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QUANTUM FIELD THEORY WITH SOLITON CONSERVATION LAWS

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A B S T R A C T

Field theories with soliton conservation laws are the most promising candidates for explicitly constructable models. We exemplify in the case of the massive Thirring model how the old S matrix bootstrap idea, supplemented with a soliton factorization property, may be used as a systematic starting point for the construction of the S matrix, form factors and (hopefully) correlation functions.

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1. Classical prerequisites

I will follow the terminology of Mc Laughlin and Scott (see talk at this workshop) and use the word soliton for lumps of classical energy which under collision retain their identity i.e. behave persistently. The deeper reason for this unexpected behaviour is in all known cases the validity of an infinite number of conservation laws which generalize and refine the energy-momentum conservation. Everybody knows the free field example for higher conservation laws:

$$(\partial_\mu \partial^\mu + m^2) \phi = 0$$

$$E = \frac{1}{2} \int (\vec{\partial} \phi \vec{\partial} \phi + m^2 \phi^2 + \partial_0 \phi \partial_0 \phi) d^{D-1} x \quad (1a)$$

$$\vec{P} = \int \partial_0 \phi \vec{\partial} \phi d^{D-1} x \quad (1b)$$

Replacing $\phi \rightarrow \frac{\partial^n}{\partial x_1^{n_1} \dots \partial x_{D-1}^{n_{D-1}}} \phi$

one obtains an infinite set of conserved quantities which represent the higher moments (around $p = 0$) of the energy - and momentum- distribution. The theories of soliton behaviour which have been constructed up to now can in some sense be viewed as a peculiar "deformation" of free fields such that certain deformed higher currents of the free field are still kept conserved as a result of the special interaction. Relativistic equations of this type have been found only for $D=2$, the difficulty of constructing conservation laws which generalize the energy momentum conservation in higher dimension is perhaps related to a curious property which on the level of relativistic S-matrices has been discussed by Coleman and Mandula ¹⁾.

For $D=2$ relativistic equations with soliton behaviour there is a very powerful geometrical principle for their construction ²⁾:

Start with a free field theory and introduce interaction via a geometrical constraint. As an example consider the so called $O(3)$ σ -model:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \partial^\mu \vec{\phi} \quad , \quad \vec{\phi}^{(x)} \text{ vector in internal space} \quad (2)$$

with $\vec{\phi}^2 = 1$

The method of Lagrangian (field)-multipliers leads to:

$$\partial_\mu \partial^\mu \vec{\phi} + (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}) \vec{\phi} = 0 \quad , \quad \vec{\phi}^2 = 1 \quad (3)$$

or in light-cone variables $u = \frac{t+x}{2} \quad v = \frac{t-x}{2}$

$$\partial_u \partial_v \vec{\phi} + (\partial_u \vec{\phi} \cdot \partial_v \vec{\phi}) \vec{\phi} = 0 \quad (4)$$

The equation can be further simplified by distorting each light-cone variable in such a way that the energy-momentum conservation following from (4)

$$\partial_v \vec{\phi}_u^2 = 0 = \partial_u \vec{\phi}_v^2 \quad (5a)$$

leads to a normalized density²⁾

$$\vec{\phi}_u^2 = 1 = \vec{\phi}_v^2 \quad (5b)$$

Further simplicity can be gained by introducing $\vec{\phi}, \vec{\phi}_u, \vec{\phi}_v$ as an intrinsically defined 3-frame, $\vec{\phi}$ is orthogonal to $\vec{\phi}_u$ and $\vec{\phi}_v$ only $\vec{\phi}_u$ and $\vec{\phi}_v$ form a nontrivial angle:

$$\cos \varphi = \vec{\phi}_u \cdot \vec{\phi}_v \quad (6a)$$

This geometrical invariant angle also appears in the volume

$$\vec{\phi}_u \times \vec{\phi}_v \cdot \vec{\phi} = \sin \varphi \quad (6b)$$

The original equation of motion (4) together with the normalization (5b) yields²⁾ the Sine-Gordon equation for φ :

$$\partial_{uv} \varphi - \sin \varphi = 0 \quad (7)$$

One can see that a solution of the Sine-Gordon-equation leads to a solution of the σ -model up to frame-rotations²⁾. Collecting our results we can say that a 0(3)-geometrically constrained (i.e. non-linear) σ model leads to precisely one geometrical invariant φ which obeys the Sine-Gordon equation. The Sine-Gordon equation can be shown to be the only relativistic one component equation with higher conservation laws.

