving the bound state problem has yet to be accomplished and it is by no means clear that this can be done without some additional assumptions, which lie outside the conventional framework.

Another point of view is to assume that a bare theory can be constructed by treating whatever is responsible for the mass spectrum separately from the interaction, by which the hadrons interact with one another. Then provided the renormalization does not alter the basic structure of the spectrum, the bootstrap idea can be realized in the latter stage. The present article describes an attempt to construct a quantum field theory of the hadronic scattering problem along these lines.

The basic idea we have in mind is a simple generalization of a multi-component field theory, where the components are associated with some discrete internal

symmetry [e.g. $SU(n)$]. In order to have an infinite mass spectrum, where the hadron states lie on Regge trajectories, we have, instead of a discrete, a continuous internal co-ordinate, which for example could be the relative vector between a quark-antiquark pair. This means that the basic field will be a bilocal object $\phi(x,r)$. The idea of such fields is an old one and was introduced originally by Yukawa⁸⁾. However, the objective at that time was somewhat different. In particular it was hoped that the internal structure would remove the renormalization problem. In our case, as far as the dependence on the centre-of-mass co-ordinate X^μ is concerned, the framework we develop here is essentially the same as an ordinary Q.F.T. In fact, by defining a local theory with respect to X^μ , we will derive a generalized Feynman graph expansion of the scattering matrix and discuss a renormalization program closely analogous to that encountered in a usual Q.F.T.

There is, however, at the outset a conceptual problem, namely, since the interaction will necessarily couple the excited states to the ground state continuum, they will be unstable. This kind of problem arises in the Weinberg-Salam⁹⁾ model, where for example the muon is unstable with respect to the decay $\mu \rightarrow e \bar{\nu}$. One might argue that if the energy, say for the scattering process $e \nu \rightarrow \mu \bar{\nu}$, is high enough so that the muon lives sufficiently long compared with the time of the interaction, one can neglect the instability problem. At sufficiently high energies this will also apply to our case. However, we shall proceed by constructing a bare theory and then argue that the instability problem can be dealt with in the renormalization program.

Although our approach will be developed along the lines of a conventional Q.F.T. as far as quantization and locality is concerned, there is one important innovation, namely in order to ensure a positive-definite mass spectrum and causal Green functions, the internal "motion" (i.e. the relative co-ordinate) must be coupled to the total momentum and the mass operator of the theory must have additional infra-red singularities. We shall show that the latter are directly linked to Regge behaviour at the tree graph level. Although within the framework we can give no simple physical interpretation of these infra-red singularities, we will speculate that this might be resolved within the renormalization program.

In Section 2 we discuss how we introduce a mass spectrum through wave functions and a mass operator and in Section 3 we construct the free quantized bilocal field for this system of hadrons and the causal Green functions governing the propagation. Hence we show why a positive-definite mass spectrum leads to the presence of additional infra-red singularities in the Green functions and that causality requires these to be included in the quantization and in the definition of the Feynman

function. The scattering problem is discussed in Section 4, where we introduce a generalized local interaction for the bilocal fields and discuss how one constructs a unitary S-operator for the bare theory, in which the instability problem is ignored. The Feynman graph expansion of the Dyson representation of this S-operator is given in Section 5. In Section 6 we examine the structure of the bare theory at the tree graph level. This we do by considering an explicit example, which, in a limiting case, corresponds to the non-relativistic oscillator mass spectrum. Here we show that the tree graphs are multi-Regge behaved and satisfy the Steinman conditions. Further we discuss in this section how tree graph unitarity links this Regge behaviour to the infra-red singularities of the mass operator. The renormalization program is outlined in Section 7, where we demonstrate that the infinite mass renormalizations lead to a renormalization of the intercept of the Regge trajectory, we also indicate that even though the basic divergences in the generalized ϕ^3 example are of higher degree due to the infinite summation over poles, the model is nevertheless super renormalizable in the sense of Weinberg power counting. Finally, we discuss how, in a non-perturbative situation, the Dyson equations of the renormalized theory, may be used to relate the Regge behaved renormalized tree graphs to higher order graphs involving loops, We find that only in this situation can we relate the s-channel discontinuity of the Regge poles to physical intermediate states.

2. MASS SPECTRUM AND MESON WAVE FUNCTIONS

We introduce the mass spectrum into the theory by assuming that a free meson state can be described by a wave function $\Psi(p, x_1, x_2)$, which depends on the total momentum and two co-ordinates x_1 and x_2 , the latter for example can be thought of as the co-ordinates of a quark-antiquark pair and describe the internal structure of the meson system (see Ref. 2)). The essential property we will ascribe to this wave function is that we can separate out its centre-of-mass system motion, so that it can be written in the form

$$\Psi(p, x_1, x_2) = e^{i p \cdot X} \psi(p, r) \quad (2.1)$$

where

$$X^\mu = \frac{1}{2} (x_1^\mu + x_2^\mu) \quad \text{and} \quad r^\mu = x_1^\mu - x_2^\mu$$

The relative wave function $\psi(p, r)$ is defined through the following eigenvalue equation, which defines the mesonic mass spectrum

$$\mathcal{M}^2(p, \partial_r, r) \varphi_\alpha(p, r) = m_\alpha^2 \varphi_\alpha(p, r) \quad (2.2)$$

Equation (2.2) may be accompanied with subsidiary and boundary conditions. α represents a set of discrete quantum numbers, which characterize the spectrum. The mass operator $\mathcal{M}^2$ could for example correspond to the non-relativistic quark model¹⁰⁾ or the spherical cavity model of confined quarks in Ref. (2). The dependence of $\mathcal{M}^2$ on the total momentum p^μ is necessary in order to guarantee a positive-definite meson mass spectrum. Furthermore in order that we can construct causal Green functions and generalized free fields, which correspond to a system of particles with definite spin, the wave functions must satisfy the following conditions for arbitrary time-like p^μ ($p^2 > 0$):

i) They should be normalizable and form an orthonormal set, i.e.,

$$\int d^4r \varphi_\alpha^*(p, r) \varphi_{\alpha'}(p, r) = \delta_{\alpha\alpha'} \quad (2.3)$$

ii) We shall also require the completeness relations

$$\sum_\alpha \varphi_\alpha(p, r) \varphi_\alpha^*(p, r') = \delta^4(r - r') \quad (2.4)$$

iii) In order that the particle states correspond to definite spin, the wave functions must be separable and have the following tensor structure

$$\begin{aligned} \varphi_\alpha(p, r) = & F_J\left(\frac{p \cdot r}{\sqrt{p^2}}, -r^2 + \frac{(p \cdot r)^2}{p^2}\right) \chi_{\mu_1 \dots \mu_J}^{\alpha_1 \dots \alpha_J}(p) r_{\alpha_1} \dots r_{\alpha_J} \\ & \times \epsilon_{\lambda}^{\mu_1 \dots \mu_J}(p) \end{aligned} \quad (2.5)$$

where $\epsilon_{\lambda}^{\mu_1 \dots \mu_J}(p)$ is the polarization tensor and $\chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(p)$ is a totally symmetric tensor projection operator of rank J satisfying

$$\begin{aligned}
 (a) \quad & p^{\mu} \chi_{\mu \mu_2 \dots \mu_J}^{\nu_1 \dots \nu_J}(p) = 0 \\
 (b) \quad & g^{\mu_1 \mu_2} \chi_{\mu_1 \mu_2 \dots \mu_J}^{\nu_1 \dots \nu_J}(p) = 0 \\
 (c) \quad & \chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(p) = \chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(p) \\
 (d) \quad & \chi_{\mu_1 \dots \mu_J}^{\alpha_1 \dots \alpha_J} \chi_{\alpha_1 \dots \alpha_J}^{\nu_1 \dots \nu_J} = \chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}
 \end{aligned}
 \tag{2.6}$$

Conditions (i) to (iii) strongly restrict the class of mass operators $\mathcal{M}^2(p, \partial_r, r)$ and we shall focus our attention on the following type

$$\begin{aligned}
 \mathcal{M}^2(p, \partial_r, r) = & \left[\lambda g_T^{\mu\nu}(p) + g_S^{\mu\nu}(p) \right] \partial_{r_\mu} \partial_{r_\nu} + \\
 & + V_T \left(\frac{p \cdot r}{\sqrt{p^2}} \right) + V_S \left(-r^2 + \frac{(p \cdot r)^2}{p^2} \right)
 \end{aligned}
 \tag{2.7}$$

where $g_{S,T}^{\mu\nu}(p)$ are internal metric tensors or projection operators and are respectively defined by

$$\begin{aligned}
 g_S^{\mu\nu}(p) &= g^{\mu\nu} - p^{\mu} p^{\nu} / p^2 \\
 g_T^{\mu\nu}(p) &= p^{\mu} p^{\nu} / p^2
 \end{aligned}
 \tag{2.8}$$

By defining the variables $\tau^2 = g_T^{\mu\nu}(p) r_\mu r_\nu = (p \cdot r)^2 / p^2$ and $\vec{r} = \vec{\epsilon}_\mu(p) r^\mu$, where $\epsilon_\mu^i(p)$ is the polarization vector associated with the momentum p^μ (i.e., $p^\mu \epsilon_\mu^i(p) = 0$, $\vec{\epsilon}_\mu(p) \cdot \vec{\epsilon}_\nu(p) = -g_{S\mu\nu}$ and $g^{\mu\nu} \epsilon_\mu^i \epsilon_\nu^j = \delta_{ij}$) we can see that $\psi(p, r)$ has a separable form $T(\tau) R(\vec{r})$ with the "space" part satisfying the Schrödinger-like equation

$$\left[-\vec{\nabla}_{\vec{r}}^2 + V_S(\vec{r}^2) \right] R(\vec{r}) = k^2 R(\vec{r}) \quad (2.9)$$

From (2.9) it is a simple matter to show the wave function ψ_α fulfils condition (iii) and to see that the projection operator $\chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}$ has the explicit form

$$\chi_{\mu_1 \dots \mu_J}^{\nu_1 \dots \nu_J}(p) = T_{j_1 \dots j_J}^{i_1 \dots i_J} \epsilon_{\mu_1}^{j_1}(p) \dots \epsilon_{\mu_J}^{j_J}(p) \epsilon_{i_1}^{\nu_1}(p) \dots \epsilon_{i_J}^{\nu_J}(p) \quad (2.10)$$

where $T_{j_1 \dots j_J}^{i_1 \dots i_J}$ is the totally symmetric traceless tensor of rank J introduced by Fierz (Ref. (11)).

The full wave function $\Psi(p, x, r)$ satisfies the wave equation

$$\left[\square_x + M^2(i\partial, \partial_r, r) \right] \Psi(p, x, r) = 0 \quad (2.11)$$

for $p^2 = m_\alpha^2$; $\Psi(p, x, r) = e^{ip \cdot x} \psi_\alpha(p, r)$. The dependence of the mass operator on the total momentum $p^\mu = i \partial^\mu$, implies that (2.11) has additional solutions other than those given by (2.1) and (2.2). The nature and rôle of these extra solutions will first become apparent when we construct causal Green functions for the propagation of this system of meson states.

