SPIN PRECESSION IN A QUADRUPOLE LENS

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ABSTRACT

In transporting a polarized proton beam through a channel consisting of the dipole magnets and the quadrupoles, one needs to calculate the depolarization effects due to the magnetic fields. Usually the dipole magnet has a homogeneous field and a spin precession is calculated straightforwardly. But an equation of motion of spin in a quadrupole lens has not yet been solved analytically. We propose here such a solution and the result obtained is applied to some interesting practical cases.

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1. INTRODUCTION

A possibility of production of a high-energy polarized proton beam from lambda decay¹⁾ raises a question on the depolarization effect when this beam passes through a channel consisting of the dipole magnets and the quadrupole lenses. An equation for spin precession of particle moving in a homogeneous electromagnetic field, for example in the dipole, has already been solved²⁾. But the equation of motion of spin in the quadrupole lens, which has been discussed by many authors for some particular cases^{3,4)}, has not been solved in an analytic form. Having the idea of transporting a polarized proton beam from the target station to an experimental apparatus, it is reasonable to find the analytic solution for the rotation of the polarization vector in the quadrupole lens. We shall therefore try to solve this problem in Section 2 and then the solution found for the general case will be applied for illustration in some particular cases in Section 3. In the appendices more details of the mathematical treatment of this problem are given.

2. SOLUTION OF THE PROBLEM

2.1 General equation for spin precession in a magnetic field

In the general case of the magnetic field $\vec{B}$ the precession of the polarization vector $\vec{P}$ is defined by a well-known expression

$$\frac{d\vec{P}}{dt} \equiv \dot{\vec{P}} = \frac{e}{m\gamma} \vec{P} \times \left[\frac{g}{2} \vec{B}_L + \left(1 + \frac{g-2}{2} \gamma \right) \vec{B}_T \right] \quad (1)$$

where $\vec{B}_L$ and $\vec{B}_T$ are components of $\vec{B}$ parallel and perpendicular to the velocity $\vec{v}$ of the particle, respectively; e , and m are the electrical charge and the mass of the particle, g is the gyromagnetic ratio; $\gamma = \epsilon/m$ is the Lorentz factor. Assuming that the particle moves in the z direction with a constant velocity, that xz is a horizontal and that yz is a vertical plane, we can obtain a matrix equation

$$\frac{d\vec{P}}{dz} \equiv \vec{P}' = \frac{e}{pc} \hat{A} \cdot \vec{P} \quad (2)$$

where $\hat{A}$ is an antisymmetric tensor defined through the components of the vector

$$\vec{A} = \frac{g}{2} \vec{B}_L + \left(1 + \frac{g-2}{2} \gamma \right) \vec{B}_T \quad (3)$$

by the formula

$$|\hat{A}| = \begin{vmatrix} 0 & A_z & -A_y \\ -A_z & 0 & A_x \\ A_y & -A_x & 0 \end{vmatrix} \quad (4)$$

The general solution of problem (2) can be presented in the form of the system of the second order inhomogeneous Volterra equation in the one-dimensional case⁵⁾.

$$P_i(z) = P_{i0} + \lambda \sum_{k=1}^3 \int_0^z A_{ik}(z, \xi) P_k(\xi) d\xi \quad (5)$$

where $\lambda = e/pc$ and a Kernel of this equation $A_{ik}(z, z)$ is known. Using a method of successive approximations, one can find the solution of the Volterra equation for any practical interesting case.

2.2 Solution for the quadrupole lens

We suppose that the following conditions are fulfilled for the quadrupole lens:

$$\vec{B}_L = 0, \quad \vec{B}_T = B_x \vec{i} + B_y \vec{k} \quad (6)$$

where $\vec{i}$ and $\vec{k}$ are unit vectors in the x and y directions, respectively,

$$B_x = G_y, \quad B_y = G_x \quad (7)$$

Then introducing a parameter

$$c = \lambda \left(1 + \frac{g^2}{2} \right) G = \lambda \alpha_0 G \quad (8)$$

we have the system of equations (see Appendix 1)