If we had started from an $O(4)$ model, the result of a similar consideration would have been two geometrical invariants which satisfy a coupled system of equations²⁾. As the Sine-Gordon theory, this generalized equation with two fields also has the soliton property.

An educated guess is that suitably defined geometrical invariants of geometrically constrained $D=2$ free field theories provide a "construction machine" for two dimensional relativistic soliton situations.

2. Quantum equivalences

Our main aim is to understand the quantum theory of solitons. The simplest relativistic case is the Sine-Gordon theory, and here some new ideas recently lead to an amazing progress. These ideas rely heavily on the use of the soliton conservation laws. The benefit of restricting one's interest to the special quantum field theory of solitons is that one obtains a much more powerful dynamical method as the collective variable method either in its functional form³⁾ or in the language of canonical formalism⁴⁾. The method cannot be applied to field theories with topological kinks which do not have the soliton property.

The quantum field theory of solitons relies heavily on "quantum equivalences". The case studied up to now is the equivalence between the Sine-Gordon quantum field theory⁵⁾⁶⁾⁷⁾ and the massive Thirring model.

The easiest and fastest way to understand this equivalence is to start from exponential free Bose fields in $D=2$ dimensions:⁸⁾⁷⁾

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx} a(p) \frac{dp}{2p_0} + h.c. = \phi^{(+)} + \phi^{(-)} \quad (8)$$

$$[a(p), a^\dagger(q)] = 2p_0 \delta(p-q) \quad p_0 = \sqrt{p^2 + m^2}$$

leading to

$$\langle \phi(x) \phi(y) \rangle = \frac{1}{2\pi} \int e^{-ip(x-y)} \frac{dp}{2p_0} = i \Delta^{(+)}(x-y) \quad (9)$$

and the vanishing of all higher point correlation functions. For $m=0$ we meet the well known two dimensional infrared divergence:

$\int \frac{dp}{|p|} \rightarrow \infty$ which tells us that ϕ does not exist as a Wightman field but rather ϕ can only be considered as an indefinite field obtained

in a regularized indefinite metric Hilbert space defined for example by

$$\langle \phi(x) \phi(y) \rangle_{reg} = \langle \phi(x) \phi(y) \rangle_m \Big|_{\text{short distance}} = -\frac{1}{4\pi} \ln \left\{ \left[\frac{\epsilon^2 + i\epsilon \epsilon_0}{\mu^2} \right] \right\} \quad (10)$$

$$= i \Delta^{(+)}(x-y)$$

and vanishing higher truncated functions

$$\xi = x - y, \quad \mu = \frac{m e^\gamma}{2}, \quad \gamma = \text{Eulers constant}$$

μ may also be considered as an independent parameter in the spirit of a "renormalization spot" in massless renormalization theory⁹⁾. This point of view will be adopted in the following.

It is now very important to realize that exponentials of ϕ

$$: e^{i\lambda \phi(x)} : = e^{i\lambda \phi^{(-)}(x)} e^{i\lambda \phi^{(+)}(x)} \quad (11)$$

can be viewed as living in a positive definite Hilbert space if a certain " λ -rule" is respected¹⁰⁾. To abstract this λ -rule consider first exponentials of the massive ϕ :

$$\langle 0 | : e^{i\lambda_1 \phi(x_1)} : \dots : e^{i\lambda_n \phi(x_n)} : | 0 \rangle \quad (12)$$

$$= \exp \left\{ - \sum_{i < j} \lambda_i \lambda_j i \Delta^{(+)}(x_i - x_j) \right\}$$

$$= \exp \left\{ - \sum_{i < j} \lambda_i \lambda_j \left[i \Delta^{(+)}(x_i - x_j) + \frac{1}{4\pi} \ln \frac{m^2 e^{2\gamma}}{4\mu^2} \right] \right\} \cdot$$

$$\times \exp \left\{ - \sum \frac{\lambda_i^2}{8\pi} \ln \frac{m^2 e^{2\gamma}}{4\mu^2} \right\} \exp \left\{ \frac{(\sum \lambda_i)^2}{8\pi} \ln \frac{m^2 e^{2\gamma}}{4\mu^2} \right\}$$