3. BILOCAL FIELDS AND GREEN FUNCTIONS

We can describe the above system of meson states in terms of a free bilocal field constructed in the following way

$$\phi(x, r) = \sum_{\alpha} \frac{1}{(2\pi)^{3/2}} \int d^4 p \, \theta(p_0) \delta(p^2 - m_\alpha^2) \times \left[\varphi_\alpha(p, r) e^{-ip \cdot x} + \varphi_\alpha^*(p, r) e^{ip \cdot x} \right] \quad (3.1)$$

This field satisfies the equation of motion (2.11), i.e.,

$$[\square_x + M^2(i\partial, \partial_r, r)] \phi(x, r) = 0 \quad (3.2)$$

Provided that the wave functions $\psi_\alpha(p, r)$ satisfy

$$\psi_\alpha^*(p, r) = \psi_\alpha(-p, r) \quad (3.3)$$

then the field $\phi(x, r)$ may be expressed in terms of a generalized Fierz decomposition¹¹⁾

$$\phi(x, r) = \sum_\alpha \psi_\alpha(i\partial, r) \phi_\alpha(x) \quad (3.4)$$

where $\phi_\alpha(x)$ is the usual free field corresponding to the meson state α .

Further, since we consider only wave functions with the form given in (2.5), each component $\phi_\alpha = \phi_{\mu_1 \dots \mu_J}^{(\bar{\alpha}, J)}$ corresponds to a free field describing a particle of mass m_α^2 and definite spin J , i.e.,

$$\begin{aligned} (a) \quad & [\square + m_\alpha^2] \phi_{\mu_1 \dots \mu_J}^{(\bar{\alpha}, J)}(x) = 0 \\ (b) \quad & \partial^{\mu_1} \phi_{\mu_1 \dots \mu_J}^{(\bar{\alpha}, J)}(x) = 0 \\ (c) \quad & g^{\mu_1 \mu_2} \phi_{\mu_1 \mu_2 \dots \mu_J}^{(\bar{\alpha}, J)}(x) = 0 \\ (d) \quad & \phi_{\dots \mu_i \dots \mu_j \dots}^{(\bar{\alpha}, J)}(x) = \phi_{\dots \mu_j \dots \mu_i \dots}^{(\bar{\alpha}, J)}(x) \end{aligned} \quad (3.5)$$

Since the states corresponding to the ϕ_α are the only observables and not the internal motion, the quantum field $\phi(x, r)$ is defined by quantizing the components $\phi_\alpha(x)$ in the conventional way by writing

$$\phi_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p \theta(p_0) \delta(p^2 - m_\alpha^2) (A_\alpha(p) e^{-ip \cdot x} + A_\alpha^\dagger(p) e^{ip \cdot x}) \quad (3.6)$$

where $A_\alpha^\dagger(p)$ and $A_\alpha(p)$ are the creation and annihilation operators of the particle with mass m_α and internal quantum numbers α , satisfying

$$\begin{aligned} [A_\alpha(p), A_{\alpha'}^\dagger(p')] &= \delta_{\alpha\alpha'} 2p_0 \delta^{(3)}(p-p') \\ [A_\alpha(p), A_{\alpha'}(p')] &= [A_\alpha^\dagger(p), A_{\alpha'}^\dagger(p')] = 0 \end{aligned} \quad (3.7)$$

The commutator of $\phi(x,r)$, i.e.,

$$i \Delta(x-x'; r, r') = [\phi(x, r), \phi(x', r')] \quad (3.8)$$

is determined by the Fierz decomposition (3.4), which leads to

$$\begin{aligned} i \Delta(x; r, r') &= \sum_{\alpha} \psi_{\alpha}(i\partial, r) \psi_{\alpha}(-i\partial, r') i \Delta(x; m_{\alpha}^2) \\ &= \frac{1}{(2\pi)^3} \int d^4p e^{-ip \cdot x} \sum_{\alpha} \psi_{\alpha}(p, r) \psi_{\alpha}^*(p, r') \\ &\quad \times \varepsilon(p_0) \delta(p^2 - m_{\alpha}^2) \end{aligned} \quad (3.9)$$

$\Delta(x; r, r')$ does not vanish outside the light cone of the centre-of-mass co-ordinate x^μ , which in one sense reflects the fact that $\phi(x, r)$ describes an extended object. However, as we shall see, it is also related to the fact that we have not quantized all the solutions of the basic equations (3.2).

In order to introduce Green functions one has to be able to analytically continue the wave functions $\psi_{\alpha}(p, r)$ to complex p^μ or more precisely the following sum

$$g(p; r, r') = \sum_{\alpha} \frac{\psi_{\alpha}(p, r) \psi_{\alpha}^*(p, r')}{p^2 - m_{\alpha}^2} \quad (3.10)$$

The commutator (3.9) can then be represented by the contour integral

$$\Delta(x; r, r') = \frac{1}{(2\pi)^4} \int_C d^4p e^{-ip \cdot x} g(p; r, r') \quad (3.11)$$

where the contour C in the p_0 integration encloses the poles in $g(p; r, r')$ at $p^2 = m_{\alpha}^2$ in a counter clockwise sense (Fig. 1).

Fig. 1

$\Delta(X; r, r')$ satisfies the homogeneous equation (3.2) and if $g(p; r, r')$ had no additional singularities in the p_0 plane other than the poles of $p_0 = \pm \sqrt{m_\alpha^2 + \vec{p}^2}$, and converged sufficiently fast at infinity, then it would vanish outside the light cone $X^2 = 0$. However, such wave functions do not exist, since they will either have singularities at finite p^2 or will be entire functions, which are singular at infinity. Thus $g(p; r, r')$ will always have additional singularities in p^2 . Causality requires all singularities to be located on the real p_0 axis, i.e., $p^2 \geq 0$ and since by assumption the complete spectrum for $p^2 > 0$ is given by the eigenvalue equation (2.2), any additional singularities must be located at $p^2 = 0$. From Fig. 1, we see that if we define a contour $C_1 = C + C_0$, where C_0 encloses the $p^2 = 0$ singularities in the p_0 plane and provided $g(p; r, r')$ converges sufficiently fast at infinity, then the function

$$\Delta_1(X; r, r') = \frac{1}{(2\pi)^4} \int_{C_1} d^4 p e^{-i p \cdot X} g(p; r, r') \quad (3.12)$$

will be causal. In order to construct advanced and retarded Green functions using Δ_1 , the extra pieces Δ_0 , defined in terms of the contour C_0 in Fig. 1 must satisfy the homogeneous equation of motion (3.2), i.e., correspond to the additional solutions mentioned in Section 2. This can only come about if the mass operator in Eqs.(2.2) and (3.2) is infra-red singular. In Section 5 we will consider an explicit example, in which this is the case. For the moment we discuss how one

constructs Green functions for any mass operator with the above properties. The advanced and retarded Green functions are defined by

$$\underline{G}_{R,A}(x;r,r') = \int_{C_{R,A}} \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} g(p,r,r') \quad (3.13)$$

where the contours C_R and C_A are shown in Fig. 1 and the contours C_1 , C_0 , C_R and C_A are related by

$$C_1 = C_A - C_R = C + C_0$$

From this definition and the assumptions we have made concerning the analyticity and convergence of $g(p;r,r')$ it is a simple matter to show that G_A and G_R have the following properties:

- (1) $\underline{G}_R(x;r,r') = -\theta(x_0) \Delta_1(x;r,r')$
- (2) $\underline{G}_A(x;r,r') = \theta(-x_0) \Delta_1(x;r,r')$
- (3) $\underline{G}_A(x;r,r') = \underline{G}_R(-x;r,r') \quad (3.14)$
- (4) $\underline{G}_R(x;r,r') - \underline{G}_A(x;r,r') = -\Delta_1(x;r,r')$
- (5) $[\square_x + M^2(i\partial_x, \partial_r, r)] \underline{G}_{R,A}(x-x';r,r') = \delta^4(r-r') \delta^4(x-x')$

Property (5) relies in addition on the completeness relation (2.4), which has to analytically continue to complex p . Since we have needed to include the extra singularities at $p^2 = 0$, in order to define strictly local Green functions and that these singularities correspond to additional solutions of the equation of motion in Eq. (3.2), we should include them in the definition of the free bilocal field $\phi(x,r)$. Δ_0 is then simply the commutator of this extra piece. We thus

introduce a new sector to the Fock space, corresponding to the infra-red part of $g(p;r,r')$. For simplicity we denote this by the state $|I.R\rangle$. This is not necessarily a one-particle state and its interpretation will only become clearer when we introduce an interaction and a unitary scattering matrix for this system of bound states. In fact, unitarity requires the inclusion of this piece and provides what seems to be a unique interpretation of $|I.R\rangle$ in terms of Regge singularities in the Born term of the four-point function. Although one could, in principle, proceed by avoiding the introduction of this extra component in the free field (i.e., the $p^2 = 0$ piece), one of the prices one would have to pay is a violation of microscopic causality coming from the absence of the part Δ_0 (the latter just cancels the part of Δ outside the light cone $X^2 = 0$). However, a more problematic point will be the effect of its absence on the unitarity relations¹²⁾.

Finally, by dividing $\Delta_1(X;r,r')$ (i.e., $\phi(X,r)$) into positive and negative frequency parts, with an appropriate generalization for the infra-red component, we can define the Feynman function for this bilocal field

$$\Delta_F(x-x';r,r') = -i \langle 0 | T \{ \phi(x,r), \phi(x',r') \} | 0 \rangle$$

with the property

$$\Delta_F(x-x';r,r') = -\Delta_A(x-x';r,r') + \Delta^{(+)}(x-x';r,r') \quad (3.15)$$

where the time order is defined with respect to the centre-of-mass co-ordinate X_0 .

4. THE SCATTERING PROBLEM

An interaction can be formally introduced in a completely analogous way to an ordinary scalar field theory, by adding a source term to the right-hand side of Eq. (3.2), i.e.,

$$[\square_x + M^2(i\partial, \partial_r, r)] \phi(x,r) = g \rho(x,r) \quad (4.1)$$

where ρ is assumed to be obtained from an action functional $W[\phi]$ of the field ϕ , i.e.,

$$g \rho(x, r) = \frac{\delta W[\phi]}{\delta \phi(x, r)} \quad (4.2)$$

Equation (4.1) can be integrated using the Green functions defined in Section 3, leading to the integral equation

$$\phi(x, r) = \phi_{out}(x, r) - g \int d^4x' d^4r' G_R(x-x'; r, r') \rho(x', r') \quad (4.3)$$

where ϕ_{in} and ϕ_{out} satisfy the homogeneous wave equation and are respectively identified as incoming and outgoing fields. Using (3.14) ϕ_{out} and ϕ_{in} are related by

$$\phi_{out}(x, r) = \phi_{in}(x, r) - g \int d^4x' d^4r' \Delta_1(x-x'; r, r') \rho(x', r') \quad (4.4)$$

where Δ_1 is the causal commutator defined in (3.12). Using these last two equations the interacting field ϕ and hence ϕ_{out} can be expressed in terms of ϕ_{in} by means of a perturbation expansion of the Yang-Feldman type¹³⁾ leading to the transformation

$$\phi_{out}(x, r) = S^\dagger \phi_{in}(x, r) S \quad (4.5)$$

where S is a functional of ϕ_{in} .

