SPIN IN THE SPECTRUM OF STATES OF DUAL MODELS

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ABSTRACT

We consider the spectra of states of dual models, studying the helicity partition functions- These functions are generalizations of modular functions for an extension of the modular group. The spectra of the known models are rederived, and some further possibilities are considered. The singularities of the helicity partition functions yield the Regge trajectories of the models, including the newly discovered sister trajectories.

1. INTRODUCTION

This paper is essentially a continuation of Ref. 1), henceforth cited as (I). We have seen in (I) that the partition functions of dual string models have simple transformation properties under the modular group. This makes it possible to construct partition functions for new models. The next logical step is to construct the Fock space for the excited states of the string. In particular one would like to know the behaviour of these states under Lorentz transformations. Now the oscillators of the known models form representations of $O(D)$, where D is the transverse dimension. If one knows these representations, one can try to construct the other Lorentz generators, which amounts to the construction of the gauge algebra. Here we restrict ourselves to $O(D)$.

The appropriate quantity to be studied then is the helicity partition function, which is defined in Section 2. In Section 3 we formulate its transformation properties. In part, they have been guessed from the known models, in part from the consideration of functional integrals, which will be treated in a separate paper. Here we formulate the transformation properties as an axiom. Unfortunately, this generalization of modular functions has not yet been studied by mathematicians. Thus I have been forced to a mixture of mathematical and physical development, and I may not have kept the right balance. The physically interesting solutions are rather simple from a mathematical point of view, thus I shall discuss topics like analytical structure at the cusps, Hecke operators, etc. in a separate, mathematical paper. Nevertheless I do not want to run the risk of missing anything physically interesting. Fortunately, already the exclusion of helicity partition functions which cannot be compatible with unitarity, makes life much simpler. Thus I derive in Sections 4 and 5 that Regge trajectories are linear even in the most general case, and I show how one can read off these trajectories from the helicity partition function, generalizing the procedure of Goldstone and Thorn ²⁾. Then models with arbitrarily high intercepts will be excluded.

However, most of the remaining helicity partition functions still seem to be too unphysical to allow the construction of reasonable amplitudes. In (I) a simplicity assumption WH 2 has been used to extract the most interesting cases, but I do not know why it works. Thus in Section 6 I give some examples, ordered from "physical" to unphysical, hoping that somebody will be able to separate chaff from wheat with some simple additional axiom. In Section 7 I discuss some possibilities to do this, and also the hints given by the experiments.

2. THE HELICITY PARTITION FUNCTION

In (I) we have studied the partition function $A(w)$, which may be defined by

$$\text{Tr} \exp(iP_- t_+) = (2\alpha' w/i)^{-\frac{1}{2}D} A(w), \quad (1)$$

with conveniently normalized trace. We use the infinite momentum frame with momentum components $p_+, p_-, p_\perp = (p_1, \dots, p_D)$. The trace is over all states of the string under consideration with given p_+ , and

$$\frac{t_+}{p_+} = 2\pi\alpha'w. \quad (2)$$

Thus

$$\text{Tr} \exp(iP_- t_+) = \text{Tr} \exp(2\pi i\alpha'w(M^2 + P_\perp^2)). \quad (3)$$

For the moment α' is just a constant with dimension mass^{-2} . To include the spin, we consider the trace

$$\text{Tr} (R \exp(iP_- t_+)), \quad (4)$$

where R is an element of the covering group of $SO(D)$. We cannot use a larger subgroup of the proper Lorentz group $SO(D+1,1)$, as the little groups for tachyons, zero mass particles, and positive mass particles are $SO(D,1)$, $E(D)$, and $SO(D+1)$, respectively. [$E(D)$ is the Euclidean group in D dimensions.]

We restrict ourselves to even D , which is not unreasonable as daily experience shows that $D = 2$ (but compare Ref. 3!). Moreover it is rather uncertain, if string models for odd D exist.

Let $\exp(2\pi i\zeta_i)$, $i = 1, \dots, D/2$ be the eigenvalues of R in the vector representation. Then the trace (4) depends only on the angular variables $\zeta = (\zeta_1, \dots, \zeta_{D/2})$. We use the normalization

$$\langle p_{\perp}, \alpha | \tilde{p}_{\perp}, \tilde{\alpha} \rangle = \delta^D(p_{\perp} - \tilde{p}_{\perp}) \delta_{\alpha \tilde{\alpha}}, \quad (5)$$

where α includes all discrete quantum numbers such as mass and helicity. Then

$$\text{Tr} (R \exp (i P_{\perp} t_{\perp})) = \prod_{i=1}^{D/2} (2 \sin \pi \zeta_i)^{-2} \sum_{\lambda, m} d(\lambda, m) \exp (2 \pi i (\sum \lambda_i \zeta_i + \alpha' m^2 w)), \quad (6)$$

where $d(\lambda, m)$ is the number of states with mass m and helicity $\lambda = (\lambda_1, \dots, \lambda_{D/2})$. The helicity is defined with respect to a subgroup

$$\underbrace{\text{SO}(2) \times \dots \times \text{SO}(2)}_{D/2 \text{ factors}}$$

of $\text{SO}(D)$; it is a weight of $\text{SO}(D)$.

The transformation properties of the traces become simpler, if one includes only the square root of the zero mode factor. For example

$$W^{-D/4} A(w) = \left(-\frac{1}{w}\right)^{-D/4} A\left(-\frac{1}{w}\right) \quad (7)$$

in the ordinary string model. This becomes transparent if one uses functional integration and will be discussed in a subsequent paper.

Here we just define the helicity partition function

$$A(\zeta, w) = \prod_{i=1}^{D/2} (2 \sin \pi \zeta_i)^{-2} \sum_{\lambda, m} d(\lambda, m) \exp (2 \pi i (\sum \lambda_i \zeta_i + \alpha' m^2 w)). \quad (8)$$

For $d(\lambda, m)$ we require the following properties:

Axiom 1:

a)

$$\sum_{\lambda} d(\lambda, m) \exp (2 \pi i \sum \lambda_i \zeta_i)$$

is a character of $\text{SO}(D+1)$ for $m^2 > 0$,

is a character of $\text{SO}(D)$ for $m^2 = 0$,

is a constant for $m^2 < 0$.

- b) There is at most a finite number of states with $m^2 < 0$.
 c) For some β_0 ,

$$\sum_{\lambda} d(\lambda, m) = o(\exp m\beta_0). \quad (9)$$

d)

$$\max \left\{ \sum_{i=1}^{D/2} \lambda_i \mid d(\lambda, m) \neq 0 \right\} = O(\alpha' m^2). \quad (10)$$

We cannot admit non-trivial representations of $SO(D,1)$ for the tachyons, as they are infinite dimensional. 1d) means that the slope of Regge trajectories is bounded.