$$P_x(z) = P_{x0} - c \int_0^z P_z(\xi) x(\xi) d\xi \quad (9)$$

$$P_y(z) = P_{y0} + c \int_0^z P_z(\xi) y(\xi) d\xi \quad (10)$$

$$P_z(z) = P_{z0} + c P_{x0} v_x(z) - c P_{y0} v_y(z) - c^2 \int_0^z K(z, \xi) P_z(\xi) d\xi \quad (11)$$

where the following notations are introduced

$$V_x(z) = \int_0^z x(\xi) d\xi \quad ; \quad V_y(z) = \int_0^z y(\xi) d\xi \quad (12)$$

and

$$K(z, \xi) = [V_x(z) - V_x(\xi)] x(\xi) + [V_y(z) - V_y(\xi)] y(\xi) \quad (13)$$

We can see from Eqs. (9) to (11) that one can find the solution for any order of value c by the method of successive approximations.

Taking into account the equation of motion of a particle in the magnetic field of the quadrupole

$$x'' + K^2 x = 0 \quad , \quad (14)$$

and

$$y'' - K^2 y = 0 \quad , \quad (15)$$

where $K^2 = eG/pc$, and substituting in Eqs. (12) and (13) one obtains

$$V_x(z) = -\frac{x'}{K^2} \quad , \quad V_y(z) = -\frac{y'}{K^2} \quad , \quad (16)$$

whilst

$$K(z, \xi) = -\frac{1}{K^2} [x'(z) - x'(\xi)] + \frac{1}{K^2} [y'(z) - y'(\xi)] \quad (17)$$

Up to the order of c^2 the final results follow from Eqs. (9) to (11) and (16) to (17):

$$P_i(z) = \sum M_{ik}(z) P_{k0} \quad (18)$$

where M is given by the following matrix (see Appendix 2):

$$\begin{aligned}
 M_{11} &= 1 + \frac{\alpha_0^2}{2} \left[K^2(x^2 - x_0^2) + 2x'_0(x' - x'_0) \right] \\
 M_{12} &= \frac{\alpha_0^2}{2} \left[K^2(xy - x_0 y_0) - (x'y' - x'_0 y'_0) + 2y'_0(x' - x'_0) \right] \\
 M_{13} &= \alpha_0(x' - x'_0) \\
 M_{21} &= -\frac{\alpha_0^2}{2} \left[K^2(xy - x_0 y_0) + (x'y' - x'_0 y'_0) - 2x'_0(y' - y'_0) \right] \\
 M_{22} &= 1 - \frac{\alpha_0^2}{2} \left[K^2(y^2 - y_0^2) - 2y'_0(y' - y'_0) \right] \\
 M_{23} &= \alpha_0(y' - y'_0) \\
 M_{31} &= -\alpha_0(x' - x'_0) \\
 M_{32} &= -\alpha_0(y' - y'_0) \\
 M_{33} &= 1 - \frac{\alpha_0^2}{2} \left[x'^2 - x_0'^2 + y'^2 - y_0'^2 \right].
 \end{aligned} \tag{19}$$

Here x_0 , x'_0 , y_0 , and y'_0 are the initial values of the particle coordinates in the input of the quadrupole lens and x , x' , y and y' are the same coordinates in the output.

The matrix M given by (19) furnishes the solution of the polarization precession problem in the approximation of the order of α_0^2 . The contributions, including a higher order of α_0^3 , can be obtained by making use of formulae (9) to (11). As an example, the term of the order of α_0^3 has been calculated in Appendix 2.

The solution given above is true for the quadrupole lens focusing the beam in the horizontal plane (xz) and defocusing it in the vertical (yz) plane (FD type). In the case of the DF type we have to change $\alpha_0 \rightarrow -\alpha_0$ and to interchange coordinates x and y in Eqs. (14) and (15).