A critical condition for S to be a unitary operator is that the commutators of the in and out fields are equal, i.e.,

$$[\phi_{out}(x, r), \phi_{out}(x', r')] = [\phi_{in}(x, r), \phi_{in}(x', r')] \quad (4.6)$$

This relation will be satisfied to all orders in the coupling constant only if the source term $\rho(X,r)$ is local in the centre-of-mass co-ordinate $X^{(14)}$, i.e.,

$$\frac{\delta \rho(x',r')}{\delta \phi(x,r)} = 0 \quad \text{for } x' \neq x \quad (4.7)$$

In addition ϕ_{in} and ϕ_{out} must contain the extra infra-red piece discussed in Section 3. This treatment, of course, can only refer to the bare theory, in which ϕ_{in} is identified with the free field constructed in Section 3, since it ignores the renormalization problem. One consequence of renormalization is that the higher-lying states in the mass spectrum become unstable and decay into the ground state continuum, so that the physical asymptotic field is made up of only the renormalized ground state sector of the Fock space. In order to proceed in perhaps the most simple way, we depart from the LSZ formulation of the renormalized scattering problem, by initially defining the bare scattering theory and tackle the instability problem in the renormalization program, which is discussed in Section 7. The advantage of considering the bare theory first lies in maintaining the separation of the binding mechanism from the interaction, by which the hadrons in the mass spectrum scatter. From the phenomenological point of view this will allow us to isolate, for example, amplitudes for $\pi\pi \rightarrow \rho\rho$ or $\pi\rho \rightarrow \pi\rho$. As far as unitarity is concerned the bare theory would correspond to approximating the resonance intermediate states by a narrow width limit. The full renormalized theory will be very similar to the treatment of Swanziger¹⁵⁾ of unstable particles in an S-matrix approach, where it was shown that these can be handled in much the same way as stable particle poles, except that they lie on unphysical sheets of the scattering amplitudes.

Using the orthonormality condition (2.3) of the internal wave functions, we can show, for each component in the Fierz decomposition (3.4), that

$$\phi_{out}^\alpha(x) = S^\dagger \phi_{in}^\alpha(x) S$$

which means that S is the scattering operator of the bare theory.

Apart from the infinite mass renormalization, which in Section 7 we will demonstrate to be a renormalization of the Regge intercept, the bare theory may be a good approximation to the full theory, since in summing over the intermediate states, the replacement of Breit-Wigner resonance forms by a narrow width approximation is often a reliable procedure.

5. GENERALIZED FEYNMAN RULES FOR THE BARE THEORY

For the bare theory, we can derive Feynman rules for calculating arbitrary scattering amplitudes, provided that we choose a quasi-local interaction, in which the interaction Lagrangian is local with respect to the external co-ordinate X^μ . In this case the S-operator defined in (4.5) has a generalized Dyson representation, namely

$$S = T \exp \{ i W [\phi_{in}] \} \quad (5.1)$$

where the time ordering is with respect to X_0 . The crucial point in the proof of (5.1) is that this time ordering is relativistic invariant. Since the generalization to higher order couplings is straightforward, for simplicity in constructing the Feynman rules, we consider here only the following generalized ϕ^3 coupling

$$W [\phi] = \frac{g}{3} \int d^4x d^4r_1 d^4r_2 d^4r_3 \tilde{O}(r_1, r_2, r_3) \phi(x, r_1) \phi(x, r_2) \phi(x, r_3) \quad (5.2)$$

where $\tilde{O}(r_1, r_2, r_3)$ is a well-behaved c-number weight function. Inserting this interaction into the Dyson representation, the time ordered products can be decomposed into normal order products and Feynman propagators according to Wick's theorem. Hence if we assume the existence of a unique vacuum state with the properties

$$|0\rangle_{in} = |0\rangle_{out} = |0\rangle ; \quad \langle 0 | 0 \rangle = 1 \quad (5.3)$$

on which the operators defined in (3.7) create and annihilate positive normed particle states, then the calculation of the S-matrix

$$S_{fi} = \langle in, f | S | in, i \rangle \quad (5.4)$$

reduces to a straightforward generalization of the Feynman graph expansion of the ordinary ϕ^3 theory. The only new innovation is the integration over relative

momentum, which is analogous to the summation over internal symmetry indices in, e.g., a gauge theory with an $SU(n)$ symmetry.

However, as we mention in Section 3, we have also to include an extra sector to the Hilbert space corresponding to the $p^2 = 0$ (infra-red) singularity. This sector is denoted by $|I.R\rangle$ and defined so that $\langle 0|I.R\rangle = \langle p, \alpha|I.R\rangle = 0$. Since we will not try to analyse $|I.R\rangle$ in terms of conventional particle states, its interpretation will only become apparent through the unitarity relations of the scattering amplitudes.

The Feynman scattering amplitude T_{fi} for the transition $i \rightarrow f$ is defined through

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^4(p_f - p_i) T_{fi} \quad (5.5)$$

where P_i^μ and P_f^μ are the initial and final total momentum, respectively. In order to write down the graphical rules for calculating T_{fi} in momentum space, we need the relevant functions in momentum space, namely:

1) The wave function. When the normal ordered expressions in the Wick expanded S-operator acts on the external states they give rise to a plane wave factor in the centre-of-mass co-ordinate multiplied by the internal wave function, i.e.,

$$\begin{aligned} \phi_{in}^{(+)}(x, r) |p, \alpha\rangle &= (2\pi)^{-3/2} \varphi_\alpha(p, r) e^{-i p \cdot x} |0\rangle \\ \langle p, \alpha | \phi_{in}^{(-)}(x, r) &= \langle 0 | (2\pi)^{-3/2} \varphi_\alpha^*(p, r) e^{i p \cdot x} \end{aligned} \quad (5.6)$$

In momentum space the wave function is given by

$$u_\alpha(p, q) = (2\pi)^{-3/2} \int d^4 r e^{-i q \cdot r} \varphi_\alpha(p, r) \quad (5.7)$$

where q^μ is the relative momentum vector.

2) The propagator. The contractions of the Wick expanded time ordered products contain the generalized Feynman propagator

$$i \Delta_F(x-x'; r, r') = \langle 0 | T \{ \phi_{in}(x, r) \phi_{in}(x', r') \} | 0 \rangle$$

which, in momentum space is given by

$$\Delta_F(p; q, q') = \int d^4x d^4r d^4r' e^{-ip \cdot x} e^{iq \cdot r} e^{-iq' \cdot r'} \times \Delta_F(x; r, r') \quad (5.8)$$

3) The relative momentum vertex function. Finally, in momentum space the integrals over relative momenta involve the weight function $\sigma(q_1, q_2, q_3)$, where

$$\sigma(q_1, q_2, q_3) = \int d^4r_1 d^4r_2 d^4r_3 e^{-iq_1 \cdot r_1 - iq_2 \cdot r_2 - iq_3 \cdot r_3} \times \tilde{\sigma}(r_1, r_2, r_3) \quad (5.9)$$

In terms of these functions an arbitrary scattering amplitude can be calculated order by order in the coupling constant using the following graphical rules.

1. For each external line corresponding to an incoming or outgoing hadron with momentum p^μ and internal excitation quantum numbers α , we have the following factor with its graphical notation

$$\begin{array}{l} \text{Diagram 1: A horizontal line with an arrow pointing right. At the left end, there is a curly bracket labeled 'p, \alpha'. At the right end, there is a curly bracket.} \rightarrow U_\alpha(p, q) \text{ incoming} \\ \text{Diagram 2: A horizontal line with an arrow pointing right. At the left end, there is a curly bracket. At the right end, there is a curly bracket labeled 'p, \alpha'.} \rightarrow U_\alpha^*(p, q) \text{ outgoing} \end{array}$$

2. For an internal hadron line we have a generalized Feynman propagator depending on the total four-momentum carried by it and the relative momenta q^μ and q'^μ associated with each end respectively

$$\text{Diagram: A horizontal line with an arrow pointing right. At the left end, there is a curly bracket labeled 'q'. At the right end, there is a curly bracket labeled 'q''. The line is labeled 'p' in the middle.} \rightarrow i \Delta_F(p, q, q')$$

3. For an arbitrary vertex connecting internal and external lines, we have the following factor involving the weight function σ and a δ -function for the four momentum conservation of the external momenta.

$$i g (2\pi)^4 \delta^4 (p_1 + p_2 + p_3) \sigma(q_1, q_2, q_3)$$

4. For each internal hadron line, we have the integral

$$\int \frac{d^4 p}{(2\pi)^4}$$

and for each relative momentum the integral

$$\int \frac{d^4 q}{(2\pi)^4}$$

5. In the Feynman amplitude defined in (5.5) the over-all four-momentum conservation δ -function

$$-i (2\pi)^4 \delta^{(4)}(P_f - P_i)$$

is factored out.

This set of rules describes essentially a local field theory, involving a mass spectrum instead of a single mass. The Feynman amplitude for each order in the coupling constant is obtained by applying these rules to each topologically distinct graph and adding these graphs together. Since the wave functions depend on the scalar product $p \cdot q$ and the Feynman propagator on $k \cdot q$ and $q \cdot q'$ the external momenta will be coupled in a highly non-trivial fashion already at the tree graph level and in order to analyse this, we now turn to an explicit example.

6. SCATTERING AMPLITUDES AT THE TREE GRAPH LEVEL

In order to demonstrate that a positive mass spectrum in a quantum field theory of the type we propose here already implies a rich structure at the tree graph level, we shall consider an explicit model for the mass operator $\mathcal{M}^2(p, \partial_r, r)$ defined in Section 2. We will show here that the Born terms are Regge behaved and that the multi-Regge behaviour of the higher point functions satisfy the Steinmann condition¹⁶⁾. Furthermore we indicate why the infra-red sector $|I.R\rangle$ is linked to this Regge behaviour through unitarity.

We shall work with a mass operator, which yields the non-relativistic oscillator spectrum. To start with we consider the following mass operator¹⁷⁾

$$\mathcal{M}^2(p, \partial_r, r) = -m_0^2 + \lambda M_T^2 + M_S^2$$

where

$$M_{T,S}^2 = m_1^2 g_{T,S}^{\mu\nu} [p^2 \partial_{r_\mu} \partial_{r_\nu} - p^{-2} r_\mu r_\nu] \quad (6.1)$$

and $g_T^{\mu\nu}$ and $g_S^{\mu\nu}$ were the internal metric tensors defined in Section 2 (Eq. 2.8). The metric coefficients $g_T^{\mu\nu}$ and $g_S^{\mu\nu}$ are simply projection operators, respectively, projecting out the relative time and space parts in the rest system $p^\mu = (\sqrt{p^2}, \vec{0})$. If $\lambda = +1$ one recovers the usual four-dimensional oscillator model¹⁸⁾ which has tachyon solutions. By choosing $\lambda \leq 0$ we can ensure a positive-definite mass spectrum. It is a simple matter to solve (2.2) for $p^2 > 0$, with this mass operator and we refer to the work of Sogami in Ref. (17) for details. The mass spectrum is given by

$$m_\alpha^2 = -m_0^2 - 2\lambda m_1^2(n_0 + \nu_2) + 2m_1^2(2n + l + 3/2) \quad (6.2)$$

where n_0, n and ℓ are the time, radial and angular momentum excitation numbers, respectively. Equation (6.2) corresponds to linear Regge trajectories with alternate daughters, i.e.

$$\alpha_{n_0 n}(\tau) = \lambda n_0 - 2n + \alpha(\tau)$$

with

$$\alpha(\tau) = \alpha_0 + \alpha' \tau$$

$$\alpha_0 = \frac{m_0^2 - (-\lambda + 3)m_1^2}{2m_1^2} \quad (6.3)$$

$$\alpha' = \frac{1}{2m_1^2}$$

where $\alpha_0 < 0$ ensures positive-definite masses. In the limit $\lambda \rightarrow -\infty$ (α_0 fixed) the time excitations are suppressed and we are left simply with the non-relativistic oscillator model. For arbitrary time-like momentum p^μ and $\lambda > 0$ the eigenfunctions can be written in the form

$$\psi_{\alpha}(p, r) = T_{n_0} \left(\frac{p \cdot r}{\rho \sqrt{p^2}} \right) R_{n\ell} \left(\frac{|r|}{\rho} \right) Y_{\ell m}(\Omega \bar{r}) \quad (6.4)$$

where $\vec{r} = \vec{\epsilon}_\mu(p) r^\mu$ and $\vec{\epsilon}_\mu(p)$ is the polarization vector associated with the momentum p^μ (i.e., $p^\mu \epsilon_\mu^i = 0$, $\vec{\epsilon}_\mu(p) \vec{\epsilon}_\nu(p) = -g_{S\mu\nu}(p)$ and $g^{\mu\nu} \epsilon_\mu^i \epsilon_\nu^j = \delta^{ij}$). The normalized wave functions are given by

$$T_{n_0}(\tau) = \frac{1}{[\sqrt{\pi} \rho 2^{n_0} n_0!]}^{1/2} e^{-\tau^2/2} H_{n_0}(\tau)$$

$$R_{n\ell}(r) = \left[\frac{n! (2\ell+1)!}{(2\pi) \rho^{3n+\ell} \Gamma(2n+\ell+3/2) \ell!} \right]^{1/2} e^{-r^2/2} \times L_n^{\ell+1/2}(r^2) \quad (6.5)$$

where H_n and L_n^α are Hermite and associated Laguerre polynomials, respectively. The functions $\Psi_\alpha(p, r)$ satisfy the orthogonality and completeness relations (2.3) and (2.4) for $p^2 > 0$. Furthermore, they are independent of λ so that we can take the limit $\lambda \rightarrow -\infty$ without effecting these properties.