By virtue of axiom 1, Eq. (8) defines for $\text{Im } \zeta = 0$, $\text{Im } w > 0$ a meromorphic function of $\zeta = (\zeta_1, \dots, \zeta_{D/2})$ and w . We write $A(\zeta, w)$ also for the analytic continuation of this function. For each index set $K \subseteq \{1, \dots, D/2\}$ we define ζ_K by leaving out the components ζ_i of ζ for which $i \notin K$. We set

$$A(\zeta_K, w) = \lim_{\zeta_{\bar{K}} \rightarrow 0} A(\zeta, w) \prod_{i \notin K} (2 \sin \pi \zeta_i). \quad (11)$$

Here $\bar{K}$ is the complement of K in $\{1, \dots, D/2\}$. For $K = \emptyset$ we re-obtain Eq. (1). $A(\zeta_K, w)$ is meromorphic for $\text{Im } \zeta_K = 0$, $\text{Im } w > 0$. Let us list some elementary properties of $A(\zeta_K, w)$:

- i) $A(\zeta_K, w)$ is symmetric in the components of ζ_K .
 ii)

$$A(-\zeta_K^*, -w^*)^* = (-)^{|K|} A(\zeta_K, w). \quad (12)$$

- iii) $A(\zeta_K, w)$ is odd in each component of ζ_K , apart from a possible contribution from $m = 0$ in case of parity violation. However, we shall see that an unsymmetric contribution from $m = 0$ is incompatible with axiom 2 below.

- iv) If the string under consideration has only bosonic states, $A(\zeta_K, w)$ is anti-periodic under the transformation $\zeta_i \rightarrow \zeta_i + \delta_{ij}$, $j \in K$, whereas it is periodic for fermionic states. We shall also consider the mixed case with bosonic plus fermionic states. Then $A(\zeta_K, w)$ is only periodic under $\zeta_i \rightarrow \zeta_i + \delta_{ij} + \delta_{ik}$, $j, k \in K$. Note that $j = k$ is possible.
- v) Let ζ_{oK} , w_0 be purely imaginary. If $A(\zeta_{oK}, w)$ is regular for purely imaginary w with $w_0/i < w/i < \infty$, then $A(\zeta_K, w)$ is regular for $|\text{Im } \zeta_i| < |\zeta_{oK}|$, $i \in K$, $\text{Im } w > w_0/i$, as long as no component of ζ_{oK} is a real integer.
- vi) If at almost all mass levels the helicities of any state fulfil the inequality

$$\sum_{i \in K} \lambda_i \leq \kappa \alpha' m^2, \quad (13)$$

then $A(\zeta_K, w)$ is regular at points where no component of ζ_K is a real integer and where

$$\sum_{i \in K} |\text{Im } \zeta_i| < \kappa \text{Im } w. \quad (14)$$

The converse is also true. Thus the slopes of the leading Regge trajectories correspond to the singularities in ζ_K with smallest non-zero imaginary parts.

3. MODULAR TRANSFORMATIONS OF THE HELICITY PARTITION FUNCTION

Now we generalize the first working hypothesis of (I):

Axiom 2:

For some subgroup G of finite index in $SL(2, \mathbb{Z})$, one has

$$A\left(\frac{\zeta}{cw+d}, \frac{aw+b}{cw+d}\right) = \chi\left(\frac{a}{c} \frac{b}{d}\right) A(\zeta, w) \exp\left(-\mu \pi i c \zeta^2 / (cw+d)\right), \quad (15)$$

if $\begin{pmatrix} ab \\ cd \end{pmatrix} \in G$. Here

$$\int^2 = \sum_{i=1}^{D/2} \int_i^2 \quad (16)$$

and χ is a character of G .

The constant μ will be discussed below. In this context, "character" means (character of a) one dimensional representation.

For $A(\zeta_K, w)$ one finds

$$A\left(\frac{\int_K}{cw+d}, \frac{aw+b}{cw+d}\right) = \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) (cw+d)^{|K| - \frac{1}{2}D} A(\int_K, w) \times \\ \times \exp(-\mu\pi i c \int_K^2 / (cw+d)). \quad (17)$$

For $K = \emptyset$ one obtains back the first working hypothesis of (I). $A(\zeta_1, w) + A(-\zeta_1, w)$ transforms in the same way as $A(\zeta_1, w)$ itself. However, according to property (iii) of Section 2, this function can only get contributions from zero mass states, thus it cannot depend on w . Consequently it must vanish identically. Thus parity violation cannot manifest itself at the level of the particle spectrum.

For odd $D/2$, χ must be non-trivial, as

$$\chi(-g) = (-1)^{\frac{1}{2}D} \chi(g) \quad (18)$$

for any $g \in G$. For even $D/2$, we set $\chi = 1$.

Here I shall not try to motivate axiom 2 physically. From a mathematical point of view one should prove that the exponential factor cannot be replaced by a different function which is holomorphic and without zeros for $\text{Im } w > 0$. From Ref. 4) one obtains at least that the coefficient of ζ^2 can have no different form.

One could formulate axiom 2 for one branch of a multiple valued function $A(\zeta, w)$, but we shall see that branch points are impossible. Equation (17) has been studied by mathematicians⁵⁾, but instead of the periodicity in ζ_K they used different restrictions.

The translations $\zeta_i \rightarrow \zeta_i + 1$, $i \in K$, together with $SL(2, \mathbb{Z})$ generate a kind of affine group F_K , with elements

$$\begin{pmatrix} a & b & 0 \\ c & d & \sigma \\ m & n & 1 \end{pmatrix} : (f_K, w) \rightarrow \left(\frac{f_K + mw + n}{cw + d}, \frac{aw + b}{cw + d} \right). \quad (19)$$

Here

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad (20)$$

$$m, n \in \mathbb{Z}^{|K|}, \quad (21)$$

and the multiplication is the usual matrix multiplication. In general, G and the relevant translations of τ_K generate a symmetry group G_K of finite index in F_K . Of course $G_\emptyset = G$. In the non-mixed case, G_K consists of the transformations (19) for which

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad (22)$$

$$n \in \mathbb{Z}^{|K|}, \quad (23)$$

$$m/c_0 \in \mathbb{Z}^{|K|}. \quad (24)$$

We denote by c_0 the greatest common divisor of all c which occur in (22). In the mixed case, one has a basis

$$n_i = \delta_{ij} + \delta_{ik}, \quad j, k \in K \quad (25)$$

for n , and correspondingly

$$m_i = (\delta_{ij} + \delta_{ik}) c_0, \quad j, k \in K \quad (26)$$

for m . G_K is a semi-direct product of $\mathbb{Z}^{2|K|}$ by G . One finds

$$\begin{aligned} A \left(\begin{pmatrix} a & b & 0 \\ c & d & \sigma \\ m & n & 1 \end{pmatrix} (f_K, w) \right) &= (-1)^{\chi(m, n)} \chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} (cw + d)^{|K| - \frac{1}{2}D} A(f_K, w) \times \\ &\times \exp \left(\frac{\mu \pi i}{cw + d} (-c f_K^2 + (md - nc)(2f_K + mw + n)) \right) \end{aligned} \quad (27)$$

and

$$\gamma(am+cn, bm+dn) = \gamma(m, n) \quad (28)$$

for all elements (19) of G_K . Moreover, $\gamma(m, n)$ can be written in the form

$$\gamma(m, n) = \mu mn + \gamma_1 \sum m_i + \gamma_2 \sum n_i, \quad (29)$$

where

$$\gamma_1 c_0 \in \mathbb{Z}. \quad (30)$$

For bosons $\gamma_2 = 1$, for fermions $\gamma_2 = 0$. In the mixed case, one may set $\gamma_1 = \gamma_2 = 0$. For consistency one has to require

$$\mu mn \in \mathbb{Z} \quad (31)$$

For $D > 4$, and for $D = 4, 2$, non-mixed case, mn can assume the value c_0 , and one has

$$\mu c_0 \in \mathbb{Z}. \quad (32)$$

For $D = 4$, mixed case, one finds $2\mu c_0 \in \mathbb{Z}$, and for $D = 2$, mixed case, $4\mu c_0 \in \mathbb{Z}$.