3. SOME PARTICULAR APPLICATIONS

3.1 On the Serpukhov accelerator it is proposed⁴⁾ to use a standard quadrupole with the parameters $G = 34.43$ Tesla/m, and an effective length $L = 1.54$ m. For an incident momentum $pc = 70$ GeV and the initial conditions $x_0 = y_0 = 5$ mm, $x'_0 = y'_0 = 1$ mrad, one can find the elements of the matrix M:

$$\|M\| = \begin{vmatrix} 0.986 & 0.045 & -0.167 \\ 0.032 & 0.935 & 0.187 \\ 0.167 & -0.187 & 0.966 \end{vmatrix}$$

So one may expect for an initially 100% polarized proton beam in the y direction, a depolarization factor of the order of $1 - 0.935 = 0.065$.

3.2 In the CERN SPS, in the North Area, the possibility of using the H2 beam for transporting the polarized proton beam has already been discussed¹⁾. In this beam a standard quadrupole lens has the parameter $G = 21.57$ Tesla/m and a length $L = 2.99$ m. For an incident momentum $pc = 300$ GeV and the initial conditions $x_0 = 2.8$ mm, $x'_0 = 0.49$ mrad, $y_0 = 3$ mm and $y'_0 = 0.24$ mrad, one can calculate the elements of the matrix M:

$$\|M\| = \begin{vmatrix} 0.992 & 0.008 & -0.013 \\ 0.008 & 0.992 & 0.013 \\ 0.013 & -0.013 & 0.997 \end{vmatrix}$$

In this case the depolarization effect is much smaller than in the previous one. To be sure that these results are not drastically changed by the contributions of a higher order term in α_0 we calculated the terms of the order of α_0^3 , using the formulae given in Appendix 2. Their contributions to the matrix elements are as follows:

$$\Delta M_{13} = 3.1 \times 10^{-3} ; \quad \Delta M_{23} = 1.3 \times 10^{-3} ; \quad \Delta M_{34} = 8.3 \times 10^{-3} ;$$

$$\Delta M_{32} = 2.8 \times 10^{-3}$$

So the conclusion about the small depolarization effect at the SPS is still valid.

3.3 For both practical cases discussed above the depolarization effect has only been calculated for a particular trace of particle, but in fact we need to average on some assembly of particles in the input of the quadrupole. We assume that the particles are distributed in a symmetrical way and independently for all coordinates (x_0 , x'_0 , y_0 , and y'_0) and the initial polarization direction is not correlating with these coordinates. Let us further suppose that the solutions of the equations (14) and (15) are presented in the form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos kL & \frac{1}{K} \sin kL \\ -K \sin kL & \cos kL \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (20)$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cosh kL & \frac{1}{K} \sinh kL \\ K \sinh kL & \cosh kL \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} \quad (21)$$

Then averaging the matrix (19) we obtain the mean polarization after a passage of beam through the quadrupole (we include term of order α_0^3 too)

$$\begin{aligned} \langle P_x \rangle &= \left\{ 1 - 2\alpha_0^2 \sin^2 \frac{kL}{2} \left(k^2 \cos^2 \frac{kL}{2} \sigma_{x01}^2 + \sin^2 \frac{kL}{2} \sigma_{x02}^2 \right) \right\} P_{x0} \\ &+ \frac{\alpha_0^3}{16} \left\{ K^4 \cos^4 kL \sigma_{x01}^4 + 6 K^2 \cos^2 kL \sin^2 kL \sigma_{x01}^2 \sigma_{x02}^2 \right. \\ &\quad \left. + \sin^4 kL \sigma_{x02}^4 - K^4 \sigma_{x01}^4 \right\} P_{z0} \end{aligned} \quad (22)$$

$$\begin{aligned} \langle P_y \rangle &= \left\{ 1 - 2\alpha_0^2 \sinh^2 \frac{kL}{2} \left(K^2 \cosh^2 \frac{kL}{2} \sigma_{y01}^2 + \sinh^2 \frac{kL}{2} \sigma_{y02}^2 \right) \right\} P_{y0} \\ &+ \frac{\alpha_0^3}{16} \left\{ K^4 \cosh^4 kL \sigma_{y01}^4 + 6 K^2 \cosh^2 kL \sinh^2 kL \sigma_{y01}^2 \sigma_{y02}^2 \right. \\ &\quad \left. + \sinh^4 kL \sigma_{y02}^4 - K^4 \sigma_{y01}^4 \right\} P_{z0} \end{aligned} \quad (23)$$