A useful representation of the Feynman propagator can be derived, starting from the generating function

$$\tilde{G}(p, r, r', z) = \sum_{\alpha} z^{M(\alpha)} \frac{\psi_{\alpha}(p, r) \psi_{\alpha}^*(p, r')}{p^2 - m_{\alpha}^2} \quad (6.6)$$

where $M(\alpha) = -\lambda n_0 + 2n + \ell$. This function satisfies the relation

$$\begin{aligned} \frac{1}{\alpha} \left(\alpha(p^2) - z \frac{d}{dz} \right) \tilde{G}(p, r, r', z) \\ = [p^2 - M^2(p, r, r')] \tilde{G}(p, r, r', z) \end{aligned} \quad (6.7)$$

and hence the equation

$$\frac{1}{\alpha} \left(\alpha(p^2) - z \frac{d}{dz} \right) \tilde{G}(p, r, r', z) = \tilde{g}(p, r, r', z) \quad (6.8)$$

where

$$\tilde{g}(p, r, r', z) = \sum_{\alpha} z^{M(\alpha)} \psi_{\alpha}(p, r) \psi_{\alpha}^*(p, r') \quad (6.9)$$

The Green function is given by (see Eq. 6.6)

$$\tilde{G}(p, r, r') = \tilde{G}(p, r, r', 1)$$

for which an integral representation can be obtained by solving the differential equation (6.8). One finds

$$\tilde{G}(p, r, r') = \frac{\alpha'}{2i \sin \pi \alpha(p^2)} \int_C \frac{dz}{z} (-z)^{-\alpha(p^2)} \tilde{g}(p, r, r', z) \quad (6.10)$$

where the contour C starts from 1 and encloses the branch point of $(-z)^{-\alpha(p^2)}$ in a counter clockwise direction, ending below the cut at 1. One can readily check that this Green function satisfies all the conditions in Section 3 (e.g. $\tilde{G}(p; r, r')$ can be extended to complex p_0 and is sufficiently convergent as $|p_0| \rightarrow \infty$).

The utility of (6.10) becomes immediately clear, when we note that for the oscillator model $G(p; r, r', z)$ can be summed up into a closed form¹⁷⁾ which in the momentum space is given by (after taking the limit $\lambda \rightarrow -\infty$)

$$\begin{aligned} g(p, q, q', z) &= \sum_{n, \ell} z^{2n+\ell} u_{n\ell}(p, q) u_{n\ell}(p, q') \\ &= \left(\frac{p^2}{\pi}\right)^2 (1-z^2)^{-3/2} e^{-\frac{p^2}{2} \left\{ \frac{(p \cdot q)^2}{p^2} + \frac{(p \cdot q')^2}{p^2} \right\}} \quad (6.11) \\ &\times \exp \left\{ \frac{p^2}{4} g_S^{\mu\nu}(p) \left[(q-q')_\mu (q-q')_\nu b(z) \right. \right. \\ &\quad \left. \left. + (q+q')_\mu (q+q')_\nu b^{-1}(z) \right] \right\} \end{aligned}$$

with

$$b(z) = \frac{1+z}{1-z}$$

The Feynman propagator then has the simple representation

$$\Delta_F(p, q, q') = \frac{\alpha'}{2i \sin \pi \alpha(p^2)} \int_C \frac{dz}{z} (-z)^{-\alpha(p^2)} g(p, q, q', z) \quad (6.12)$$

where the $i\epsilon$ prescription is understood. In (6.11) and (6.12) the nature of the infra-red singularity manifests itself as an essential singularity at $p^2 = 0$.

The ground state wave function is given by

$$u_0(p, q) = (2\pi)^{1/2} p^2 e^{-p^2 \left[\frac{(p \cdot q)^2}{m^2} - \frac{1}{2} q^2 \right]} \quad (6.13)$$

The wave functions for excited states can be generated from (6.13) in the usual way for an oscillator model. Finally we choose the simplest interaction of the type (5.1) which has the following weight function

$$\tilde{\sigma}(r_1, r_2, r_3) = \delta^4(r_1 + r_2 + r_3) \quad (6.14)$$

which in the momentum space is given by

$$\sigma(q_1, q_2, q_3) = (2\pi)^8 \delta^4(q_1 - q_2) \delta^4(q_2 - q_3) \quad (6.15)$$

according to (5.9).

Using the Feynman rules given in Section 5, we shall now calculate the four- and five-point functions at the tree graph level. However, in order to directly calculate amplitudes valid for arbitrary (complex) external momenta we have to define the integrals over the relative momenta, which involve Gaussian integrands. These will be defined for arbitrary complex momenta by

$$\int d^4q \exp \{ -q^\mu M_{\mu\nu} q^\nu \} = \frac{\pi^2}{[\det M_{\mu\nu}]^{1/2}} \quad (6.16)$$

where $M_{\mu\nu}$ contains the external momenta. Since the left-hand side does not converge for all values of the momenta, there will appear new kinds of singularities. As far as we have investigated they always lie outside the physical region and do not pose any direct interpretational problem. These singularities as well as the Regge behaviour are a reflection of the infra-red singularity in the Green function. We shall return to this point later.

The four-point function

In the case of the four-point function $h(p_1) + h(p_2) \rightarrow h(p_2) + h(p_3)$, the Born terms are the second order graphs shown in Fig. 2.

Fig. 2

The s-graph in Fig. 2 corresponds to the expression

$$T_s(s, t) = \frac{s^2}{(2\pi)^8} \int d^4q d^4q' \mathcal{U}(p_1, q) \mathcal{U}(p_2, q) \times \Delta_F(p_1 + p_2, q, q') \mathcal{U}^*(p_3, q') \mathcal{U}^*(p_4, q') \quad (6.17)$$

Using ground state wave functions and the representation (6.12) for the Feynman propagator we obtain after performing the relative momentum integrals

$$T_s(s, t) = \frac{(g\rho^2)^2 \alpha'}{i 8 (2\pi)^4 \sin \pi \alpha(s)} \int_C \frac{dz}{z} (-z)^{-\alpha(s)} (q - z^2)^{-1/2} \times \frac{1}{\left[\frac{s}{m^2} - 1\right]} \frac{1}{\left[\left[3\left(\frac{s}{m^2} - 1\right) - z^2\left(\frac{s}{m^2} - 3\right)\right]^2 - \frac{4z^2}{m^4}(t - u)^2\right]^{1/2}} \quad (6.18)$$

(The q^μ and q'^μ integrations are most easily performed by choosing the frame

$$P_{1,3}^\mu = (E, \pm Q/2, 0, P) \quad , \quad P_{2,4}^\mu = (E, \pm Q/2, 0, -P)$$

and changing the variables according to $U = (q - q')$ and $V = (q + q')$.)

The t -graph is given by

$$\begin{aligned} T_t(s, t) &= \frac{g^2}{(2\pi)^8} \int d^4q d^4q' \, u(p_1, q) u^*(p_3, q) \\ &\quad \times \Delta_F(p_1 - p_3, q, q') u(p_2, q') u^*(p_4, q') \\ &= \frac{(g p^2)^2 \alpha'}{(2\pi)^4 8 i \sin \pi \alpha(t)} \int_C \frac{dz}{z} (-z)^{-\alpha(t)} [1 - t/m^2]^{-1} \\ &\quad \times \frac{(q - z^2)^{-1/2}}{\left[\left[3 \left(\frac{t}{m^2} - 1 \right) - z^2 \left(\frac{t}{m^2} - 3 \right) \right]^2 - \frac{4 z^2}{m^4} (u-s)^2 \right]^{1/2}} \end{aligned} \quad (6.19)$$

By expanding around $z = 0$, one can readily check that the s -graph has the particle poles with positive factorizable residues. However, at the unphysical point $s = m^2$, there is a dipole and a triple pole. The extra singularity is a property of the vertex coupling the composite hadrons together and will occur in all residue functions. These are manifestations of the new kind of singularities mentioned above.

As $s \rightarrow \infty$ the s -graph is determined by the behaviour near $z = 1$ in the z integration of the Feynman propagator. For $s \rightarrow \infty$, t fixed by rotating the contour C to infinity except near $z = \pm 1$, where it surrounds the branch points from the denominator factor in the integrand, we can write (6.18) in the form:

$$\begin{aligned} T_s(s, t) &= \lim_{s \rightarrow \infty} \frac{(g p^2)^2 \alpha'}{8 i \sin \pi \alpha(s)} e^{i\pi \alpha(s)} s^{-5/2} \\ &\quad \times \left\{ \int_{C_1} dy \frac{(1+y/s)^{-\alpha' s}}{\sqrt{2m^2 - t - y}} - e^{i\pi \alpha(s)} \int_{C_{-1}} dy' \frac{(1+y'/s)^{-\alpha' s}}{\sqrt{2m^2 - t - y'}} \right\} \\ &\sim (g p^2)^2 \alpha' s^{-5/2} \int_C dy e^{-\alpha' y} \frac{1}{\sqrt{2m^2 - t - y}} \end{aligned} \quad (6.20)$$

where in the neighbourhood of $z = 1$ and $z = -1$ we have made the scale transformation $z = 1 + y/s$ and $z = -1 - y/s$, respectively (C_1 and C_{-1} represent the integrations from 0 to ∞ above the cut minus the same integral below the cut). The fact that the s-graph falls so fast asymptotically lies in the additional s-dependence at the vertices, which gives an extra $(s^{-1})^2$ factor. If we were to rewrite the s-graph as a sum of poles with fixed residues except for the spin factors, then the remaining term, which is associated with the infra-red singularities, will have negative norm.