Equation (28) yields an additional restriction for μ . If, for example, $G_K = F_K$, one finds $\gamma_1 = \gamma_2 = 1$ and

$$\mu \in 2\mathbb{Z} + 1. \quad (33)$$

Note that fermions are impossible in this case. We shall see below that $A(\zeta_K, w)$ is meromorphic for $\text{Im } w > 0$ in the physically interesting cases. Restricting Eq. (27) to $\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{pmatrix} 10 \\ 01 \end{pmatrix}$, we see that $A(\zeta_K, w)$ is essentially an elliptic function with respect to each component of ζ_K . Let $v = 1, 2$ for the non-mixed and mixed cases, respectively. Then the pair $(vc_0 w, v)$ spans a fundamental parallelogram of $A(\zeta_K, w)$, regarded as a function of ζ_1 . Integrating the logarithmic derivative of this function around a fundamental parallelogram one finds

$$\mu v^2 c_0 = \text{number of poles} - \text{number of zeros} \quad (34)$$

of $A(\zeta_K, w)$ within a fundamental parallelogram.

4. WHY THE HELICITY PARTITION FUNCTION SHOULD BE MEROMORPHIC

Let us consider the fundamental region of G_K . If we keep ζ_K fixed, w varies over the fundamental region of G , and if w is kept fixed, ζ_K varies over the fundamental region of an elliptic function. As for the modular group, we can compactify this fundamental region by including the cusps of G . There, ζ_K loses its meaning, in the same fashion as the polar co-ordinate angle for vanishing radius. The topological structure near cusps is that of a torus shrinking to a point. Thus the compactified fundamental region is not everywhere locally homeomorphic to the complex space $C^{D/2+1}$, it is not a complex manifold. Nevertheless, it can be endowed with the structure of an analytic space, but here this will not be discussed. We just note that with

$$\zeta_i = \xi_i + i\eta_i, \quad i \in K, \quad (35)$$

$$w = x + iy, \quad (36)$$

we find that the measure

$$y^{-|K|-2} dx dy \prod_{i \in K} d\xi_i d\eta_i \quad (37)$$

is invariant, and that the total volume of the fundamental region is finite.

Thus we have a close analogy to the special case $K = \emptyset$, and we can expect that the everywhere meromorphic solutions of Eq. (27) form a finite dimensional vector space, once the pole structure of $A(\zeta_K, w)$ is known.

Thus our next task is the determination of the singularities of $A(\zeta, w)$, which will turn out to be intimately related to the Regge trajectories.

In particular, we should exclude branch points and essential singularities. For branch points this will be done below. But in general, we can only make plausible that $A(\zeta, w)$ is meromorphic. At first, let us consider special cases, where one can give a proof.

Let $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$. This is true for the conventional and the Neveu-Schwarz strings. As has been shown in (I), for such groups G one necessarily has a tachyon. We restrict ourselves to the non-mixed case, i.e., to purely bosonic spectra.

Then $A(\zeta_K, w)$ is meromorphic. Its poles lie in the hyperplanes

$$P_{imn} = \{s, w \mid s_i = mw + n\}, \quad (38)$$

where $i \in K$, $m, n \in \mathbb{Z}$. These poles are simple.

Proof: We have $c_0 = v = 1$. Thus we can restrict ourselves to the region where

$$|\operatorname{Re} s_i| \leq \frac{1}{2}, \quad (39)$$

$$|\operatorname{Im} s_i| \leq \operatorname{Im} w / 2 \quad (40)$$

for all $i \in K$. For real ζ_K , the poles are given by the zeros of $\prod \sin \pi \zeta_i$. Now take w purely imaginary and apply the transformation $w \rightarrow -1/w$, $\zeta_K \rightarrow \zeta_K/w$. One finds that there is no pole, if

$$0 < |\operatorname{Im} s_i| < w/i \quad (41)$$

for all $i \in K$. By property (v) of Section 2, there is no pole for

$$0 < |\operatorname{Im} s_i| < \operatorname{Im} w, \quad (42)$$

where w is arbitrary. Thus the pole structure in one fundamental region is completely known. For other fundamental regions, the pole structure is that of

$$\prod_{i \in K} (\sin \pi (s_i + m_i w))^{-1}, \quad (43)$$

where $m_i \in \mathbb{Z}$ for all $i \in K$. Q.E.D.

Now let $A(\zeta, w)$ be modular for some group $G_{D/2}$. If we transform this function by any element of $F_{D/2}$ then the transformed function

$$\tilde{A}(s, w) = A\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix} (s, w)\right) \exp\left(\frac{\mu \pi i}{c w + d} (c s^2 - (md - nc)(2s + mw + n))\right) \quad (44)$$

is modular for the conjugated subgroup

$$\tilde{G}_{D/2} = \begin{pmatrix} a & b & \sigma \\ c & d & \sigma \\ m & n & 1 \end{pmatrix}^{-1} G_{D/2} \begin{pmatrix} a & b & \sigma \\ c & d & \sigma \\ m & n & 1 \end{pmatrix}. \quad (45)$$

The singularities of these functions are obviously in one-to-one correspondence. Note in particular that the system of hyperplanes P_{imn} is invariant under F_K .

We have seen in (I) that the partition functions of the Neveu-Schwarz and the Ramond model are related by a modular transformation. We shall find below that this stays true for the helicity partition functions. Thus it seems possible to generalize a conjecture which D'Adda formulated for the partition functions⁶⁾.

d'Adda's conjecture:

For any dual string model consider the subspaces for which the multiplicative quantum numbers of the model assume definite values. Let P_s denote the projection operators on these subspaces. Take in analogy to the Eqs. (6), (8)

$$A_s(p, w) = \prod_{i=1}^{D/2} (2 \sin \pi p_i) \cdot \text{Tr} (P_s R \exp(i p_- t_+)). \quad (46)$$

Then the $A_s(\zeta, w)$ span a finite dimensional vector space, which is invariant and irreducible under $F_{D/2}$.

Models for which this conjecture is true have a tachyon in some sector. Then it is rather natural to assume that this sector is invariant under $w \rightarrow -1/w$, such that the pole structure (38) applies.

Moreover, we shall see in the next section that this pole structure is the most natural from the point of view of Regge behaviour. In contrast, even multiple poles yield unreasonable results, and essential singularities are probably worse.

5. REGGE POLES

Let $P_K(w)$ be the $(2|K|-2)$ -dimensional surface of singularities of $A(\zeta_K, w)$, where w is kept fixed for the moment. From the axioms we know that $A(\zeta_K, w)$ has no singularities, if

$$p_K \in R_K(w) = \left\{ p(cw+d) + mw+n \mid p \in R^{|K|}, \begin{pmatrix} a & b & \sigma \\ c & d & \sigma \\ m & n & 1 \end{pmatrix} \in G_K \right\}, \quad (47)$$

and if no component of ρ is integer. $R_K(w)$ is dense in $C^{|K|}$. This leaves no room for branch points, and constitutes a strong restriction for $P_K(w)$. From now on we assume that $P_K(w)$ contains only poles. For simplicity, we exclude the hyperplanes (38) from $P_K(w)$.