$$\begin{aligned}
 \langle P_z \rangle = & \frac{\alpha_0^3}{8} \left[k^4 \cos^4 KL \sigma_{x01}^4 + 6k^2 \cos^2 KL \sin^2 KL \sigma_{x01}^2 \sigma_{x02}^2 + \sin^4 KL \sigma_{x02}^4 \right. \\
 & \left. - k^4 \sigma_{x01}^4 \right] P_{x0} + \frac{\alpha_0^3}{8} \left[k^4 \cosh^4 KL \sigma_{y01}^4 + 6k^2 \cosh^2 KL \sigma_{y01}^2 \sigma_{y02}^2 + \right. \\
 & \left. + \sinh^4 KL \sigma_{y02}^4 - k^4 \sigma_{y01}^4 \right] P_{y0} + \left[1 - \frac{\alpha_0^2}{2} \left[\sin^2 KL (k^2 \sigma_{x01}^2 - \sigma_{x02}^2) \right] \right. \\
 & \left. + (k^2 \sigma_{y01}^2 + \sigma_{y02}^2) \sinh^2 KL \right] P_{z0}
 \end{aligned} \tag{24}$$

Here σ_{x01}^2 and σ_{x01}^4 are the second and fourth momenta distributions for x_0 , σ_{x02}^2 and σ_{x02}^4 are the corresponding distributions for x'_0 . Similar definitions are introduced for y_0 and y'_0 distributions.

For a Gaussian distribution with the parameters given above (see Section 2) one can find the average polarization after the passage of the SPS quadrupole

$$\begin{vmatrix} \langle P_x \rangle \\ \langle P_y \rangle \\ \langle P_z \rangle \end{vmatrix} = \begin{vmatrix} 0.995 & 0 & 1.6 \cdot 10^{-7} \\ 0 & 0.994 & 8.1 \cdot 10^{-7} \\ 0.8 \cdot 10^{-7} & 4 \cdot 10^{-7} & 0.994 \end{vmatrix} \begin{vmatrix} P_{x0} \\ P_{y0} \\ P_{z0} \end{vmatrix} \tag{25}$$

One can see that in these cases all the polarization components are almost decoupled from one another and the depolarization effects are very small.

3.4 There is a very important question from the point of view of practical application. What is the dependence of the depolarization effect on the polarized beam dimension? In order to answer this question we assumed that the beam has a uniform distribution versus x_0 and y_0 with fixed angular divergences x'_0 and y'_0 , namely, $\sigma_{x02} = \sigma_{y02} = 2$ mrad (full width).

Using the formulae (22) to (24), which are still valid, the polarization was calculated for different beam dimensions Q (full width). The results are presented in the following table for different components of polarization, assuming that the initial polarization component was equal to 1. It is obvious from this table that a strong depolarization effect arises when the beam has a large dimension (as well as when the beam has a large angular divergency).

Table

$\begin{matrix} Q \\ P \end{matrix}$	2 (cm)	4 (cm)	6 (cm)	8 (cm)	10 (cm)
$\langle P_x \rangle$	0.978	0.914	0.807	0.657	0.463
$\langle P_y \rangle$	0.975	0.902	0.780	0.609	0.390
$\langle P_z \rangle$	0.953	0.815	0.586	0.266	0.015

4. CONCLUSIONS

The solution of the polarization precession problem in the field of a quadrupole lens is given in an analytic form by the method of successive approximations. The results are illustrated by particular applications to the standard quadrupole lenses and show that the depolarization effect is not big, especially at high energy. Some precaution has to be made in the case of the large beam dimension or angular divergency.