On the other hand, the t-graph is Regge behaved with a leading Regge trajectory $j = \alpha(t)$. This is most simply seen by writing (6.19) in the form

$$T_t(s, t) \underset{s \rightarrow \infty}{\sim} \frac{(g\rho^2)^2 \alpha'}{12(2\pi)^4 (\frac{t}{m^2} - 1)^2} \int_0^1 dz \frac{z^{-\alpha(t)-1}}{\sqrt{1 - (\frac{4zs}{3(m^2-t)})^2}} \quad (6.21)$$

where we have evaluated the discontinuity across the cut from 0 to 1. Making the scale transformation $x = 4zs/3(m^2-t)$ we obtain the leading Regge behaviour

$$T_t(s, t) \underset{s \rightarrow \infty}{\sim} \beta^2(t) \xi_\alpha(t) \Gamma(-\alpha(t)) \left(\frac{s}{s_0}\right)^{\alpha(t)}$$

where

$$\beta(t) = \sqrt{\frac{\alpha'}{12(2\pi)^4}} \frac{(g\rho^2)}{(t/m^2 - 1)^{1+\alpha(t)/2}} \Gamma(\frac{1}{2}(1+\alpha(t))) \quad (6.22)$$

and

$$\xi_\alpha(t) = [1 + e^{i\pi\alpha(t)}]$$

The non-leading behaviour can be found by taking the Mellin transform of (6.9) and systematically extracting the lower-lying singularities (see Appendix A). One finds alternate daughters at $j = \alpha(t) - 2, \alpha(t) - 4, \dots$. However, there are also odd order fixed poles at $j = -1, -3, \dots$. The $j = -1$ fixed pole in fact dominates the next-to-leading behaviour since $\alpha < 0$. As $\alpha(t) \rightarrow -1$ the s^{-1} behaviour will smoothly take over so $\alpha_{\text{eff}}(t) \rightarrow -1$. This situation is reminiscent of the Regge behaviour in ladder diagrams of an ordinary ϕ^3 theory, where the large angle behaviour is determined by just such a fixed pole¹⁹⁾. In the latter case this corresponds to the dominance at large t of the s-channel Born term,

which is simply the kernel in the ladder iteration. One additional feature of the Regge behaviour in (6.22) is the singularity in the residue functions at $t = m^2$. This extra singularity appears in all vertices and is again a reflection of the infra-red singularity in the Green functions.

The poles at $\alpha(t) = -1, -3 \dots$ are only apparent since it means that the Regge pole collides with the fixed poles giving rise to double poles at these points in the j -plane.

At first sight the emergence of Regge behaviour at the Born level, which is of second order in the coupling constant g , seems to be in direct conflict with unitarity. However, we have to remember that in addition to the particle states we also have an infra-red sector, which will appear in the right-hand side of the unitarity relation. To this order in the coupling constant the unitarity relation should read

$$T_s^\dagger - T_s + T_t^\dagger - T_t = \langle p_3, p_4 | T^\dagger | \alpha \rangle \langle \alpha | T | p_1, p_2 \rangle \\ + \langle p_3, p_4 | T^\dagger | I.R. \rangle \langle I.R. | T | p_1, p_2 \rangle$$

which at high energies reduces to

$$\text{disc}_s T_{\text{Regge}}(s, t) = \langle p_3, p_4 | T^\dagger | I.R. \rangle \langle I.R. | T | p_1, p_2 \rangle \quad (6.23)$$

The appearance of the sector $|I.R.\rangle$ as intermediate "states" in the above relation provides an interpretation of the Regge pole discontinuity in the s -channel. We will try to provide a possible physical interpretation of this phenomenon in Section 8.

We shall now turn to the higher point functions where our main purpose will be to see if their multi-Regge behaviour satisfies the Steinman condition, which is essentially a self-consistency check, since the latter is closely related to the causality of the theory¹⁶⁾. In order to simplify the calculations, we shall use a truncated form of the Feynman propagator, which picks out the leading behaviour near $z = 0$ in the integral representation (6.17). This is given by

$$\Delta_F^{\text{Regge}}(p, q, q') = \frac{\alpha'}{2i \sin \pi \alpha(p^2)} \left(\frac{p^2}{\pi}\right)^2 \int \frac{dz}{z} (-z)^{-\alpha(p^2)} \times e^{-\frac{p^2}{p^2} [(p \cdot q)^2 + (p \cdot q')^2]} \times e^{\frac{p^2}{2} (q^2 + q'^2)} \times e^{2p^2 z \left(\frac{(p \cdot q)(p \cdot q')}{p^2} - q \cdot q' \right)} \quad (6.24)$$

and can be used to extract the leading Regge behaviour, exactly reproducing (6.22). It also determines the poles in the s -graph; however, to obtain the complete residues one needs the non-leading terms in z .

The five-point function

We shall consider the five-point function in Fig. 3, which is one of the five tree graphs to order g^3 , and compute its behaviour in the double Regge region:

Fig. 3

The amplitude corresponding to Fig. 3 is given by

$$\begin{aligned}
 T_5(s, s_1, s_2, t_1, t_2) = & \frac{-g^3}{(2\pi)^{12}} \int d^4q d^4q' d^4q'' \\
 & \times U(p_1, q) U(p_3, q) \Delta_F(p_1 - p_3, q, q') U(p_4, q') \\
 & \times \Delta_F(p_2 - p_5, q', q'') U(p_2, q'') U(p_5, q'')
 \end{aligned} \tag{6.25}$$

It is convenient to evaluate this expression by considering the four frames introduced by Toller²⁰⁾ for the purpose of analyzing a five-point function which simplifies the upper, middle and lower vertices, respectively.

For the upper vertex we have the frame F_1 in which

$$\begin{aligned}
 k_1^\mu &= p_3^\mu - p_1^\mu = (0, 0, 0, \sqrt{-t_1}) \\
 p_1^\mu &= (E_1, 0, 0, a_1) \\
 p_3^\mu &= (E_1, 0, 0, -a_1)
 \end{aligned}$$

where

$$E_1 = \frac{1}{2} \sqrt{4m^2 - t_1}$$

and

$$a_1 = \frac{1}{2} \sqrt{-t_1}$$

Similarly for the lower vertex we define the frame F_2 .

For the middle vertex we choose two frames F'_1 and F'_2 , where in F'_1 we have

$$K_1^\mu = (0, 0, 0, \sqrt{-t_1})$$

$$K_2^\mu = (W_1, 0, 0, V_1)$$

$$P_4^\mu = (W_1, 0, 0, u_1)$$

with

$$W_1 = \lambda^{1/2} (t_1, t_2, m^2) / 2 \sqrt{-t_1}$$

$$V_1 = (m^2 - t_1 - t_2) / 2 \sqrt{-t_1}$$

and

$$u_1 = (m^2 + t_1 - t_2) / 2 \sqrt{-t_1}$$

while for F'_2 we have

$$K_1^\mu = (-W_2, 0, 0, V_2)$$

$$K_2^\mu = (0, 0, 0, \sqrt{-t_2})$$

$$P_4^\mu = (W_2, 0, 0, -u_2)$$

with

$$W_2 = \lambda^{1/2} (t_1, t_2, m^2) / 2 \sqrt{-t_2}$$

$$V_2 = (m^2 - t_1 - t_2) / 2 \sqrt{-t_2}$$

and

$$u_2 = (m^2 + t_2 - t_1) / 2 \sqrt{-t_2}$$

These frames are related by the following Lorentz transformations:

$$\begin{aligned}
 F_1 \rightarrow F_1' : L(\alpha_1, \xi_1, \gamma_1) &= R_z(\alpha_1) B_x(\xi_1) R_z(\gamma_1) \\
 F_2 \rightarrow F_2' : L(\alpha_2, \xi_2, \gamma_2) &= R_z(\alpha_2) B_x(\xi_2) R_z(\gamma_2) \\
 F_1' \rightarrow F_2' : L(\beta) &= B_z(\beta)
 \end{aligned} \tag{6.26}$$

For large s_1 and s_2 the boost parameters ξ_1 and ξ_2 are given by

$$\begin{aligned}
 \cosh 2 \xi_1 &\approx s_1 / 2 E_1 W_1 \\
 \cosh 2 \xi_2 &\approx s_2 / 2 E_2 W_2
 \end{aligned}$$

while β is given by

$$\cosh \beta = \frac{W_1 W_2 - u_1 u_2}{m^2}$$

The Toller angle is simply given by $\omega = \gamma_1 + \gamma_2$ which is related to the Toller variable by

$$\eta = (\cosh \beta + \cos \omega) / 2 W_1 W_2 ; \quad [\eta \sim \frac{s}{s_1 s_2}] \tag{6.27}$$

Making use of the truncated propagator (6.24) and evaluating the integrals over q and q'' in (6.25) in the frames F_1 and F_2 , respectively, and using the Lorentz transformations (6.26) to write the result in the frame F_1' we obtain in a straightforward way

$$\begin{aligned}
 T_S &= \frac{(g p^2)^3 \alpha'^2}{9 (2\pi)^{11/2} \pi^4} a(t_1) a(t_2) \int_0^1 \frac{dz_1}{z_1} z_1^{-\alpha(t_1)} \int_0^1 \frac{dz_2}{z_2} z_2^{-\alpha(t_2)} \\
 &\times \int d^4 q' \exp \left\{ \frac{3}{2} q'^2 + q_3'^2 - \frac{1}{m^2} (W_1 q_0' - u_1 q_3')^2 \right. \\
 &\quad + (\sinh \beta q_0' - \cosh \beta q_3')^2 \\
 &\quad + \lambda_1 (q_0' + q_1' \cos \omega + q_2' \sin \omega)^2 \\
 &\quad \left. + \lambda_2 (q_0' \cosh \beta - q_1' - q_3' \sinh \beta)^2 \right\}
 \end{aligned} \tag{6.28}$$

where

$$\lambda_i = \frac{s_i^2 z_i^2 a(t_i)}{6 m^2 w_i^2}$$

and

$$a(t) = [1 - t/m^2]^{-1}$$

The remaining vertex integral can be evaluated by first making the variable change

$$\begin{aligned} X &= q_0' & ; & \quad Y = q_3' \\ U &= q_0' + \cos \omega q_1' + \sin \omega q_2' \\ V &= \cosh \beta q_0' - q_1' - q_3' \sinh \beta \end{aligned}$$

the inverse of which is given by

$$\begin{aligned} q_1' &= \cosh \beta X - V - \sinh \beta Y \\ q_2' &= -(\operatorname{cosec} \omega + \cot \omega \cosh \beta) X \\ &\quad + \operatorname{cosec} \omega U + \cot \omega V + \cot \omega \cosh \beta Y \end{aligned}$$

and the Jacobian by $(\sin \omega)^{-1}$.

Finally, the (X,Y) and (U,V) integrations can be decoupled through the transformations

$$\begin{aligned} \bar{X} &= X - \frac{3}{(g^2 - h^2) \cos \omega} [(g \cosh \beta + h \sinh \beta) U + g V] \\ \bar{Y} &= Y - \frac{3}{(g^2 - h^2) \cos \omega} [(g \sinh \beta + h \cosh \beta) U + h V] \end{aligned}$$

in the limit of large $\cos \omega$, so that we obtain

$$\begin{aligned} &(\sin \omega)^{-1} \int d\bar{X} d\bar{Y} \exp \left\{ -g(t_1, t_2) (\bar{X}^2 + \bar{Y}^2) - 2h(t_1, t_2) \bar{X} \bar{Y} \right\} \\ &\times \int du dv \exp \left\{ (\lambda_1 - f(t_1, t_2) \sec^2 \omega) u^2 \right. \\ &\quad \left. + 3 \sec \omega u v \right. \\ &\quad \left. + (\lambda_2 - f(t_2, t_1) \sec^2 \omega) v^2 \right\} \end{aligned} \quad (6.29)$$

In this way we obtain for the five-point function the expression

$$T_5 = c (g p^2)^3 \alpha'^2 a(t_1) a(t_2) \bar{V}(t_1, t_2) \times \int_0^1 \frac{dz_1}{z_1} z_1^{-\alpha(t_1)} \int_0^1 \frac{dz_2}{z_2} z_2^{-\alpha(t_2)} \quad (6.30)$$

$$\times \sqrt{[-\lambda_1 \lambda_2 \cos^2 \omega + f(t_2, t_1) \lambda_1 + f(t_1, t_2) \lambda_2 + (\frac{3}{2})^2 - f(t_1, t_2) f(t_2, t_1)]^{1/2}}$$

where

$$\bar{V}(t_1, t_2) = [g^2(t_1, t_2) - h^2(t_1, t_2)]^{-1/2}$$

and is just the (X,Y) integral. The functions g, h and f can be read-off from the above integrals and are simple rational functions of t_1 and t_2 .