$R_K(w)$ is the union of $|K|$ -dimensional hyperplanes. Any hyperplane

$$\{\rho(\alpha w + \beta) + \gamma \mid \rho \in R^{|K|}\} \quad (48)$$

with arbitrary $\alpha, \beta \in R$, $\gamma \in C^{|K|}$, can be locally approximated with arbitrary accuracy by hyperplanes of $R_K(w)$. To avoid intersections of $P_K(w)$ with $R_K(w)$, the tangential hyperplanes of $P_K(w)$ must be parallel to all hyperplanes of $R_K(w)$. This is only possible if they have the form

$$\sum_{i \in K} \lambda_i \rho_i = \rho_0, \quad (49)$$

where $\lambda_i \in R$ for all $i \in K$. But analytic functions which are everywhere real are constant. Thus ρ_0 and the λ_i can be made constant, too, and $P_K(w)$ itself consists of hyperplanes of the form (49). Now the hyperplanes (38) can be included again as special cases.

For $\text{Im } \zeta_0 = 0$, ζ can become real. Thus for such ζ_0 one obtains only the solutions of $\prod \sin \pi \zeta_i = 0$. Using the transformations of G_K which leave w invariant we see that the λ_i must be integer multiples of some λ_0 , otherwise one obtains an accumulation of pole hyperplanes. We normalize λ_0 in such a way that the translations of ζ in G_K induce the invariance transformations

$$\rho_0 \rightarrow \rho_0 + c_0 m w + n, \quad m, n \in Z, \quad (50)$$

of $P_K(w)$. Then

$$\lambda_i v \in Z, \quad (51)$$

$$\lambda_i - \lambda_j \in Z \quad (52)$$

for all $i, j \in K$.

Now take $K = \{1\}$. The points of $P_1(w)$ are locked within $R_1(w)$, which forms a dense union of lines in the complex plane. Varying w one finds that the lines of singularity of $A(\zeta_1, w)$ must have the form

$$\zeta_1 = \alpha w + \beta, \quad (53)$$

where α, β are real and independent of w .

From axiom 2 we know that with $\zeta_1 = (\alpha\beta) \binom{w}{1}$ also any $\zeta_1 \in (\alpha, \beta + v c_0 Z) G \binom{w}{1}$ is a singularity of $A(\zeta_1, w)$. Thus α and β must be rational, otherwise the imaginary part of ζ_1 could become arbitrarily small (for $\alpha = 0$ we have of course $\zeta_1 \in Z$). Inserting this result into Eq. (49), we see that the poles of $A(\zeta, w)$ lie in sets of hyperplanes of the form

$$\sum \lambda_i \zeta_i + \alpha w + \beta \in Z \oplus c_0 w Z, \quad (54)$$

where the λ_i fulfil the relations (51) and (52), and α, β are rational. We choose α, β such that

$$0 \leq \alpha \leq \frac{1}{2} c_0, \quad (55)$$

$$0 \leq \beta < 1. \quad (56)$$

Because of the properties (i-iii) of $A(\zeta, w)$ listed in Section 2, one obtains other pole hyperplanes from (54), if one changes the sign of some λ_i , permutes the λ_i , or replaces β by $1-\beta$. One can also replace $(\alpha\beta)$ by $(\alpha\beta)g$ with any $g \in G$.

A set of pole hyperplanes (54) is generated by a factor

$$\prod_{n=1}^{\infty} \left[1 - \exp 2\pi i (\sum \lambda_i \zeta_i + (\alpha + n c_0) w + \beta) \right]^{-1} \quad (57)$$

of $A(\zeta, w)$ (times a similar factor which takes care of $-\infty < n \leq 0$). For $\beta = 0$ this corresponds to a set of bosonic generators of excitations of the string which changes the helicity by $\lambda = (\lambda_1, \dots, \lambda_{D/2})$ and $\alpha' m^2$ by $\alpha + n c_0$. Remember that $\beta = 0$, if $\alpha = 0$. For $\beta \neq 0$ there seems to be only one natural way to ensure the

positivity of $d(\lambda, m)$ required by axiom 1: All multiples of $\beta \bmod 1$ have to occur, with α unchanged. Then we can take these β as multiples of some $1/L$, $L \in \mathbb{Z}$, and obtain the factor

$$\prod_{n=1}^{\infty} \left[1 - \exp 2\pi i / L \left(\sum \lambda_i f_i + L (\alpha + n c_0) w \right) \right]^{-1}. \quad (58)$$

According to relation (51), bosonic excitations with half odd helicity can only occur in the mixed case, as it has to be. We have seen in Section 4 that for a large class of models only helicities of the form $\lambda_i = \pm \delta_{ij}$ with $\alpha = \beta = 0$ occur. They correspond to vector excitations $A_{n\mu}^+$, $n = 1, 2, \dots$. Experimentally only $A_{1\mu}^+$ is "observed" up to now.

From a field theoretic point of view it is difficult to imagine bosonic excitations of the string in addition to the changes of shape which are expressed by the $A_{n\mu}^+$. In particular, all internal symmetries should be compact, such that the oscillations of the corresponding quanta cannot have arbitrarily large amplitudes.

If λ is a highest weight, the factors of (58) correspond obviously to Regge trajectories with slope $\alpha' \lambda / (\alpha + n c_0)$. For a double pole one would obtain degeneracies along this trajectory which increase linearly with $\alpha' m^2$. For a triple pole this increase would be quadratic, etc. Thus we have good reason to admit only simple poles for $A(\zeta, w)$.

Goldstone and Thorn²⁾ have shown that all Regge trajectories of the ordinary string model (including the sister trajectories⁷⁾ with slope α'/n) can be obtained from a partial fraction expansion of the helicity partition function. This procedure can easily be generalized. Consider a Mittag-Leffler expansion of $A(\zeta, w)$ with denominators combined to

$$\sin \pi L \left(\sum \lambda_i f_i + \alpha + n c_0 \right). \quad (59)$$

Then an expansion of the sine yields a linear Regge trajectory for which the leading weight changes by

$$\Delta \lambda = L (\lambda_1, \dots, \lambda_{D/2}), \quad (60)$$

$$\Delta \alpha' m^2 = L (\alpha + n c_0). \quad (61)$$

The residue yields the daughter structure. For $L = 1$ the trajectories are exchange degenerate, because the greatest common divisor of the $v\lambda_i$ and the $\lambda_i - \lambda_j$ is 1.

Now let us consider the poles of $A(\zeta_1, w)$ at $\zeta_1 = mvc_0w + nv$, $m, n \in \mathbb{Z}$. There residues are related by axiom 2. In the non-mixed case one obtains for bosons a contribution

$$\sum_{n=-\infty}^{\infty} \left(\frac{(-1)^{n\gamma_1} \exp(\mu\pi i n^2 c_0^2 w)}{2 \sin \pi(\beta_1 + nc_0 w)} - \text{subtractions} \right) A(w) \quad (62)$$

to the Mittag-Leffler expansion of $A(\zeta_1, w)$, and for fermions

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2} \exp(\mu\pi i n^2 c_0^2 w) \cot \pi(\beta_1 + nc_0 w) - \text{subtractions} \right) A(w). \quad (63)$$

In the mixed case one finds

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2} \exp(4\mu\pi i n^2 c_0^2 w) \cot \pi\left(\frac{1}{2}\beta_1 + nc_0 w\right) - \text{subtractions} \right) A(w). \quad (64)$$

Expanding in powers of $\exp 2\pi i(\zeta_1/v + nc_0 w)$ one sees that for $\mu < 0$ the sister trajectories have arbitrarily high intercept

$$\alpha_{n0} \sim -\mu n/2. \quad (65)$$

As in the ordinary string model, one expects that in the tree graph approximation only the trajectories with highest slope contribute to the four-point function, but unitarity corrections will cause a coupling of the sister trajectories. Thus $\mu < 0$ is incompatible with unitarity. If furthermore the only bosonic excitations of the string are the changes of shape, then Eq. (34) yields $\mu \leq 1$.