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APPENDIX I

Equation (2) of the main text can be written in the following forms

$$P'_x = \lambda (P_y A_z - P_z A_y) \quad (A.1)$$

$$P'_y = \lambda (P_z A_x - P_x A_z) \quad (A.2)$$

$$P'_z = \lambda (P_x A_y - P_y A_x) . \quad (A.3)$$

We assume that for the quadrupole

$$A_z = 0 \quad (A.4)$$

$$A_x = (1 + \frac{q-2}{2} \gamma) B_x ; \quad A_y = (1 + \frac{q-2}{2} \gamma) B_y \quad (A.5)$$

where B_x and B_y are given by formulae (7). Then integrating Eqs. (A.1) to (A.3) we arrive at the following expressions

$$P_x(z) = P_{x0} - c \int_0^z P_z(\bar{z}) x(\bar{z}) d\bar{z} \quad (A.6)$$

$$P_y(z) = P_{y0} + c \int_0^z P_z(\bar{z}) y(\bar{z}) d\bar{z} \quad (A.7)$$

$$P_z(z) = P_{z0} + c \left\{ \int_0^z P_x(\bar{z}) x(\bar{z}) d\bar{z} - \int_0^z P_y(\bar{z}) y(\bar{z}) d\bar{z} \right\} \quad (A.8)$$

Substituting P_x and P_y from Eqs. (A.6), (A.7) in (A.8), and introducing

$$V_x(z) = \int_0^z x(\bar{z}) d\bar{z} \quad (A.9)$$

$$V_y(z) = \int_0^z y(\bar{z}) d\bar{z} \quad (A.10)$$

we can rewrite Eq. (A.8) as the formula:

$$P_z(z) = P_{z0} + c [P_{x0} V_x(z) - P_{y0} V_y(z)] - c^2 \int_0^z d\tilde{z} \left\{ x(\tilde{z}) \int_0^{\tilde{z}} P_z(\tilde{z}') x(\tilde{z}') d\tilde{z}' + y(\tilde{z}) \int_0^{\tilde{z}} P_z(\tilde{z}') y(\tilde{z}') d\tilde{z}' \right\}. \quad (A.11)$$

The double integral can be transformed into a single integral by making use of the formula

$$I(x) = \int_0^x d\xi \phi(\xi) \int_0^\xi f(\xi') d\xi' = \int_0^x [V(x) - V(\xi)] f(\xi) d\xi, \quad (A.12)$$

where

$$V(x) = \int_0^x \phi(\xi) d\xi \quad (A.13)$$

so

$$\int_0^z d\tilde{z} x(\tilde{z}) \int_0^{\tilde{z}} P_z(\tilde{z}') x(\tilde{z}') d\tilde{z}' = \int_0^z [V_x(z) - V_x(\tilde{z})] P_z(\tilde{z}) x(\tilde{z}) d\tilde{z} \quad (A.14)$$

$$\int_0^z d\tilde{z} y(\tilde{z}) \int_0^{\tilde{z}} P_z(\tilde{z}') y(\tilde{z}') d\tilde{z}' = \int_0^z [V_y(z) - V_y(\tilde{z})] P_z(\tilde{z}) y(\tilde{z}) d\tilde{z} \quad (A.15)$$

Using these transformations the Eq. (A.11) can be rewritten as

$$P_z(z) = P_{z0} + c [P_{x0} V_x(z) - P_{y0} V_y(z)] - c^2 \int_0^z [V_x(z) - V_x(\tilde{z})] x(\tilde{z}) P_z(\tilde{z}) d\tilde{z} - c^2 \int_0^z [V_y(z) - V_y(\tilde{z})] y(\tilde{z}) P_z(\tilde{z}) d\tilde{z} \quad (A.16)$$

By introducing a notation

$$K(z, \bar{z}) = \left\{ V_x(z) - V_x(\bar{z}) \right\} x(\bar{z}) + \left\{ V_y(z) - V_y(\bar{z}) \right\} y(\bar{z}) \quad (\text{A.17})$$

we obtain instead of Eq. (A.16)

$$P_z(z) = P_{z0} + c P_{x0} V_x(z) - c P_{y0} V_y(z) + \\ - c^2 \int_0^z K(z, \bar{z}) P_z(\bar{z}) d\bar{z} \quad (\text{A.18})$$

This is expression (11) given in the main text.