The leading double Regge behaviour of T_5 in the limit $s_1, s_2 \rightarrow \infty$, t_1, t_2 and η fixed is obtained by making the scale transformations

$$y_i = z_i^2 s_i^2 C_i(t_1, t_2) \quad i=1, 2$$

where the function C_i is used to scale out the t_1 and t_2 dependence. The result can be written in the form

$$T_5 \simeq \beta(t_1) \beta(t_2) s_1^{\alpha(t_1)} s_2^{\alpha(t_2)} v(t_1, t_2, \eta)$$

where in the limit of large η

$$v(t_1, t_2, \eta) = (\alpha' g p^2) V(t_1, t_2) \times \int_0^\infty \frac{dy_1 dy_2 y_1^{-1-\alpha(t_1)/2} y_2^{-1-\alpha(t_2)/2}}{\sqrt{-y_1 y_2 c \eta^2 + y_1 + y_2 + 1}} \quad (6.31)$$

with

$$c = 2 w_1 w_2 [1 - (3/2)^2 / f(t_1, t_2) f(t_2, t_1)]^{1/2}$$

[All additional t-dependent factors are absorbed into the definition of $v(t_1, t_2)$.]

The integrals over y_1 and y_2 can be explicitly evaluated giving ($\alpha_i = \alpha(t_i)$)

$$\begin{aligned}
 V(t_1, t_2, \eta) = & (\alpha' g \rho^2) \vee B(-\alpha_1/2, 1/2 + \alpha_1/2) B(-\alpha_2/2, 1/2 + \alpha_2/2) \\
 & \times \left\{ \frac{\Gamma(1/2) \Gamma(\frac{\alpha_2 - \alpha_1}{2})}{\Gamma(-\frac{\alpha_1}{2}) \Gamma(1/2 + \alpha_1/2)} (c\eta)^{\alpha_2} F(-\frac{\alpha_2}{2}, \frac{1}{2} - \frac{\alpha_2}{2}, 1 + \frac{\alpha_1 - \alpha_2}{2}; (c\eta)^{-2}) \right. \\
 & \left. + \frac{\Gamma(1/2) \Gamma(\frac{\alpha_1 - \alpha_2}{2})}{\Gamma(-\frac{\alpha_2}{2}) \Gamma(1/2 + \alpha_1/2)} (c\eta)^{\alpha_1} F(-\frac{\alpha_1}{2}, \frac{1}{2} - \frac{\alpha_1}{2}, 1 + \frac{\alpha_2 - \alpha_1}{2}; (c\eta)^{-2}) \right\}
 \end{aligned}
 \tag{6.32}$$

From (6.32) we see immediately that the Steinman conditions are fulfilled, since T_5 will only have discontinuities in either s_1 or s_2 [corresponding to the two terms in (6.32)], but not for both. Furthermore, we can express the double Regge-behaviour of T_5 in the form

$$T_5^{\text{D.R.}} = \beta(t_1) \Gamma(-\alpha_1) \xi_{\alpha_1} s_1^{\alpha_1} [\gamma^{\alpha_1} v_{12} + \eta^{\alpha_2} v_{21}] \Gamma(-\alpha_2) \xi_{\alpha_2} s_2^{\alpha_2} \beta(t_2)
 \tag{6.33}$$

where

$$v_{12} = \frac{\Gamma(\alpha_1 - \alpha_2)}{\Gamma(-\alpha_2)} \xi_{\alpha_2}^{-1} v_0(t_1, t_2, \eta^{-1})$$

and

$$\begin{aligned}
 v_0(t_1, t_2, \eta^{-1}) = & \frac{\alpha' g \rho^2 [f(t_1, t_2)]^{\alpha_1} [f(t_2, t_1)]^{\alpha_2} c^{\alpha_1}}{[g^2 - h^2]^{\frac{1}{2}(1 + \alpha_2 - \alpha_1)}} \\
 & \times \Gamma(\frac{1}{2} + \frac{\alpha_1 - \alpha_2}{2}) \{ 1 + a_1 \eta^{-2} + a_2 \eta^{-4} \dots \}
 \end{aligned}
 \tag{6.34}$$

(6.33) has the same general structure for the double Regge behaviour of the five-point function as that obtained from the dual five-point function²¹⁾ or from more general considerations such as analyticity²²⁾. In principle one could write down rules for calculating the multi-Regge behaviour of arbitrary higher-point functions

at the tree graph level. These will be similar in character to those given in Ref. (23), however the residue and vertex functions will have additional singularities outside the physical region. The latter seem to be characteristic of the kind of theory we have presented here and are associated with the extra infrared singularities in the Green function, which are a manifestation of the internal structure of the hadrons.

7. RENORMALIZATION

In many respects the model we have presented here is very similar to an ordinary Q.F.T. involving a single self-interacting particle. This also holds for the renormalization question. In particular, once one has a Feynman graph expansion for the bare theory one must ask if the theory is finite or at least can be made finite. The most obvious source of divergence is loop insertions in the propagator (Fig. 4). Here the latter is particularly interesting, since we have an infinite number of poles described by Regge trajectories instead of a single mass parameter.

In this section we shall briefly investigate the degree and structure of the basic divergences and give some arguments, which suggest that the generalized ϕ^3 example we have considered is super renormalizable, even though the basic divergences turn out to be of higher degree than an ordinary ϕ^3 theory, due to the infinite summation over particle poles. There are also important finite mass renormalization effects, which shift the higher lying poles off the real axis, since the interaction couples the higher lying states to the ground state continuum. We shall briefly discuss how in principle the renormalized Green function can be constructed and how they can be used to calculate the scattering amplitudes and quasi-scattering amplitudes involving unstable particles.

We first consider the infinite mass renormalization by example of the single loop insertion in Fig. 4b. This corresponds to the expression

$$\Delta_F^{(2)}(P, q, q') = g^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \Delta_F(P, q, q_1) \int \frac{d^4 \kappa}{(2\pi)^4} \Delta_F(\kappa, q_1, q_2) \Delta_F(P - \kappa, q_1, q_2) \Delta_F(P, q_2, q')$$

(7.1)

Fig. 4

In order to study the renormalization of the pole positions, we need only to determine the effect of this loop insertion near $z = 0$ in the integral representation (6.12), so we can simplify the calculation using the form (6.24) for the propagators outside the loop. In the loop integration the divergence comes from the region $K^2 \sim (p - K)^2 \rightarrow \infty$, which is determined by the behaviour near $z = 1$ (see Section 6) and since the suppression of the time excitations in (6.12), by taking the limit $\lambda \rightarrow -\infty$, depended on $|z| < 1$, we should keep λ initially finite. This could have been anticipated because, when taking the $\lambda \rightarrow -\infty$ limit in the mass formula (6.2), we have to make an infinite shift of the mass constant m_0^2 in order to keep α_0 finite. Concentrating only on the divergent piece in (7.1) we replace it by

$$\Delta_F^{(2)} \sim \alpha'^4 (g p^2)^2 \int \frac{d\bar{z}}{\bar{z}} \frac{d\bar{z}'}{\bar{z}'} (\bar{z}\bar{z}')^{-\alpha(p^2)} e^{-[\frac{(p,q)^2}{p^2} - \frac{1}{2}q^2]} \Delta M^2 e^{-[\frac{(p,q')^2}{p^2} - \frac{1}{2}q'^2]} \quad (7.2)$$

where

$$\begin{aligned} \Delta M^2 = & \int \frac{d^4 \kappa}{(2\pi)^4} \int \frac{d\bar{z}_1}{\bar{z}_1} \bar{z}_1^{-\alpha(\kappa^2)} \int \frac{d\bar{z}_2}{\bar{z}_2} \bar{z}_2^{-\alpha((p-\kappa)^2)} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \\ & \times (1-\bar{z}_1^2)^{-3/2} (1-\bar{z}_1^2)^{-1/2} (1-\bar{z}_2^2)^{-3/2} (1-\bar{z}_2^2)^{-1/2} \\ & \times \exp \left[\frac{1}{2} (q_1^2 + q_2^2) - \frac{1}{p^2} ((p,q_1)^2 + (p,q_2)^2) + 2\bar{z} \left(\frac{(p,q_1)(p,q)}{p^2} - q \cdot q_1 \right) \right. \\ & \quad \left. + 2\bar{z}' \left(\frac{(p,q_2)(p,q')}{p^2} - q_2 \cdot q' \right) \right] \\ & \times \exp \left[\frac{1}{4} (q_1 - q_2)^2 (b_1 + b_2) + \frac{1}{4} (q_1 + q_2)^2 (b_1^{-1} + b_2^{-1}) \right. \\ & \quad \left. - \frac{1}{4} \left\{ \frac{(\kappa \cdot (q_1 - q_2))^2}{\kappa^2} (a_1 + b_1) + \frac{(\kappa \cdot (q_1 - q_2))^2}{\kappa'^2} (a_2 + b_2) \right. \right. \\ & \quad \left. \left. + \frac{(\kappa \cdot (q_1 + q_2))^2}{\kappa^2} (a_1^{-1} + b_1^{-1}) + \frac{(\kappa \cdot (q_1 + q_2))^2}{\kappa'^2} (a_2^{-1} + b_2^{-1}) \right\} \right] \end{aligned}$$

where

$$a_i = \frac{1+z_i^\lambda}{1-z_i^\lambda} ; \quad b_i = \frac{1+z_i}{1-z_i} \quad \text{and} \quad k' = p-k$$

We evaluate the q_1 and q_2 integrals by making the transformations

$$\bar{q}_1 = \frac{q_1 - q_2}{2}$$

$$\bar{q}_2 = \frac{q_1 + q_2}{2}$$

Working in the limit $z_1, z_2 \rightarrow 1$ (i.e. by defining $z_1 = 1 + y_1$ and $z_2 = 1 + y_2$ with $y_1, y_2 \sim 0$ so that $a_i = +1/\lambda y_i$ and $b_i = -1/y_i$) and after carrying out the $\bar{q}_1$ and $\bar{q}_2$ integrations, bearing in mind that the divergent part corresponds to $K^2 \sim K'^2 \rightarrow \infty$, we obtain

$$\Delta M^2 \sim \alpha' e \left[\frac{(p \cdot (z q + z' q'))^2}{p^2} - (z q + z' q')^2 \right]$$

$$\times \frac{\pi}{\sqrt{2}} \int \frac{d^4 k}{(2\pi)^4} \int_0^{\Delta \ll 1} \frac{dy_1 dy_2}{(y_1 + y_2)^2} e^{-\alpha' (y_1 k^2 + y_2 (p-k)^2)}$$
(7.3)

$$= e^{[\dots]} \frac{\pi}{\lambda^{1/2}} \int_0^{\Delta} \frac{dy_1 dy_2}{(y_1 + y_2)^4} e^{-\frac{y_1 y_2}{y_1 + y_2} p^2}$$

From (7.3) we see that we have a quadratic divergence in the loops, although the integral is very similar to that encountered in a ϕ^3 theory. The fact that we have a higher divergence here is a reflection of the infinite number of poles in the Feynman propagator. In this respect this is similar to the divergences of the dual model⁽²⁴⁾. If we write this above divergent piece in the form

$$\Delta M^2 = e \left[\frac{(p \cdot (z q + z' q'))^2}{p^2} - (z q + z' q')^2 \right] \alpha' s m^2$$
(7.4)

and insert it into (7.2) we obtain for the leading divergence

$$\begin{aligned} \Delta_{F \text{ div.}}^{(1)} &= \int \frac{dz dz'}{(z z')} (z z')^{-\alpha(p^2)} \alpha' 8\pi^2 \exp\left[-p^2 \left\{ \frac{(p \cdot q)^2}{p^2} - \frac{1}{2} q^2 \right\} (1+z^2) \right] \\ &\quad \times \exp\left[-p^2 \left\{ \frac{(p \cdot q')^2}{p^2} - \frac{1}{2} q'^2 \right\} (1+z'^2) \right] \\ &\quad \times \exp\left[p^2 \left\{ \frac{(p \cdot q)(p \cdot q')}{p^2} - q \cdot q' \right\} z z' \right] \end{aligned} \quad (7.5)$$