For $\mu \geq 0$ it is rather easy to construct the solutions of Eq. (27) once the pole hyperplanes (54) are known. Consider $A(\zeta, w)$ as a function of ζ_1 . For $\mu \geq 0$, this function is determined by its residues. Otherwise one could construct a solution of Eq. (27) without poles, in contradiction to Eq. (34). Note that a constant function of ζ_1 cannot occur, as $A(\zeta, w)$ is odd.

For $\mu > 0$, a glance at the expressions (62)-(64) shows that one needs no subtractions in the Mittag-Leffler expansion. At least in the most important cases one also needs no entire part. For $\mu = 0$, $A(\zeta, w)$ is an odd elliptic function. Thus again one needs neither subtractions nor an entire part.

Now one can consider the residues as functions of ζ_2 . They are again determined by their own residues, etc. Thus $A(\zeta, w)$ is determined by its multiple residues at the intersections of $D/2$ hyperplanes of the form (54). These multiple residues are essentially modular functions. Thus the solution of Eq. (27) is reduced to the determination of the pole structure, and of a finite number of modular functions.

6. EXAMPLES

a) The ordinary string

Let us first consider the case with pole structure (38). For $\mu = 1$, Eq. (34) tells us that the solutions of Eq. (27) have no zeros as functions of ζ . Thus the quotient of two of these solutions is an elliptic function without poles, i.e., it does not depend on ζ . We obtain

$$\begin{aligned} A(p_K, w) &= A(w) \prod_{i \in K} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n \exp(\mu \pi i n^2 w)}{2 \sin \pi(p_i + n w)} \right) = \\ &= A(w) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n w))^{3/|K|} \prod_{i \in K} \theta_1(p_i/w)^{-1}. \end{aligned} \quad (66)$$

The equality follows from the fact that both sides transform as theta functions and have identical residues.

Equation (66) yields the spin structure of the ordinary string, but of course $A(w)$ may contain additional factors, in particular factors arising from internal symmetries.

There can be no tachyon with $\alpha' m^2 < -1$, as otherwise one would also obtain a tachyon with non-vanishing helicity, in contradiction to Lorentz invariance.

Let us apply these results to the case $G_K = F_K$, i.e. the non-mixed case with $G = \text{SL}(2, \mathbb{Z})$. In Section 4 we have derived the pole structure (38) for this case, thus Eq. (34) yields $\mu \leq 1$. Unitarity yields $\mu \geq 0$, and Eq. (33) leaves only the possibility $\mu = 1$. In this simple case, the partition function can be determined

without additional assumptions. Because of the restriction for tachyons, $A(w)$ has just a single pole at the only cusp $w = i\infty$. From Eq. (15) of (I) we find that $A(w)$ can only have zeros at $w = i$ and $w = \exp \pi i/3$, where they count with multiplicity 1/2 and 1/3, respectively. But a zero at $w = i$ is impossible, because in this case $A(w)$ would contain a factor

$$E_3(w) = 1 - 540 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad (67)$$

$$q = \exp 2\pi i w, \quad (68)$$

$$\sigma_3(n) = \sum_{k|n} k^3, \quad (69)$$

and $d(\lambda, m)$ could not be non-negative. Thus there are only three possibilities:

i) No zero, $D = 24$,

$$A(w) = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24}. \quad (70)$$

This is the conventional string spectrum. By insertion into Eq. (66) we obtain the helicity partition function of this model.

ii) Simple zero at $w = \exp \pi i/3$, $D = 16$,

$$A(w) = E_2(w) q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24}. \quad (71)$$

iii) Double zero at $w = \exp \pi i/3$, $D = 8$,

$$A(w) = E_2(w)^2 q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24}. \quad (72)$$

Here

$$E_2(w) = 1 + 240 \sum_{n=1}^{\infty} \sigma_2(n) q^n, \quad (73)$$

$$\sigma_2(n) = \sum_{k|n} k^2. \quad (74)$$

The possibilities (ii)-(iii) had been excluded in (I) by WH(2). Here we see that they are very unreasonable from a physical point of view (note in particular the huge number of massless scalars).

In all cases one obtains sister trajectories with slope α'/n , which correspond to the usual excitations $A_{n\mu}^+$. Their intercepts are

$$\alpha_{n0} = \frac{1}{n} - \frac{n-1}{2}, \quad n=1, 2, \dots \quad (75)$$

b) The Neveu-Schwarz-Ramond string, and related spectra

Now let us consider the case $\mu = 0$, but still with pole hyperplanes (38). Here we have many more possibilities. Let us consider the simplest case where all residues are related by G_K . Then

$$A(p+mw+n, w) = (-)^{\gamma(m,n)} A(p, w). \quad (76)$$

For $\gamma(m,n)$ we have the following possibilities:

i)

$$\gamma(m,n) = \sum (m_i + n_i). \quad (77)$$

This yields

$$\begin{aligned}
 A(p_K, w) &= A(w) \prod_{i \in K} \sum_{n=-\infty}^{\infty} \frac{(-)^n}{2 \sin \pi(p_i + nw)} = \\
 &= A(w) \prod_{i \in K} \left(\frac{\theta_3(p_i/w)}{\theta_1(p_i/w)} / \frac{\theta_3(0/w)}{(2\pi)^{-1} \theta_1'(0/w)} \right). \quad (78)
 \end{aligned}$$

Thus we have obtained the spin structure of the Neveu-Schwarz model. For this model, $A(\zeta, w)$ has a very simple form, as

$$A(w) = \left(\frac{\theta_3(0/w)}{(2\pi)^{-1} \theta_1'(0/w)} \right)^8. \quad (79)$$

ii)

$$y(m, n) = \sum n_i. \quad (80)$$

This case can be reduced to (i) by the transformation $(\zeta, w) \rightarrow (\zeta, w + 1)$. The corresponding $A(\zeta, w)$ is not consistent with axiom 1, as one obtains $d(\lambda, m)$ of both signs. But it can be interpreted according to D'Adda's hypothesis. For the Neveu-Schwarz model states with different G-parities are counted with different signs.

iii)

$$y(m, n) = \sum m_i. \quad (81)$$

This case can be reduced to (i) by the transformation $(\zeta, w) \rightarrow (\zeta/(w+1), w/(w+1))$. One finds

$$\begin{aligned}
 A(p_K, w) &= A(w) \prod_{i \in K} \sum_{n=-\infty}^{\infty} \frac{1}{2} (-)^n \cot \pi(p_i + nw) = \\
 &= A(w) \prod_{i \in K} \left(\frac{\theta_2(p_i/w)}{\theta_1(p_i/w)} / \frac{\theta_2(0/w)}{(2\pi)^{-1} \theta_1'(0/w)} \right). \quad (82)
 \end{aligned}$$

This is the spin structure of the Ramond model, for which

$$A(w) = \left(\frac{\theta_2(\sigma/w)}{(2\pi)^{-1} \theta_1'(\sigma/w)} \right)^8. \quad (83)$$

$\gamma(m,n) = 0$ is impossible, as the sum of the residues of an elliptic function must vanish. The invariance groups G_{NS} and G_R , which characterize according to (I) the Neveu-Schwarz and the Ramond models, are the largest subgroups of $SL(2,Z)$ which are compatible with the spin structures (78) and (82), respectively. If one wants to include internal symmetries into $A(w)$, one has to use subgroups of G_{NS} or G_R .