Therefore by a multiplicative renormalization

$$B_{\lambda_i}(x) = \left(\frac{m^2 e^{2\gamma}}{4\mu^2} \right)^{\frac{\lambda_i^2}{8\pi}} : e^{i\lambda_i \phi(x)} : \quad (13)$$

we can represent the zero mass exponentials as the limit $m \rightarrow 0$ in the following way.

$$\lim_{m \rightarrow 0} \langle 0 | B_{\lambda_1}(x_1) \dots B_{\lambda_n}(x_n) | 0 \rangle = \delta_{\sum \lambda_i, 0} \exp - \sum_{i < j} \lambda_i \lambda_j \ln \left[(x_i - x_j)^2 + i\epsilon(x_{i0} - x_{j0}) \right] / \mu^2 \quad (14)$$

A limit of positive definite Wightman functions is again positive definite and therefore we obtain.

Theorem: The zero mass exponential fields $:e^{i\lambda\phi}:$ computed with the rule (14) lead to a positive definite Hilbert space if the charge conservation rule $\sum \lambda_i = 0$ is maintained.

Note that derivatives of ϕ as $\partial_\mu \phi$ which are infrared-finite, may be incorporated into the Hilbert space without spoiling its positive definiteness.

Now as with classical fields the zero mass ϕ decomposes into the two independent light-cone parts. ϕ contains creation and annihilation operators for right and left-hand movers.

$$\phi(x) = \phi(u) + \phi(v) \quad (15)$$

$$\langle \phi(u) \phi(u') \rangle = -\frac{1}{4\pi} \ln(u - u' - i\epsilon) \mu \quad (16)$$

$$\langle \phi(u) \phi(v) \rangle = 0, \quad \mathcal{H} = \mathcal{H}_u \otimes \mathcal{H}_v$$

Therefore $:e^{i\lambda_1 \phi(u)}:$ and $:e^{i\lambda_2 \phi(v)}:$ (17)

are independent positive definite fields if the λ_i 's fulfill the charge conservation rule separately. These boson fields are the building blocks for all solvable 2 dimensional models involving massless fermions. Let us define two fields:

$$\psi_1 = \sqrt{\frac{\mu}{2\pi}} : \exp i(\delta_1 \phi(u) + \delta_2 \phi(v)) : \quad (18a)$$

$$\psi_2 = \sqrt{\frac{\mu}{2\pi}} : \exp -i(\delta_2 \phi(u) + \delta_1 \phi(v)) : \quad (18b)$$

Defining in addition the currents

$$j_{\mu 5} = - \frac{\beta}{2\pi} \frac{\partial \phi}{\partial x^{\mu}} \quad (19a)$$

$$j_{\mu} = \varepsilon_{\mu\nu} j_5^{\nu} \quad , \quad \varepsilon'^0 = 1 \quad (19b)$$

we may now consider operator products of $\underline{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with the current :

$$: j_{\mu} \underline{\psi} : = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \{ j_{\mu}(x+\varepsilon) \underline{\psi}(x) + \underline{\psi}(x) j_{\mu}(x-\varepsilon) \} \quad (20)$$

It is convenient at this point to introduce two dimensional Dirac-matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (21)$$

Consider now the combination

$$: \gamma^{\mu} j_{\mu} \underline{\psi} : = \beta : \begin{pmatrix} 0 & \partial_u \phi \\ -\partial_v \phi & 0 \end{pmatrix} \underline{\psi} : = \beta \begin{pmatrix} : \partial_u \phi \psi_2 : \\ - : \partial_v \phi \psi_1 : \end{pmatrix} \quad (22)$$

Up to the factor $-g = \frac{2\pi\delta_2}{\beta}$ this is precisely what we obtain if we apply the Dirac operator $i\gamma^{\mu} \partial_{\mu}$ to $\underline{\psi}$:

$$i\gamma^{\mu} \partial_{\mu} \underline{\psi} = \delta_2 \begin{pmatrix} : \partial_u \phi \psi_2 : \\ - : \partial_v \phi \psi_1 : \end{pmatrix} \quad (23a)$$

Hence we read off the equation:

$$i\gamma^{\mu} \partial_{\mu} \underline{\psi} = - \Theta \gamma^{\mu} : j_{\mu} \underline{\psi} : \quad (23b)$$

It is convenient to normalize the current such that

$$[j^0(x), \underline{\psi}(y)]_{t_x=t_y} = - \delta(x-y) \underline{\psi}(y) \quad (24)$$