If we make the variable change

$$Z = z z' \quad \text{and} \quad \eta = z$$

We obtain for $z \sim 0$, to leading order

$$\Delta_{F \text{ div.}}^{(2)} = \int \frac{dZ}{Z} Z^{-\alpha(p^2)} \alpha' 8\pi^2 \log Z \varphi(p, q, q', Z) \quad (7.6)$$

where

$$\begin{aligned} \varphi(p, q, q', Z) &= \exp\left[-p^2 \left\{ \frac{(p \cdot q)^2}{p^2} + \frac{(p \cdot q')^2}{p^2} - \frac{1}{2} (q^2 + q'^2) - \right. \right. \\ &\quad \left. \left. - \left(\frac{(p \cdot q)(p \cdot q')}{p^2} - q \cdot q' \right) Z \right\} \right] \end{aligned}$$

The full iteration in Fig. 4 can be evaluated recursively, making use of the transformation

$$\int \frac{dz_1}{z_1} \frac{dz_2}{z_2} \dots \frac{dz_n}{z_n} = \int_0^1 \frac{dZ}{Z} \int_Z^1 \frac{d\eta_1}{\eta_1} \int_{\eta_1}^1 \frac{d\eta_2}{\eta_2} \dots \int_{\eta_{n-2}}^1 \frac{d\eta_{n-1}}{\eta_{n-1}} \quad (7.7)$$

where

$$Z = z_1 z_2 \dots z_n, \quad \eta_1 = z_2 z_3 \dots z_n, \quad \dots, \quad \eta_{n-1} = z_n$$

For the leading divergence this leads to the result

$$\Delta_F^R \sim \int_0^1 \frac{dz}{z} z^{-\alpha(P^2)} \left[1 + \frac{\alpha' \delta m^2 \log z}{1!} + \frac{(\alpha' \delta m^2 \log z)^2}{2!} + \dots \right] \times \varphi(P, q, q', z) \quad (7.8)$$

Hence this iteration of leading divergences simply leads to an infinite shift of the Regge intercept α_0 and from Eqs. (6.7) we see this corresponds to the addition of a piece δm^2 in the mass operator. This looks completely analogous to the usual infinite mass shifts in a Q.F.T. There are, however, non-leading divergences, which are likely, in addition to lead to a renormalization of the Regge slope parameter α' .

A general vertex involves three relative momenta and the simplest loop insertion (Fig. 5) corresponds to the expression

$$\Gamma^{(3)}(P_1, P_2, q, q', q'') = g^3 \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k, q, q') \Delta_F(k-P_2, q', q'') \times \Delta_F(k+P_1, q'', q) \quad (7.9)$$

Fig. 5

Again the most divergent piece of (7.9) comes from the region $K^2 \rightarrow \infty$ and $z \sim 1$ part of the integral representation of the Feynman propagators. Defining the variables $y_i = 1 - z_i$, $i = 1, 3$ we can write the divergent piece in the form

$$\Gamma_{\text{div}}^{(3)} \sim \frac{g^3 \rho^{1/2} \alpha^1}{\lambda^{1/2}} \int \frac{d^4 \kappa}{(2\pi)^4} \int \frac{dy_1 dy_2 dy_3}{y_1^2 y_2^2 y_3^2} e^{-\alpha'(\kappa^2 y_1 + (\kappa + p_1)^2 y_2 + (\kappa - p_3)^2 y_3)} \\ \times \text{Lt}_{\kappa^2 \rightarrow \infty} \frac{\pi}{\text{Perm}_{1,2,3}} \exp \left[\frac{p^2}{2y_1} (\vec{q}_3 - \vec{q}_k)^2 - \frac{p^2}{2y_2} \frac{(\kappa \cdot (q_1 - q_k))^2}{\kappa^2} \right] \quad (7.10)$$

Using the fact that the leading divergence corresponds to $y_1 \sim y_2 \sim y_3 \sim 0$, so that the following identity can be implemented

$$\text{Lt}_{y_1, y_2, y_3 \rightarrow 0} \frac{1}{2\pi} \left(\frac{1}{y_2 y_3} + \frac{1}{y_3 y_1} + \frac{1}{y_1 y_2} \right)^{1/2} \exp \left[-\frac{(q_1 - q_3)^2}{2y_1} - \frac{(q_3 - q_2)^2}{2y_2} - \frac{(q_1 - q_2)^2}{2y_3} \right] \\ = \delta(q_1 - q_2) \delta(q_2 - q_3) \quad (7.11)$$

After the loop integration we obtain

$$\Gamma_{\text{div}}^{(3)} \sim (2\pi)^8 \delta g \delta^4(q_1 - q_2) \delta^4(q_2 - q_3)$$

where

$$\delta g = \frac{g^3 \rho^{1/2} \alpha^1}{4 \lambda^{1/2} (2\pi)^{12}} \text{Lt}_{\lambda \rightarrow 0} \int \frac{dy_1 dy_2 dy_3}{(\rho \lambda)^{-1} (y_1 + y_2 + y_3)^4} \\ \text{i.e., } \rho^2 \delta g = \frac{(g \rho^2)^3 \alpha^1}{4 \lambda^{1/2} (2\pi)^{12}} \int \frac{d\sigma}{\sigma^2} \quad (7.12)$$

Hence we have a linear divergence in the coupling constant renormalization. One way of seeing that these higher order divergences are due to the infinite summations over the poles, is to see what happens in the limit $\alpha' \rightarrow 0$, in which in principle we can recover a ϕ^3 theory with a single self-interacting particle. This limit can be realized for the case $\lambda = +1$, for which the internal and external momentum becomes uncoupled (see Section 6).

The ϕ^3 limit is obtained by requiring

- i) $\lambda = -1$
- ii) $\text{Lt}_{\alpha' \rightarrow 0} \alpha(m^2) = 0$, i.e. $m^2 = -\alpha_0/\alpha'$ fixed as $\alpha' \rightarrow 0$
- iii) $\rho^2 \rightarrow 0$ with $g\rho^2 \rightarrow g'$ fixed.

In this limit the propagator becomes

$$\begin{aligned} \Delta_F &= \lim_{\substack{\alpha' \rightarrow 0 \\ p^2 \rightarrow 0}} \left\{ \frac{\alpha'}{2i \sin \pi \alpha(p^2)} \left(\frac{\rho^2}{\pi} \right)^2 \int_C \frac{dz}{z} \right\} \\ &= \frac{1}{\pi^2} \frac{1}{p^2 - m^2} \end{aligned} \quad (7.13)$$

and an arbitrary tree graph has the form

$$(g\rho^2)^{n+1} \prod_{i=1}^{n-1} \frac{1}{s_i - m^2} \quad (7.14)$$

Hence $g' = g\rho^2$ is the usual ϕ^3 coupling constant, which has the dimensions of a mass. The mass divergence discussed above is given by

$$8m^2 = g'^2 \alpha' \Lambda^2 \quad (7.15)$$

Hence by keeping $\alpha' \Lambda^2$ fixed as $\alpha' \rightarrow 0$, where Λ is the cut-off parameter, we lose the quadratic divergence. Similarly the vertex divergence, i.e.,

$$8g' = g'^3 (\alpha' \Lambda^2) p/\Lambda \rightarrow 0 \quad (7.16)$$

disappears in this limit. Even though the basic divergences in our generalized ϕ^3 example are of higher degree than the usual ϕ^3 theory, the following argument indicates that the model is still super renormalizable and can be made finite by the BPH⁽²⁵⁾ renormalization program. We shall exploit the close similarity between the expressions giving the most divergent piece of the skeleton diagrams with the corresponding Feynman parameter integrals of a ϕ^3 theory.

The most divergent piece of an arbitrary skeleton graph involving N external lines, n propagators, v vertices and ℓ loops ($n = r + \ell - 1$), is given by the following expression (see Fig. 6)

$$\begin{aligned}
 I_{n\ell}(p_1, \dots, p_N) = & \int \prod_{i=1}^r d^4 q_i f(p_1, \dots, p_N, q_1, \dots, q_r) \\
 & \times \int \frac{dy_1 \dots dy_n}{y_1^2 \dots y_n^2} \int \prod_{\ell} \frac{d^4 k_\ell}{(2\pi)^4} e^{-\alpha' \sum_{j=1}^n y_j k_j^2} \\
 & \times \lim_{\substack{\ell \rightarrow \infty \\ k' \rightarrow \infty}} \prod_{k=1}^n \exp \left\{ - \left[- \frac{(q_{i_k} - q_{j_k})^2}{y_k} + \frac{((q_{i_k} - q_{j_k}) \cdot k'_k)^2}{\lambda y_k} \right] \right\}
 \end{aligned} \tag{7.17}$$

where $p_1, \dots, p_N$ are the external momenta and f is simply a product of the external wave functions

Fig. 6

Since we are interested in the behaviour near $y_i \sim 0$ ($i = 1, n$), we can generalize Eq. (7.8), which is simply a convenient trick for carrying out the Gaussian integrals over the $\{q_i\}$ and picking out the leading terms as $y_i \rightarrow 0$. When we have more than one loop, we must take into account all topological permutations, corresponding to cutting the graph, so that it is in the form of a tree graph. The appropriate generalization is given by

$$\begin{aligned}
 \text{Lt}_{\{y_i \rightarrow 0\}} \pi^{1/2(n-r)} \left(\sum_{\substack{\text{Topological} \\ \text{Permutations}}} \frac{1}{y_{i_1} \dots y_{i_{n-r}}} \right)^{1/2} e^{-\sum_k (q_{i_k} - q_{j_k})^2 / y_k} \\
 = \prod_{\substack{i < j \\ i=1, n-r}} \delta(q_i - q_j)
 \end{aligned} \tag{7.18}$$

where each term in the summation over the topological permutations represents a product of y_i , corresponding to a string of propagators left after the graph is reduced to a tree graph by cutting ℓ propagators. The sum represents all possible ways this can be done. The arguments involved here are identical to the topological recipe given by Symanzik²⁶⁾ for calculating the Chisholm determinants for an arbitrary graph in a ϕ^3 theory. By noting that

$$\sum_{\substack{\text{Topological} \\ \text{Permutations}}} \left(1/y_{i_1} \dots y_{i_{n-r}} \right) = \frac{D(y_1, \dots, y_n)}{y_1 \dots y_n} \tag{7.19}$$

we can write (7.17) in the form

$$I_{n,\ell}^{\text{div.}}(p_1, \dots, p_n) \propto \int \frac{dy_1 \dots dy_n}{D(y_1, \dots, y_n)} I_{n,\ell}^0(y_1, \dots, y_n, p_1, \dots, p_n) \tag{7.20}$$

where $I_{n,\ell}^0$ is the corresponding ϕ^3 integrand and $D(y_1 \dots y_n)$ is the same determinant as that entering in $I_{n,\ell}^0$ (see Ref. 26)) and it is the latter, which determines the divergence character of the graph. We notice that, since the additional factor of D^{-2} coming from the infinite summation over poles and the $\{q_i\}$ integrals is fixed and independent of the number of insertions, the divergence degree of the skeleton graphs decreases with the number of insertions. Hence the theory appears

to be super renormalizable and should be made finite by suitable subtraction of the basic divergences. One can also express the renormalization in terms of the Dyson equations shown in Figs. 7a and 7b