In (I) it has been proposed to describe mesons in zeroth approximation by Eq. (78) with

$$\begin{aligned} A(w) &= \prod_{\substack{n=1,2,\dots \\ r=\frac{1}{2},\frac{3}{2},\dots}} (1-q^n)^{-2} (1+q^n)^3 (1+q^{\frac{n}{2}})^2 (1+q^{2n})^2 = \\ &= \frac{1}{2} (\theta_4(\sigma/\frac{w}{2}) \theta_3(\sigma/2w))^{-1}. \end{aligned} \quad (84)$$

Here we see that $A(\zeta,w)$ has indeed reasonable transformation properties. However, it is probably not possible to exclude the transformed function

$$\begin{aligned} A(w) &= \frac{1}{2} q^{-\frac{7}{4}} \prod_{\substack{n=1,2,\dots \\ r=\frac{1}{2},\frac{3}{2},\dots}} (1-q^n)^{-2} (1+q^n)^3 (1+q^{\frac{n}{2}})^2 (1+q^{2n})^2 = \\ &= \frac{1}{2} (\theta_4(\sigma/\frac{w}{2}) \theta_2(\sigma/2w))^{-1}, \end{aligned} \quad (85)$$

with corresponding helicity partition function again given by Eq. (78). Furthermore, D'Adda's conjecture yields additional sectors with fermions. One also has problems with the internal quantum numbers, which will be discussed below.

Expanding the terms of the sums of Eqs. (78) and (82) into powers of $\exp 2\pi i(\zeta_i + nw)$, one sees that in both cases all leading sister trajectories cross at the lowest mass level and helicity one half. In particular, the zero mass spinor of the Ramond model lies on all leading sister trajectories. For the Neveu-Schwarz model one has the intercepts

$$\alpha_{n0} = \frac{1}{2n} + \frac{1}{2}, \quad n = 1, 2, \dots \quad (86)$$

If Eq. (78) has something to do with experiment, one should observe sister trajectories with

$$\alpha_{n\sigma} = \frac{1}{2}, \quad n = 1, 2, \dots \quad (87)$$

It is interesting that the partition functions (84) and (85) can also arise from a quite different type of helicity partition function: We maintain of course the interpretation of the bosonic excitation, which corresponds to the factors $(1-q^n)^{-1}$. They are vectors and correspond to the changes of shape. But we take the $(1+q^r)$ -oscillators as scalars, the $(1+q^{r/2})$ -oscillators as spinors, and the $(1+q^{2n})$ - and $(1+q^{2r})$ -oscillators as vectors. Furthermore, we have a spinor ground state for Eq. (84). Then we have the same symmetry groups $G \subset SL(2, \mathbb{Z})$ as before. The G_K are of course smaller, as we now are in the mixed case.

We have the same pole planes as before, but the residues at $\zeta = mw+n$, $m, n \in \mathbb{Z}$ are now for (84)

$$\begin{aligned} & \left(\theta_4(\sigma/\frac{w}{2}) \theta_3(\sigma/2w) \right)^{-1}, & \text{if } (m, n) \equiv (0, 0) \pmod{2}, \\ & \left(\theta_3(\sigma/\frac{w}{2}) \theta_3(\sigma/2w) \right)^{-1}, & \text{if } (m, n) \equiv (0, 1) \pmod{2}, \\ & - \left(\theta_4(\sigma/\frac{w}{2}) \theta_2(\sigma/2w) \right)^{-1}, & \text{if } (m, n) \equiv (1, 0) \pmod{2}, \\ & \left(\theta_3(\sigma/\frac{w}{2}) \theta_2(\sigma/2w) \right)^{-1}, & \text{if } (m, n) \equiv (1, 1) \pmod{2}. \end{aligned}$$

For (85) the residues are just shifted. We have $\mu = 0$ also in this case. If we would take the $(1+q^r)$ -oscillators as vectors, too, then the G_K would not change, but we would obtain $\mu = -1$, which is incompatible with unitarity, as we have seen. In general, one cannot have too many fermionic oscillators which transform non-trivially under $O(D)$, as they yield zeros in the helicity partition function. According to Eq. (34), too many zeros make μ negative.

c) Pathological cases

Now let us consider a helicity partition function for which the pole hyperplanes are not given by (38) alone. We use the partition function

$$A^0(w) = \prod_{\substack{n=1,2,\dots \\ r=\frac{1}{2}, \frac{3}{2}, \dots}} (1-q^n)^{-8} (1+q^r)^{16}, \quad (88)$$

which has been introduced in (I). The bosonic excitations are treated in the usual way. But the fermionic excitations are $O(8)$ -spinors, such that

$$A^0(f, w) \sim \prod_r \prod_{\pm} (1 + (\exp \pi i \sum_{i=1}^4 (\pm f_i)) q^r). \quad (89)$$

The product is over all sign combinations, it has 16 factors. Here we have in addition to (38) the pole hyperplanes

$$\sum_{i=1}^4 (\pm \frac{1}{2} f_i) \in (Z + \frac{1}{2})w \oplus Z \quad (90)$$

and $\mu = 3$. The residues are proportional to

$$\begin{aligned} \theta_4(w)^{-2}, & \quad \text{if } (m,n) \equiv (0,0) \pmod{2}, \\ \theta_3(w)^{-2}, & \quad \text{if } (m,n) \equiv (0,1) \pmod{2}, \\ \theta_2(w)^{-2}, & \quad \text{if } (m,n) \equiv (1,1) \pmod{2}. \end{aligned}$$

For $(m,n) \equiv (1,0) \pmod{2}$, not only eight hyperplanes of (38) intersect, but also eight hyperplanes of (90). Here the residue is proportional to

$$\eta(w)^{-24} = q^{-1} \prod_{n=1}^{\infty} (1-q^n)^{-24}. \quad (91)$$

$A^0(\zeta, w)$ does not fulfil D'Adda's conjecture, as $A(\zeta/w, -1/w)$ has an unacceptable pole structure for real ζ .

So far we have considered helicity partition functions which could be written in a neat factorized form, allowing a construction of the excited states in terms of oscillators. For such functions not only the poles but also the zeros lie in hyperplanes of the form (54). We could equally well characterize each $A(\zeta, w)$ by the location of its poles and zeros as by pole hyperplanes and residues. G_K can be characterized as an invariance group of the pole and zero surfaces, in complete analogy to WH 3 of (I). However, from our axioms we cannot conclude that the surface $A(\zeta, w) = 0$ must be a union of hyperplanes. I am somewhat reluctant to require factorization as an additional axiom, because the experimental situation is not quite clear. Let us consider as a last example the case $G = SL(2, Z)$, $v = 2$. From Eq. (32) we have $\mu \in Z$. The symmetry group is still large enough to restrict the pole hyperplanes to (38), thus $\mu = 0$ or 1 . The case $\mu = 1$ has been discussed above, thus we assume $\mu = 0$.