This requires $\frac{\beta}{4\pi} (\delta_1 - \delta_2) = 1$

For the two point function one obtains:

$$\begin{aligned} \langle \psi_1^\dagger(x) \psi_1(x') \rangle &\sim \left[\frac{1}{(u-u'-i\varepsilon)(v-v'-i\varepsilon)} \right]^{\frac{\delta_1^2 + \delta_2^2}{8\pi}} \left[\frac{u-u'-i\varepsilon}{v-v'-i\varepsilon} \right]^{\frac{\delta_1^2 - \delta_2^2}{8\pi}} \\ \langle \psi_2^\dagger(x) \psi_2(x') \rangle &\sim \left[\text{"} \right]^{\frac{\delta_1^2 + \delta_2^2}{8\pi}} \left[\text{"} \right]^{-\frac{\delta_1^2 - \delta_2^2}{8\pi}} \end{aligned}$$

i.e. $d_\psi = \frac{\delta_1^2 + \delta_2^2}{8\pi},$

dimension of ψ

$s_\psi = \frac{\delta_1^2 - \delta_2^2}{8\pi},$

spin of ψ (to be read off from the Lorentz transformation of the two-point function).

Taking the conventional Thirring value

we obtain

$$\delta_1 + \delta_2 = -\beta, \quad \delta_2 - \delta_1 = \frac{4\pi}{\beta}$$

The total charges $Q = \int j_0 dx$, and $Q_5 = \int j_0 \gamma_5 dx$ can be written as surface terms

i.e. $Q_5 = \phi(u=\infty) - \phi(v=-\infty)$

Neglecting these surface-terms in (which in technical language means performing a Klein-Transformation, changing the relative commutation of ψ_1 and ψ_2 to an anticommutation) we may write also ⁶⁾:

$$\psi = \sqrt{\frac{\mu}{2\pi}} e^{\frac{i\pi}{4} \gamma^5} : \exp i \left\{ \frac{\beta}{2} \gamma^5 \phi(x) + \frac{2\pi}{\beta} \int_x^\infty \dot{\phi}(x') dx' \right\} : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (25)$$

That ψ transforms like a $s=\frac{1}{2}$ spinor is not true, if we interpret this formula classically; it is a true quantum phenomenon.

The missing step in the complete construction of the Thirring model is the realization that j_μ can be obtained by a suitable space-time limiting procedure as a composite field from the ψ 's. ¹¹⁾

$$j_\mu(x) \sim \lim_{\eta, \tilde{\eta}} \frac{1}{2} \sum Z(\eta) \frac{1}{2} [\bar{\psi}(x) \gamma_\mu \psi(x+\eta)] \quad (26)$$

γ_μ = spacelike unit vector

$$\tilde{\gamma}_\mu = \varepsilon_{\mu\nu} \gamma^\nu$$

The invariant function $\bar{Z}(\eta)$ is just the excess of the true two point-function over the free two-point function.

$$\bar{Z}(\eta) \sim \frac{\langle \psi_1^+(x) \psi_1(x+\eta) \rangle}{u_x - u_{x+\eta} - i\varepsilon} = \frac{\langle \psi_2^+(x) \psi_2(x+\eta) \rangle}{v_x - v_{x+\eta} - i\varepsilon} \quad (27)$$

(for $S_\psi \neq \frac{1}{2}$ $Z(\eta)$ would have a γ_5 -matrix structure).

With this apparatus we can now look at the mass perturbation on the Thirring Lagrangian. Our boson representation converts this into (the line integral part in (25) drops out)

$$N[\bar{\psi}\psi] = \frac{1}{\pi} : \cos \beta \phi : \quad (28)$$

Since we already saw that the Thirring Lagrangian corresponds to a free Boson field we obtain the correspondence ($\dim \bar{\psi}\psi < \dim \mathcal{L}_0 = 2$, in terms of $\beta : \beta^2 < 8\pi$)

$$T_{M.Th}^{\mu\nu} = : \frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x_\nu} : - g^{\mu\nu} \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - M g^{\mu\nu} : \cos \beta \phi : \quad (29)$$