APPENDIX II

Our aim is to obtain an analytic expression for the matrix M [see formulae (18) in the main text] giving a transformation of the initial polarization vector $\vec{P}(0)$ to a final vector $\vec{P}(z)$ after the passage of a quadrupole.

In Appendix I the equation for $P_z(z)$ was given [see expression (A.18)]. Neglecting all terms higher than those of the order of c^2 we arrive at the following formula

$$P_z(z) = P_{z0} + c P_{x0} V_x(z) - c P_{y0} V_y(z) - c^2 \int_0^z K(z, \xi) P_{z0} d\xi \quad . \quad (A.19)$$

The integral

$$I(z) = \int_0^z K(z, \xi) d\xi \quad , \quad (A.20)$$

can be found directly as follows:

$$\begin{aligned} I(z) &= \int_0^z [V_x(z) - V_x(\xi)] x(\xi) d\xi + \int_0^z [V_y(z) - V_y(\xi)] y(\xi) d\xi = \\ &= V_x(z) V_x(z) - \frac{1}{2} V_x^2(z) + V_y(z) V_y(z) - \frac{1}{2} V_y^2(z) - I_0 \\ &= \frac{1}{2} [V_x^2(z) + V_y^2(z)] - I_0 = \frac{1}{2} [V_x^2(z) + V_y^2(z) - V_{x0}^2 - V_{y0}^2] \end{aligned} \quad (A.21)$$

Substituting expression (16) of the main text for $I(z)$ and then in (A.19) we have the following equation for $P_z(z)$

$$\begin{aligned} P_z(z) &= -\frac{c}{K^2} [x' - x'_0] P_{x0} - \frac{c}{K^2} [y' - y'_0] P_{y0} + \\ &+ \left[1 - \frac{c^2}{2K^4} (x'^2 - x_0'^2 + y'^2 - y_0'^2) \right] P_{z0} \quad . \end{aligned} \quad (A.22)$$

This is the last line of the matrix M.

Having found $P_z(z)$ it is not difficult to calculate the other components of polarization. For example, substituting $P_z(z)$ in expression (A.6) we get

$$\begin{aligned} P_x(z) = P_{x0} - c \int_0^z P_z(\bar{z}) x(\bar{z}) d\bar{z} = P_{x0} - c P_{z0} \int_0^z x(\bar{z}) d\bar{z} + \\ + \frac{c^2}{K^2} P_{x0} \int_0^z [x'(\bar{z}) - x'_0] x(\bar{z}) d\bar{z} + \frac{c^2}{K^2} P_{y0} \int_0^z [y'(\bar{z}) - y'_0] x(\bar{z}) d\bar{z} \end{aligned} \quad (A.23)$$

In a similar way we find $P_y(z)$ from expressions (A.7) and (A.22):

$$\begin{aligned} P_y(z) = P_{y0} + c \int_0^z P_z(\bar{z}) y(\bar{z}) d\bar{z} = P_{y0} + c P_{z0} \int_0^z y(\bar{z}) d\bar{z} + \\ - \frac{c^2}{K^2} P_{x0} \int_0^z [x'(\bar{z}) - x'_0] y(\bar{z}) d\bar{z} - \frac{c^2}{K^2} P_{y0} \int_0^z [y'(\bar{z}) - y'_0] y(\bar{z}) d\bar{z} \end{aligned} \quad (A.24)$$

The following formulae, can be proved directly by partial integration:

$$\begin{aligned} \int x'(\bar{z}) y(\bar{z}) d\bar{z} &= \frac{1}{2} (xy + \frac{1}{K^2} x'y') \\ \int y'(\bar{z}) x(\bar{z}) d\bar{z} &= \frac{1}{2} (xy - \frac{1}{K^2} x'y') \\ \int x'(\bar{z}) x(\bar{z}) d\bar{z} &= \frac{1}{2} x^2 ; \quad \int y'(\bar{z}) y(\bar{z}) d\bar{z} = \frac{1}{2} y^2 \end{aligned} \quad (A.25)$$