The possible partition functions are still given by Eqs. (70)-(72), but now we have

$$A(w) = (A_B + A_F)(w), \quad (92)$$

where A_B, A_F are the partition functions for bosons and fermions. The residues are for $\zeta_i = m_i w + n_i$, $m_i, n_i \in Z$

$$(A_B + A_F)(w), \quad \text{if } (\sum m_i, \sum n_i) \equiv (0, 0) \pmod{2},$$

$$(A_B - A_F)(w), \quad \text{if } (\sum m_i, \sum n_i) \equiv (0, 1) \pmod{2},$$

$$w^{\frac{D}{2}} (A_B - A_F)\left(-\frac{1}{w}\right), \quad \text{if } (\sum m_i, \sum n_i) \equiv (1, 0) \pmod{2},$$

$$(w+1)^{\frac{D}{2}} (A_B - A_F)\left(\frac{w}{w+1}\right), \quad \text{if } (\sum m_i, \sum n_i) \equiv (1, 1) \pmod{2}.$$

They are modular of weight $-D/2$ with respect to $\Gamma, \Gamma_0(2), \Gamma^0(2), G_\theta$, respectively. Multiplying by $\eta(w)^{24}$ one obtains modular forms, i.e. modular functions without poles. They have weight $12-D/2$. The space of modular forms with weight k for G and its conjugated subgroups $\Gamma_0(2), \Gamma^0(2)$ is $(1+k/4)$ -dimensional⁸⁾. As $A(\zeta, w)$ is an elliptic function, the sum of residues has to vanish. Thus $A(w)$ can be

expressed in terms of $(A_B - A_F)(w)$. If this condition is fulfilled we can explicitly construct an elliptic function with the given residues, e.g. by a Mittag-Leffler expansion.

Lorentz invariance yields the additional condition that $\Lambda_F(w)$ cannot contain a tachyon term. With this condition we obtain a $(3-D/8)$ -dimensional solution space.

For $D = 24$ there is no solution. For $D = 16$ the solutions are proportional to

$$A(\mathcal{F}, w) = \left(\prod_{i=1}^8 \theta_1(\mathcal{F}_i | w)^{-1} \right) \sum_{j=2}^4 \prod_{i=1}^8 \theta_j(\mathcal{F}_i | w) \quad (93)$$

At mass zero, the representation content with respect to $O(16)$ is $2 \times (1, 1, 0, \dots) \oplus 2 \times (1, 0, \dots) \oplus (\frac{1}{2}, \frac{1}{2}, \dots)$. But at $\alpha' m^2 = 1$ there is no $SO(17)$ invariance, as there are too few states with helicity zero. Thus Lorentz invariance is violated, too.

For $D = 8$ a basis of the solution space is given by

$$\begin{aligned} A(\mathcal{F}, w) = & \frac{1}{2} \left(\prod_{i=1}^4 \theta_1(\mathcal{F}_i | w)^{-1} \right) \left(\prod_{n=1}^{\infty} (1 - q^n)^{-12} \right) E_2(w) \times \\ & \times \left[\prod_{i=1}^4 \theta_2(\mathcal{F}_i | 2w)^2 + \prod_{i=1}^4 \theta_3(\mathcal{F}_i | 2w)^2 + \frac{1}{16} \sum_{k=0}^3 \prod_{i=1}^4 \theta_3\left(\frac{\mathcal{F}_i}{2} \middle| \frac{w+k}{2}\right)^2 \right. \\ & \left. + \frac{1}{16} \sum_{k=0}^7 \underbrace{\left(\theta_3\left(\frac{\mathcal{F}_1}{2} \middle| \frac{w+k}{2}\right)^2 \theta_3\left(\frac{\mathcal{F}_2}{2} \middle| \frac{w+k}{2}\right)^2 \theta_4\left(\frac{\mathcal{F}_3}{2} \middle| \frac{w+k}{2}\right)^2 \theta_4\left(\frac{\mathcal{F}_4}{2} \middle| \frac{w+k}{2}\right)^2 \right)}_{6 \text{ permutations}} \right] \\ & + \underbrace{\theta_2(\mathcal{F}_1 | 2w)^2 \theta_2(\mathcal{F}_2 | 2w)^2 \theta_3(\mathcal{F}_3 | 2w)^2 \theta_3(\mathcal{F}_4 | 2w)^2 + \dots}_{6 \text{ permutations}}, \end{aligned} \quad (94)$$

$$\begin{aligned} A(\mathcal{F}, w) = & \prod_{i=1}^4 \theta_1(\mathcal{F}_i | w)^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-12} \left[\sum_{j=1}^4 \prod_{i=1}^4 \theta_j\left(\frac{\mathcal{F}_i}{2} \middle| w\right)^4 + \right. \\ & \left. + \sum_{j < k} \underbrace{\left(\theta_j\left(\frac{\mathcal{F}_1}{2} \middle| w\right)^4 \theta_j\left(\frac{\mathcal{F}_2}{2} \middle| w\right)^4 \theta_k\left(\frac{\mathcal{F}_3}{2} \middle| w\right)^4 \theta_k\left(\frac{\mathcal{F}_4}{2} \middle| w\right)^4 + \dots \right)}_{6 \text{ permutations}} \right] \\ & + \underbrace{\theta_1\left(\frac{\mathcal{F}_1}{2} \middle| w\right)^4 \theta_2\left(\frac{\mathcal{F}_2}{2} \middle| w\right)^4 \theta_3\left(\frac{\mathcal{F}_3}{2} \middle| w\right)^4 \theta_4\left(\frac{\mathcal{F}_4}{2} \middle| w\right)^4 + \dots}_{24 \text{ permutations}}. \end{aligned} \quad (95)$$

The referee of (I) asked for a physical motivation of WH 2. I still cannot give any, but look what happens if you do not use it!

The solutions can be constructed by starting with a function of slightly lower symmetry, applying all symmetry transformations, and summing. Mathematically speaking, one uses Hecke operators.

At mass zero one finds for Eq. (94) the $O(8)$ representation content $(1,1,0,0) \oplus 9 \times (1,0,0,0) \oplus 8 \times (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \oplus 329 \times (0,0,0,0)$. For Eq. (95) one has $(1,1,0,0) \oplus 25 \times (1,0,0,0) \oplus 76 \times (0,0,0,0)$. The particle spectrum is probably consistent with Lorentz invariance, but the tensor and the huge number of scalars at mass zero are prohibitive. Surely, these solutions will be excluded by some axiom which has not yet been formulated. Note in particular that they have no product expansion. Thus the space of states cannot be constructed using oscillators in the usual way.

7. DISCUSSION

Let us summarize: The partition functions of relativistic strings are modular functions. Extending the modular group one can obtain the spin distribution at each mass level. When one includes angular variables into the partition function, one finds the transformation law (27).

If the helicity partition function $A(\zeta, w)$ is meromorphic, one obtains linear Regge trajectories. I have not found any non-meromorphic $A(\zeta, w)$ which fulfils the axioms, moreover one can prove meromorphicity in some cases.

The simplest possible pole system of $A(\zeta, w)$ is given by Eq. (38). It corresponds to the Regge trajectories which describe arbitrary changes of shape of the string. This pole system is realized in the known models, and it is consistent with the few things one knows about the real meson spectrum.

If the invariance group acts transitively on the pole system, one obtains essentially only two possibilities for the spin structure of the model, corresponding to the ordinary and to the Neveu-Schwarz-Ramond strings. However, in both cases one can introduce additional internal symmetries. The two possibilities correspond to the two values $\mu = 1, 0$ of a parameter which enters into the transformation law. This parameter determines directly the behaviour of the sister trajectories. Whereas for $\mu = 1$ only a few trajectories have high intercepts, for $\mu = 0$ all leading sister trajectories should yield important contributions to the amplitudes near $t = 0$.