This is the Coleman equivalence.⁵⁾ The argument is convincing though somewhat formal since the word "mass perturbation" does not mean perturbation theory in the mass parameter, which would be horribly infrared divergent. Another way of understanding this equivalence would be to define the axial current and its corresponding quantum potential ϕ in the actual massive Thirring model. Although this model is not known explicitly, the structural knowledge on leading and nonleading short distance behaviour on the basis of the Callan-Symanzik equation (which simplify because $\beta(g) \equiv 0$ ¹²⁾) is sufficient to assert that $N[\bar{\psi}\psi]$ can be defined by just a multiplication renormalization (space-time limiting) procedure, as long as there are no nonleading singularities. This assertion holds for $\frac{\beta^2}{4\pi} < 1$ or $\dim \bar{\psi}\psi < 1$. The general theory gives for $\dim \bar{\psi}\psi \equiv 1$ (i.e. the free field) a nonleading logarithmic divergence for short distances which gains strength

(i.e. a non-integer power) as $\dim \bar{\Psi}\Psi > 1$ and one needs an additive renormalization as well, in order to define $N[\bar{\Psi}\Psi]$. Although the existence of Wick-ordered powers for ϕ i.e. $:\phi^n:$ is unaffected as long as $\dim \bar{\Psi}\Psi < 2$, there will be divergences in $:e^{i\beta\phi}:^7$ as soon as $\dim \bar{\Psi}\Psi \geq 1$. These divergences, which have been called "cumulative mass effects"⁷⁾, change the Coleman form of the equivalence and render the Mandelstam formula⁶⁾ for Ψ invalid¹³⁾. For $\beta^2 < 4\pi$, however, one can give a short distance argument that $N[\Psi_1 + \Psi_2]$ is proportional to $:e^{-i\beta\phi}:^7$. Below the "critical" value $\beta^2 = 4\pi$ the Mandelstam formula is consistent in the following sense:

Theorem (Swieca⁸⁾) The Ψ defined à la Mandelstam (25) transforms like an $s = \frac{1}{2}$ quantum field as a result of the quantum Sine-Gordon equation for ϕ .

One can also see⁸⁾ that the field equation for Ψ (for $\beta^2 < 4\pi$) is the massless Thirring equation (this differs from Mandelstam's claim⁶⁾) if one defines the term $N[j_\mu \Psi]$ by short distance expansion from

$\mathcal{H}(x) \Psi(y)$ which appears in the Heisenberg equation of motion $\frac{\partial \Psi}{\partial t} = i[H, \Psi]$. The mass term emerges if one defines a new composite field (by subtracting a multiple of Ψ) with $\langle 0 | N[j_\mu \Psi] | 1, \Psi \rangle = 0$

The fact that exponentials involving line integrals over quantum fields are in fact independent on the integration path is a new phenomenon of "quantum cohomology". As the Coleman equivalence, it is definitely not true for classical fields. In the Coleman equivalence we have obtained local fermion fields Ψ as describing interpolating fields for the Sine-Gordon quantum solitons. It is easy to see that we can also find an interpolating local Bose-field which also carries the soliton charge and lives in the same Hilbert-space. This expresses the ambiguity of statistics of two dimensional kinks ; even the principle of locality does not distinguish between the two possibilities¹³⁾. It is an open question whether any of these statistics ambiguities persist for higher dimensional quantum kinks (monopoles !).

3. Towards a construction of the Quantum massive Thirring (Sine-Gordon) model.

This construction is very specific and makes use of the infinite number of conservation laws. It is important to work on the Thirring side of the Coleman equivalence in order to have a natural local interpolating field for the solitons. The starting observation is that the massive Thirring model possesses an infinite set of conservation laws ¹⁴⁾ which survive quantization. Asymptotically these conserved quantities go over into the corresponding free field expression in terms of the in and out fields. As a consequence ¹⁵⁾ the S-matrix is purely elastic. The two particle S-matrix for solitons can be written in terms of three amplitudes t, r, u :

$$\langle s(p'_1) \bar{s}(p'_2) | S | s(p_1) \bar{s}(p_2) \rangle = \delta(p'_1 - p_1) \delta(p'_2 - p_2) t - \delta(p'_2 - p_2) \delta(p'_1 - p_1) r \quad (30)$$

$$\langle s(p'_1) s(p'_2) | S | s(p_1) s(p_2) \rangle = \left(\delta(p'_1 - p_1) \delta(p'_2 - p_2) - \delta(p'_1 - p_2) \delta(p'_2 - p_1) \right) u \quad (31)$$

The δ function factors in front come about as a consequence of an identity for $\delta^{(2)}(p' - p)$. Instead of the variable $\nu = \frac{p_1 p_2}{m^2}$ we use the uniformization variable called "rapidity" (hyperbolic angle) $\cosh \theta = \nu$, $\theta = \theta_1 - \theta_2$, which maps the physical sheet into the strip $0 < \text{Im } \theta < \pi$:

We make the following technical assumption:

Assumption: t, r , and u are meromorphic for all θ .