Fig. 7

By defining the propagator and vertex renormalizations

$$\begin{aligned}\Gamma' &= Z_1^{-1} \Gamma^R \\ \Delta_F' &= Z_2 \Delta_F^R\end{aligned}\tag{7.21}$$

where Γ' and Γ_F' are defined through the bare theory and Γ^R and Δ_F^R are defined by a suitable boundary condition (e.g. by choosing a subtraction point), then the Dyson equations in Fig. 7 can be written in the form

$$\begin{aligned}\Delta_F^R(p, q, q') &= Z_2^{-1} \Delta_F(p, q, q') + g_R^2 \int \frac{d^4 q''}{(2\pi)^4} \Delta_F(p, q, q'') \\ &\times \int \frac{d^4 q'''}{(2\pi)^4} \frac{d^4 q'''}{(2\pi)^4} \frac{d^4 q'''}{(2\pi)^4} \int \frac{d^4 \kappa}{(2\pi)^4} \Delta_F^R(p + \kappa, q'', q''') \\ &\times \Delta_F^R(\kappa, q'', q''') \Gamma^R(p + \kappa, \kappa, q'', q''', q''') \Delta_F^R(p, q''', q')\end{aligned}\tag{7.22}$$

and

$$\begin{aligned} \Gamma^R(p_1, p_2, q, q', q'') &= Z_1 (2\pi)^8 \delta^4(q - q') \delta^4(q' - q'') \\ &+ \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2''}{(2\pi)^4} K_4^R(p_1, p_1 + p_2, k, q, q'', q_1, q_1'') \\ &\times \int \frac{d^4 q_1'}{(2\pi)^4} \int \frac{d^4 q_2'}{(2\pi)^4} \Delta_F^R(k, q_1, q_2) \Delta_F^R(p_2 - k, q_1'', q_2'') \\ &\times \Gamma^R(k, p_2, q_1', q_2', q_2'') \end{aligned} \quad (7.23)$$

where the Kernel K_4^R is expressed in terms of the renormalized propagator and vertices and the renormalized coupling constant $g_R = Z_2^{3/2} Z_1^{-1} g_0$.

In addition to the infinite mass renormalization, the loops in the present model will shift the higher lying poles off the real axis. In the bare theory we can consider Green functions involving the propagator $\Delta_F(p, q, q')$ as an "external leg". However because of the coupling of the internal momentum, such a Green function corresponds to an expression of the form

$$\underline{G}(p, q, p_1, \dots, p_N) = \int \frac{d^4 q'}{(2\pi)^4} \Delta_F(p, q, q') \tilde{\underline{G}}(p, q', p_1, \dots, p_N) \quad (7.24)$$

where $\tilde{\underline{G}}$ does not contain the poles in the variables p^2 . We can project out an arbitrary pole by multiplying by its corresponding internal wave function $U_\alpha(p, q)$ and integrating over $d^4 q$, i.e.,

$$\begin{aligned} \underline{G}_\alpha(p, p_1, \dots, p_N) &= \int \frac{d^4 q}{(2\pi)^4} U_\alpha(p, q) \int \frac{d^4 q'}{(2\pi)^4} \Delta_F(p, q, q') \\ &\quad \times \tilde{\underline{G}}(p, q', p_1, \dots, p_N) \\ &= \frac{1}{p^2 - m_\alpha^2} \int \frac{d^4 q'}{(2\pi)^4} U_\alpha(p, q') \tilde{\underline{G}}(p, q', p_1, \dots, p_N) \end{aligned} \quad (7.25)$$

where we have used the orthonormality of the internal wave functions. We see from (7.25) that $G_\alpha(p, p_1, \dots, p_N)$ is the Green function involving the off-shell leg corresponding to particle α . If we replace Δ_F by Δ_F^R , the same projection will in part give us an amplitude involving the unstable particle state α . However, because of the presence of the interaction, one should re-diagonalize the mass operator and define new wave functions $\tilde{U}_\alpha(p, q)$. The latter could, for example, be calculated perturbatively, in the same way that one diagonalizes the Schrödinger Hamiltonian in non-relativistic quantum mechanics.

Finally, we briefly consider the problem of the additional sector of states due to the infra-red singularities in the propagator, which appeared in the bare theory. One hope is that, after renormalization, they either disappear from the theory or are themselves unstable to the ground state continuum. The dilemma we face can be summarized as follows. On the one hand, in ensuring a causal theory with a positive-definite mass spectrum, we obtain a very physical looking multi-Regge structure at the tree graph level, which is similar to that obtained in the dual model. On the other hand, these singularities do not appear to correspond to physical intermediate states. We have not yet found a satisfactory answer to this problem and shall only list two possibilities.

i) $Z_1 = 0$ (i.e. the vertex renormalization factor vanishes). This means that the bare vertex disappears from the theory and by virtue of the Dyson equations, the renormalized Regge behaved tree graphs satisfy the following equation

The diagrammatic equation (7.26) illustrates the relationship between a tree graph and its iterative expansion. On the left, a tree graph is shown with two external legs, each consisting of a double line and a circle labeled Γ^R . These legs are connected by a vertical chain of three elements: a wavy line, a shaded circle, and another wavy line. This is followed by an equals sign. To the right of the equals sign, the first term of the iterative expansion is shown. It features a similar top and bottom structure, but the central vertical chain is more complex, starting with a wavy line, followed by a shaded circle, then a vertex labeled Γ^R connected to a shaded circle, which is further connected to another shaded circle and a wavy line. This is followed by a plus sign and the word "iterations", indicating that the full expression is a sum of this term and its higher-order iterations. The equation is labeled (7.26) on the far right.

This would mean that the s-channel discontinuity of the Regge poles will be related to physical intermediate states at the expense of losing the perturbative nature of the theory. In fact such a situation is essentially a bootstrap and the usefulness of our whole approach may be open to question. However, if the structure of the renormalized tree graphs is the same as in the bare theory, as indeed the above discussion of the propagator renormalization seems to indicate, then we may be able to combine the two apparently opposing ideas discussed in Section 1, namely a quark model spectrum and a bootstrap dynamics. For practical computations we will have to content ourselves with a knowledge of the renormalized tree graphs and iterations governed by the Dyson equations and unitarity, which is not an unfamiliar situation in strong interaction physics. However, from the point of view of field theory, such a solution seems questionable, because in general the value of Z_1 depends on the subtraction point chosen, where as the physics should not. This simply means that one ought not draw any physical conclusions from a particular value of Z_1 . This leads us to consider the second possibility.

ii) $Z_1 \neq 0$: In this case the theory remains perturbative and the appearance of Regge singularities at the tree graph level conflicts with physical unitarity (i.e. unitarity with observable physical intermediate states). If the mass operator could be represented in terms of a sub-field theory of the constituents (e.g. quarks and gluons) then one would expect the Regge discontinuity to correspond to Fig. 8a.

Fig. 8

However, since we do not expect to see quarks and gluons in the final state one must require some kind of final state phenomena, in which Fig. 8a is replaced by Fig. 8b. However, now we have done a full circle, since the only sensible interpretation of the latter is the bootstrap solution above with $Z_1 = 0$, essentially because we can re-express Fig. 8b in terms of renormalized tree graphs and

derive the Dyson equation (7.23). It thus appears even though, from a field theoretic point of view, that possibility (i) is on a weak footing, it seems to be necessary from the point of view of having a physically interpretable theory.

8. SUMMARY

We have tried to describe the scattering of a system of hadrons, which lie on Regge trajectories in terms of a quantum field theory, by assuming the spectrum corresponds to a mass operator involving some internal co-ordinate (e.g. as in a quark model). The free field theory was constructed from a knowledge of the mass operator and the scattering problem was discussed in terms of the Yang-Feldman formalism. In ensuring that the mass spectrum was tachyon free, we saw that the mass operator had to be infra-red singular and further that causality required the inclusion of these singularities in the definition of the Green function and the Feynman propagator. For the bare theory we derived generalized Feynman rules for calculating the scattering matrix and showed that the tree graphs were Regge behaved, with the multi-Regge behaviour satisfying the Steinman conditions.

We examined the question of renormalization, where we found that, due to the summation over hadron states, the basic divergences were of higher degree than a theory involving a single particle pole. Despite this we found that the generalized ϕ^3 example appears to be super renormalizable in the sense of Weinberg (i.e., the divergence degree of the skeleton graph decreases with the number of insertions). Furthermore we gave an indication that the renormalized propagator has the same form as in the bare theory. We finally turned to the problem of interpretability of the theory. We found that only if we enter a non-perturbative situation, in which the vertex rescaling factor $Z_1 \rightarrow 0$, could we relate the Regge discontinuities of the renormalized tree graphs to physical intermediate states. In such a situation we have a theory, in which the structure of the tree graphs is determined by the bare theory and the iterations (i.e., loops) are governed by Dyson equations and unitarity. Furthermore, the quark model and bootstrap hypothesis are both realized in the same theory. However, if it can be shown that such a solution is wrong or leads to inconsistencies, we must conclude that, although the model we have presented has many attractive features, it is unacceptable physically, since the Regge behaviour at the tree graph level would be related to unphysical states in the theory.

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APPENDIX

The Mellin transform of the t-graph (6.19)

The t-graph (6.19) may be written

$$T_t(s, t) = \frac{(q^2)^2 \alpha'}{4 (2\pi)^4} \frac{1}{[1-t/m^2]} \int_0^1 \frac{dz}{z} z^{-\alpha(t)} \times (q-z^2)^{-1/2} M(s, t, z) \quad (A.1)$$

where

$$M(s, t, z) = (-i) \left(\frac{4zs}{m^2} + \alpha(t, z) \right)^{-1/2} \left(\frac{4zs}{m^2} - b(t, z) \right)^{-1/2} \quad (A.2)$$

with

$$\alpha(t, z) = 3(1-t/m^2) - z^2(3-t/m^2) - 2z(4-t/m^2) \quad (A.3)$$

$$b(t, z) = 3(1-\frac{t}{m^2}) - z^2(3-t/m^2) + 2z(4-t/m^2)$$

The Mellin transform of $T_t(s, t)$ is given by

$$\tilde{T}_t(j, t) = \frac{(q^2)^2 \alpha'}{4 (2\pi)^4} \frac{1}{[1-t/m^2]} \int_0^1 \frac{dz}{z} z^{-\alpha(t)} \frac{\tilde{M}(j, t, z)}{(q-z^2)^{1/2}} \quad (A.4)$$

where

$$\begin{aligned}
 \tilde{M}(j, t, z) &= \int_0^{\infty} ds s^{-j-1} M(s, t, z) \\
 &= \left(\frac{4z}{m^2}\right)^j b^{-1/2-j} a^{-1/2} B(1/2, -j) \\
 &\quad \times F(1/2, -j, 1/2-j, -\frac{b}{a}) \\
 &\quad - i \left(\frac{4z}{m^2}\right)^j b^{-j-1} B(1/2, 1+j) \\
 &\quad \times F(1/2, 1+j, 3/2+j, -\frac{a}{b})
 \end{aligned} \tag{A.5}$$

A full j -plane analysis of $\tilde{T}_t(j, t)$ can be made by expanding $(9-z^2)^{-1/2} M(j, t, z)$ around $z = 0$. The leading behaviour is readily shown to agree with (6.22).

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Fig.1

Fig. 2

Fig.3

Fig. 4

Fig. 5

Fig. 6

(a)

(b)

Fig. 7

Fig. 8