The axioms 1,2 still admit a huge set of spectra for which it seems difficult or impossible to construct amplitudes. For the moment, three additional restrictions seem plausible:

- i) $A(\zeta, w)$ must have the pole system (38). This allows no bosonic excitations of the string besides the changes of shape. For example, the function (89) is excluded.
- ii) $A(\zeta, w)$ must factorize to allow the usual Fock space representation of the string states. In particular not only the poles of $A(\zeta, w)$ must lie on hyperplanes, but also the zeros. This excludes, e.g., the functions (94) and (95). But also the functions (71) and (72) are excluded because factors like $E_2(w)$ cannot be accommodated to a Fock space.
- iii) D'Adda's conjecture. This conjecture can be interpreted physically, it works in the known models, and allows a nice description of the Neveu-Schwarz-Ramond model, including the G-parity. However, it requires a tachyon in some sector. Of course, this may just be an indication that string models without explicit introduction of quarks are bound to stay unphysical.

The model which was introduced in (I) as a possible candidate for the "correct" model has the pole system (38) and the usual Fock space representation. But D'Adda's conjecture requires the existence of additional sectors. In particular the sector with partition function (85) has a tachyon. Moreover, the internal quantum numbers cause a problem. It seems reasonable to include them by replacing (4) by

$$\text{Tr} (RS \exp(iP \cdot t_+)), \quad (96)$$

where S is an element of the internal symmetry group. Then we obtain a generalized modular function

$$A(\zeta, \theta, w) = \prod_{i=1}^{D/2} (2 \sin \pi \zeta_i)^{-1} \sum_{\lambda, \nu, m} d(\lambda, \nu, m) \exp \left(2\pi i \left(\sum_{i=1}^{D/2} \lambda_i \zeta_i + \sum_{j=1}^N \theta_j \nu_j + \alpha' m^2 w \right) \right), \quad (97)$$

where N is the rank of the internal symmetry group, and ν varies over the weights of this group. The new angular variables θ_j transform in the same way as the ζ_i . For the spectrum under consideration we have a $U(1) \times U(1)$ symmetry group, interpreted in (I) as "strangeness" $\times$ "charm". The $(1+q^{r/2})$ -oscillators should carry

"strangeness", the $(1+q^{2n})$ -oscillators "charm". If one requires that $A(\zeta, \theta_S, \theta_C, w)$ transforms as a generalized modular function, one finds the following results:

- i) "Strangeness" and "charm" are dual to each other, i.e. θ_S and θ_C can be exchanged by modular transformations.
- ii) The unit of "charm" is twice the unit of "strangeness".
- iii) At $m = 0$ one has a "charm" doublet.
- iv) The internal symmetry group seems to be $SU(2) \times SU(2)$, not only $U(1) \times U(1)$.

Of course, property (iii) ruins the similarity to nature.

In the conjugate sector (85), the ground state is a singlet. Here the $(1+q^{2r})$ -oscillators carry "charm", and the $(1+q^{r/2})$ -oscillators carry "strangeness" as before.

Now let us look at the experimental meson spectrum. One would like to obtain at the tree level $m_\pi^2 = 0$, $m_\eta^2 \neq 0$, $m_\rho^2 = m_\omega^2$. But then the zeros of $A(\zeta, w)$ cannot lie on straight lines. Indeed, for $w \rightarrow i\infty$, $A(\zeta, w)$ is dominated by the contributions from π, ρ , and ω . One obtains a line of zeros of the form

$$\mathcal{J} = \frac{w}{2} + c + O(\exp 2\pi i \alpha' m_\eta^2 w), \quad (98)$$

with c irrational. Because c is irrational, the correction term cannot vanish, as otherwise the modular transformations would yield a dense set of zeros. Thus $A(\zeta, w)$ cannot factorize in the way required for a Fock space representation. There are two ways out. Either one considers the helicity partition functions for each isospin separately or one gives up one of the conditions $m_\eta^2 \neq 0$, $m_\rho^2 = m_\omega^2$. In (I) we considered the possibility

$$\alpha' m_\omega^2 = \alpha' m_\rho^2 + \frac{1}{2} \quad (99)$$

at the tree level, but we have seen that the unrealistic features of that model are too numerous. A $SU(3)$ -symmetric spectrum has more to recommend it. However, it would indicate an almost total independence of gluonic strings and quarks. Indeed, one has to note that the spectrum of ψ -states apparently has not much to do with dual relativistic string models. Unfortunately, explicit introduction of quarks might invalidate the concept of critical dimension and the whole approach using the modular group.

Nevertheless, there are also some encouraging features of the experimentally known meson spectrum which indicate that string models are relevant. Of crucial importance is of course that all states seem to lie on exchange degenerate linear Regge trajectories of slope α' , which can be described by a bosonic oscillator $A_{1\mu}^+$, $\mu = \pm 1$. This corresponds to a factor

$$(1 - \exp 2\pi i (f+w))^{-1} (1 - \exp 2\pi i (-f+w))^{-1} \quad (100)$$

of the helicity partition function. Moreover, for isospin different from zero we seem to have a factor

$$(1 + \exp 2\pi i (f + \frac{w}{2})) (1 + \exp \pi i w) (1 + \exp 2\pi i (-f + \frac{w}{2})), \quad (101)$$

corresponding to a fermionic oscillator $B_{\frac{1}{2}\mu}^+$, $\mu = \pm 1, 0$. It yields the relations

$$\alpha' (m_\rho^2 - m_\pi^2) = \alpha' (m_{K^*}^2 - m_K^2) = \alpha' (m_{D^*}^2 - m_D^2) = \frac{1}{2}. \quad (102)$$

In relativistic string models most trajectories fill up only at higher masses. If e.g. the isospin one mesons have the spin structure of the Neveu-Schwarz model, one expects along the pion trajectory multiplicity one at $\alpha' m^2 = 0$, two at $\alpha' m^2 = 1$ three at $\alpha' m^2 = 2, 3, \dots$. This is certainly not ruled out experimentally: π at $\alpha' m^2 = 0$, A_1 and B at $\alpha' m^2 = 1$, A_3 , F_1 , and one lacking state at $\alpha' m^2 = 2$ may, at the tree level, be a reasonable description of reality. For isospin zero the pair D, E may indicate the same phenomenon, if the spin of the E is really 1^+ . It would be particularly important to observe glue excitations, which go beyond the limit of the simplest quark model, to see if they can be described by oscillators $B_{3/2\mu}^+$, $A_{2\mu}^+$, etc. As the generalization of Eq. (102) to the pair η, ω is badly violated, the isospin zero mesons present special problems, possibly because of the Adler anomaly and the corresponding coupling to instantons. For the moment it is impossible to judge, if this can be reproduced at the tree level in string models.

Baryons are still more complicated. They fulfil many linear mass relations which have no natural explanations for the string models under discussion. Moreover, exchange degeneracy is badly broken for their Regge trajectories. If this is true already at the tree level, the transformation $(\zeta, w) \rightarrow (\zeta + w, w)$ cannot belong to the symmetry group G_1 , and the residues of $A(\zeta, w)$ for $\zeta = w$ and $\zeta = 0$ must be independent. But the tree approximation may be very bad, or the whole approach may fail.

In conclusion, dual string models still seem to be rather far from reality. Nevertheless, there is still hope to achieve a better approximation to reality than for the Neveu-Schwarz model. In particular, despite its shortcomings, it would be remunerative to construct the model with $D = 2$ and partition functions (84) and (85).

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