This assumption goes beyond the known analytic properties of S ⁽²⁾ following from general field theoretical principles.

Unitarity $S^+ S = 1$ and crossing-symmetry are equivalent to the following 5 relations :

$$t(-\theta) t(\theta) + r(-\theta) r(\theta) = 1 \quad (32a)$$

$$t(-\theta) r(\theta) + r(-\theta) t(\theta) = 0 \quad (32b)$$

$$u(-\theta) u(\theta) = 1 \quad (32c)$$

$$t(\theta) = u(i\pi - \theta) \quad (33a)$$

$$r(\theta) = r(i\pi - \theta) \quad (33b)$$

Assuming $r \neq 0$ which is consistent with lowest order perturbation theory¹⁶⁾ we may introduce $h(\theta) = \frac{t(\theta)}{r(\theta)}$ which because of (32b) is antisymmetric in θ .

Knowing h we may reconstruct t using

$$t(\theta) t(i\pi + \theta) = \frac{h(\theta)}{h(i\pi - \theta)} \quad (34)$$

and using dispersion relation for $\ln t$ ¹⁷⁾ (32a), (33a) and (33b) lead to

$$\begin{aligned} -h^2(\theta) + 1 &= \frac{1}{r(-\theta) r(\theta)} = \frac{1}{r(i\pi + \theta) r(i\pi - \theta)} \\ &= h(i\pi + \theta) h(i\pi - \theta) \frac{1}{t(i\pi + \theta) t(i\pi - \theta)} \end{aligned} \quad (35)$$

and with $t(i\pi + \theta) t(i\pi - \theta) = 1$ to

$$h(i\pi + \theta) h(i\pi - \theta) + h^2(\theta) = 1 \quad (36)$$

There are many solutions of this functional relation, e.g.,

$$h(\theta) = \frac{\sinh \theta f(\theta)}{\sinh i\pi f(\theta)}$$

with f = even of period $i\pi$

It is truly amazing that the consistency of the n particle factorization of the S-matrix leads to another functional relation

$$\frac{h(i\pi + \theta) - h(i\pi - \theta)}{h(\theta)} = \text{const.} = 2 \cosh \pi \lambda \quad (37)$$

Before we explain the factorization property let us discuss the consequences of these two functional relations.

The unique solution is

$$h(\theta) = \frac{\sinh \lambda \theta}{\sinh i \lambda \pi} \quad (38)$$

With another mild technical assumption we can now solve for $t(\theta)$. The property needed in order to obtain a simple dispersion representation for $\ln t$ is that $\ln t$ is analytic (i.e. t has no zeros or poles) and falls off sufficiently rapidly at $\theta \rightarrow \infty$. If t describes the massive Thirring model scattering then we expect the existence of a λ range without bound state poles and in this region there is also no necessity for zeros. With these assumptions we write

$$\begin{aligned} \ln t &= \frac{1}{2\pi i} \oint \frac{dz}{\sinh(z-\theta)} \ln t(z) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx}{\sinh(x-\theta-i\epsilon)} \underbrace{\ln t(x) \cdot t(x+i\pi)}_{\frac{h(x)}{h(i\pi-x)}} \end{aligned} \quad (39)$$

hence

$$\ln t = \frac{1}{2\pi i} \int \frac{dx}{\sinh(x-\theta-i\epsilon)} \ln \frac{\sinh \lambda x}{\sinh \lambda(i\pi-x)} \quad (40)$$

and using tables of Fourier transforms

$$\ln t = i \int_0^{\infty} \frac{dx}{x} \frac{\sin \frac{x}{\pi}(i\pi-\theta) \sinh \frac{x}{2} \left(\frac{1}{\lambda}-1\right)}{\sinh \frac{1}{\lambda} \frac{x}{2} \cosh \frac{x}{2}} \quad (41)$$