Making use of these formulae we arrived at the following expressions for $P_x(z)$ and $P_y(z)$

$$\begin{aligned} P_x(z) = \left\{ 1 + \frac{c^2}{K^2} \left[\frac{x^2 - x_0^2}{2} + \frac{x'_0 (x' - x'_0)}{K^2} \right] \right\} P_{x0} \\ + \frac{c^2}{2K^2} \left[xy - \frac{x'y'}{K^2} - x_0 y_0 + \frac{x'_0 y'_0}{K^2} + \frac{2y'_0 (x' - x'_0)}{K^2} \right] P_{y0} \\ + \frac{c}{K^2} (x' - x'_0) P_{z0} \end{aligned} \quad (A.26)$$

$$\begin{aligned}
 P_y(z) = & -\frac{c^2}{2K^2} \left[(xy - x_0 y_0) + \frac{1}{K^2} (x'y' - x'_0 y'_0) - \frac{2x'_0}{K^2} (y' - y'_0) \right] P_{x_0} \\
 & + \left[1 - \frac{c^2}{2K^2} \left[y^2 - y_0^2 - \frac{2y'_0}{K^2} (y' - y'_0) \right] \right] P_{y_0} \\
 & + \frac{c}{K^2} (y' - y'_0) P_{z_0} .
 \end{aligned} \tag{A.27}$$

Similar calculations lead to the following contributions of the terms of the order of c^3 or α_0^3 in the elements of the M matrix

$$\begin{aligned}
 \Delta M_{13} = & \frac{\alpha_0^3}{2} \left[\left(-\frac{1}{3} x'^3 - \frac{3}{5} x' y'^2 + \frac{2}{5} K^2 x y y' + \frac{2}{5} K^2 x' y^2 + (x_0'^2 + y_0'^2) x' \right) \right. \\
 & \left. + \left(\frac{1}{3} x_0'^3 + \frac{3}{5} x_0' y_0'^2 - \frac{2}{5} K^2 x_0 y_0 y'_0 - \frac{2}{5} K^2 x_0' y_0'^2 - (x_0'^2 + y_0'^2) x'_0 \right) \right]
 \end{aligned} \tag{A.28}$$

$$\begin{aligned}
 \Delta M_{23} = & -\frac{\alpha_0^3}{2} \left[\frac{1}{3} y'^3 + \frac{3}{5} x'^2 y' + \frac{2}{5} K^2 x y x' + \frac{2}{5} K^2 y' x^2 - (x_0'^2 + y_0'^2) y' \right. \\
 & \left. - \frac{1}{3} y_0'^3 - \frac{3}{5} x_0'^2 y'_0 - \frac{2}{5} K^2 x_0 y_0 x'_0 - \frac{2}{5} K^2 y'_0 x_0'^2 + (x_0'^2 + y_0'^2) y'_0 \right]
 \end{aligned} \tag{A.29}$$

$$\begin{aligned}
 \Delta M_{31} = & \alpha_0^3 \left[\frac{1}{8} K^4 x^4 + \frac{3}{10} K^2 x y y' + \frac{3}{10} x' y'^2 - \frac{1}{5} K^2 y^2 x' \right. \\
 & \left. - \frac{1}{8} K^4 x_0^4 - \frac{3}{10} K^2 x_0 y_0 y'_0 - \frac{3}{10} x_0' y_0'^2 + \frac{1}{5} K^2 y_0'^2 x'_0 \right]
 \end{aligned} \tag{A.30}$$

$$\begin{aligned}
 \Delta M_{32} = & \alpha_0^3 \left[\frac{1}{8} K^4 y^4 - \frac{3}{10} K^2 x y x' + \frac{3}{10} x'^2 y' + \frac{1}{5} K^2 x^2 y' \right. \\
 & \left. - \frac{1}{8} K^4 y_0^4 + \frac{3}{10} K^2 x_0 y_0 x'_0 - \frac{3}{10} x_0'^2 y'_0 + \frac{1}{5} K^2 x_0'^2 y'_0 \right] .
 \end{aligned} \tag{A.31}$$

These expressions have been used in the text to calculate the contributions to the depolarization effects.