Introducing the convenient function

$$f(y) = \exp. - \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{x}{2} (1-\lambda) \sinh x \lambda y}{\sinh \frac{x}{2} \cosh \frac{x}{2} \lambda} \quad (42)$$

with $f^*(y) f(y) = f(-y) f(y) = 1$

one finds the unified representation:

$$u(y) = f(y) \quad (43a)$$

$$t(y) = \frac{\sin \pi \lambda y}{\sin \pi \lambda (1-y)} f(y) \quad (43b)$$

$$r(y) = \frac{\sin \pi \lambda}{\sin \pi \lambda (1-y)} f(y) \quad (43c)$$

and therefore the parity $\pm$ amplitudes:

$$S_+ = - \frac{\sin \frac{\pi \lambda}{2} (y+1)}{\sin \frac{\pi \lambda}{2} (y-1)} f(y) \quad (43d)$$

$$S_- = - \frac{\cos \frac{\pi \lambda}{2} (y+1)}{\cos \frac{\pi \lambda}{2} (y-1)} f(y) \quad (43e)$$

the t, r , and u fulfill (by construction) all unitarity and crossing relations. The S matrix constructed under this special meromorphy and high energy assumption ¹⁸⁾ has the following remarkable properties:

1. It is the only S -matrix of solitons and antisolitons fulfilling (under the technical assumption) the absence of pair creation and the factorization property.

2. It agrees up to 3rd order perturbation theory (only up to this order the computation has been done) with the result of renormalized perturbation theory of the massive Thirring model.

3. It reproduces for $\lambda \rightarrow \infty$, $\frac{m^2}{\lambda} = \text{fixed}$ the known semiclassical behaviour of scattering ¹⁹⁾.

4. The quasiclassically known bound-state spectrum emerges as an exact spectrum for the localization of poles in $S_{s\bar{s}}$.

This last result is certainly the most impressive result.

The last consequence deserves to be studied in some details. The best way to see the spectrum is to use Malmsten's representation for $\log \Gamma$ ¹⁷⁾¹⁹⁾. With its help one obtains an infinite product representation of t :

$$t(\theta) = \frac{F(\theta)}{F(2\pi i - \theta)} \quad (42)$$

$$F(i\pi\varphi) = \prod_{\substack{k=0 \\ \ell=0}}^{\infty} \frac{(2\ell + \frac{k+1}{\lambda} + \varphi)(2\ell + \frac{k}{\lambda} + \varphi)}{(2\ell - 1 + \frac{k+1}{\lambda} + \varphi)(2\ell + 1 + \frac{k}{\lambda} + \varphi)} \quad (43)$$

F has clearly poles at

$$\varphi = 1 - 2\ell - \frac{k+1}{\lambda}, \quad \varphi = -1 - 2\ell - \frac{k}{\lambda},$$

and zeros at

$$\varphi = -2\ell - \frac{k+1}{\lambda}, \quad \varphi = -2\ell - \frac{k}{\lambda}.$$

In the first sheet $0 < \varphi < 1$ one just obtains the $\varphi = 1 - \frac{k}{\lambda}$

$k=1, \dots, [\lambda]$ poles in t . Half of them, namely k even belong to $t+r = e^{2i\delta_+}$ (positive residue) whereas the odd ones occur in the negative C-parity amplitude $t-r = e^{2i\delta_-}$ (negative residue). Note that the corresponding mass values can be read off from

$$(p_1 + p_2)^2 = 2m^2(1 + \cosh \theta)_k \quad \text{i.e.} \quad m_k = 2m \sin \frac{\pi}{2} \frac{k}{\lambda}$$

The above form of t was guessed by Zamolodchikov ¹⁸⁾ on the basis of quasiclassical behaviour in particular the quasiclassical breather spectrum.

From $S^{(2)}$ one can easily compute with the help of Watson's formula the soliton formfactor ²⁰⁾

which reads:

$$\frac{G_-(\theta)}{G_-(-\theta)} = e^{2i\delta_-(\theta)} \quad (45)$$

assuming that G has the same meromorphy θ -picture as t,r,u, Watson's statement may be (by taking the logarithm) converted into a dispersion relation.

For

$$\langle s(p_1) | j_r | s(p_1) \rangle = G_-(i\pi - \theta) \bar{u} \gamma_r u \quad (46)$$

one obtains in this way

$$G(i\pi - \theta) = \frac{\cosh \theta/2}{\cosh \theta/2 \lambda} e^{T(\theta)} \quad (47)$$

$$T(\theta) = \int_0^\infty \frac{dt}{t \sinh t} \frac{\sin^2 \frac{\theta t}{2\pi} \cdot \sinh \frac{t}{2} (1 - \frac{1}{\lambda})}{\sinh \frac{t}{2} \frac{1}{\lambda} \cdot \cosh \frac{1}{2} t}$$

Again we have taken the minimally subtracted dispersion relation.

Amazingly, this result in powers of λ agrees with the lowest order massive Thirring perturbation. For $m\lambda = \text{const.}$, $\lambda \rightarrow 0$ it also reproduces the semi-classical result ²⁰⁾.

Furthermore the Taylor coefficients of G_- at $\theta = i\pi$ i.e the charge radius and its generalization agree with their quasiclassical form. However for large momentum transfer we have a dramatic deviation:

$$G_-(t) = (-t)^{-\frac{\lambda-1}{4}} \quad (48)$$

Let us now turn to a discussion of the factorization property. For the scattering of one type of neutral particle (i.e., lowest state of sine-Gordon) factorization reads:

$$\langle p_1' \dots p_n' | S | p_1 \dots p_n \rangle = \langle p_1' \dots p_n' | p_1 \dots p_n \rangle \prod_{i < j} e^{2i\delta(p_i, p_j)} \quad (49)$$

This formula is known from the nonrelativistic δ -function interaction ²¹⁾ and it expresses the build-up of n-particle scattering by two body interactions.

Note that the suppressed momenta always have the order $|p_1, \dots, p_n\rangle$

Consistency of factorization now means the following requirement:

the result of $\prod S_{ik} | \dots \rangle$ must be independent of the order in which the S_{ik} are applied.

This consistency requirement becomes effective for the first time for $n=3$:

$$\begin{aligned} S |ss\bar{s}\rangle &= S_{23} S_{13} S_{12} |ss\bar{s}\rangle = S_{23} (t_{13} u_{12} |ss\bar{s}\rangle + r_{13} u_{12} |\bar{s}ss\rangle) \\ &= t_{23} t_{13} u_{12} |ss\bar{s}\rangle + r_{23} t_{13} u_{12} |s\bar{s}s\rangle \\ &\quad + u_{23} r_{13} u_{12} |\bar{s}ss\rangle \end{aligned}$$

On the other hand interchanging the application of S_{12} with S_{23} one obtains:

$$S |ss\bar{s}\rangle = S_{12} S_{13} S_{23} |ss\bar{s}\rangle \quad (50a)$$

$$\begin{aligned} &= S_{12} (t_{13} t_{23} |ss\bar{s}\rangle + u_{13} r_{23} |s\bar{s}s\rangle \\ &\quad + r_{13} t_{23} |\bar{s}ss\rangle) \end{aligned} \quad (50b)$$

$$\begin{aligned} &= u_{12} t_{13} t_{23} |ss\bar{s}\rangle + t_{12} u_{13} r_{23} |s\bar{s}s\rangle \\ &\quad + r_{12} u_{13} r_{23} |\bar{s}ss\rangle + t_{12} r_{13} t_{23} |s\bar{s}s\rangle \\ &\quad + r_{12} r_{13} t_{23} |s\bar{s}s\rangle \end{aligned}$$

This leads to the additional functional relation:

$$h(\theta_{23} + \theta_{12}) = h(i\pi + \theta_{31})h(\theta_{12}) + h(\theta_{31})h(i\pi - \theta_{12}) \quad (51)$$

from which follows the relation (37)

$$\frac{h(i\pi + \theta_{31}) - h(i\pi - \theta_{31})}{h(\theta_{31})} = \frac{h(i\pi + \theta_{12}) - h(i\pi - \theta_{12})}{h(\theta_{12})} \quad (52)$$

The three particle consistency requirement can be easily expressed in a diagrammatic way. Let $S_{23}, S_{13}, S_{21} / s\bar{s}\bar{s} \rangle$ correspond to having 12 interact first, then 13 and finally 23

Fig.1a

$S_{21}, S_{13}, S_{23} / s\bar{s}\bar{s} \rangle$ corresponds then to the diagram in which the p_3 line has been shifted below the 12 intersection

Fig.1b

Consistency of factorization in this case means "shift invariance" in diagrammatic language.

It is easy to see that the special case of shift invariance for $S / s\bar{s}\bar{s} \rangle$ is typical: the general consistency is fulfilled if:

$$S_{ij} S_{ik} S_{jk} = S_{jk} S_{ik} S_{ij} \quad (53)$$

i, j, k pairwise unequal.

So the functional relation for h obtained from our special case is already the general restriction.

This graphical representation is also very useful if one tries to determine the complete S-matrix where some of the incoming or outgoing particles are not solitons, but rather mesons (breathers).

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