Reconstruction of general scalar-field dark energy models

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The reconstruction of scalar-field dark energy models is studied for a general Lagrangian density $p(\phi, X)$, where $X$ is a kinematic term of a scalar field $\phi$. We implement the coupling $Q$ between dark energy and dark matter and express reconstruction equations using two observables: the Hubble parameter $H$ and the matter density perturbation $\delta_m$. This allows us to determine the structure of corresponding theoretical Lagrangian together with the coupling $Q$ from observations. We apply our formula to several forms of Lagrangian and present concrete examples of reconstruction by using the recent Gold dataset of supernovae measurements. This analysis includes a generalized ghost condensate model as a way to cross a cosmological-constant boundary even for a single-field case.

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I. INTRODUCTION

Observations suggest that our universe has entered a stage of an accelerated expansion with a redshift $z \lesssim 1$. This has been supported by a number of recent astrophysical data including supernovae (SN) Ia, Cosmic Microwave Background (CMB), and large-scale structure. These observations show that about 70% of the total energy density of the present universe consists of dark energy responsible for an accelerated expansion (see Refs. for recent works). The simplest candidate of dark energy is a cosmological constant, but this scenario suffers from a severe fine-tuning problem of its energy scale from the viewpoint of particle physics. Hence it is natural to pursue alternative possibilities to explain the origin of dark energy.

So far a wide variety of scalar-field dark energy models has been proposed— including quintessence, k-essence, tachyons, phantoms and ghost condensates. These scenarios are distinguished from a cosmological constant because of their dynamical nature of the equation of state of scalar fields. A typical approach is to predict the evolution of the Hubble parameter theoretically and to check the consistency of models by comparing it with observations. An alternative approach is to start from observational data and to reconstruct corresponding theoretical Lagrangian. The latter is more efficient to find out best-fit models of dark energy from observations.

This reconstruction is simple for a minimally coupled scalar field with a potential $V(\phi)$ and $\phi, X$ in the Lagrangian density $L$. In fact one can reconstruct the potential and the equation of state of the field by parametrising the Hubble parameter $H$ in terms of the redshift $z$ from the luminosity distance $D_L(z)$. This method can be generalized to scalar-tensor theories, $f(R)$ gravity, and also a dark-energy fluid with viscosity terms. In scalar-tensor theories a scalar field called dilaton is coupled to gravity, which means that an additional function $F(\phi)$ exists in front of the Ricci scalar $R$. If the evolution of matter perturbations $\delta_m$ is known observationally in addition to the Hubble parameter $H(z)$, one can even determine the function $F(\phi)$ together with the potential $V(\phi)$ of the scalar field.

In this paper we will provide a reconstruction program for a very general scalar-field Lagrangian density $p(\phi, X)$, where $X \equiv -\frac{1}{2}(\nabla \phi)^2$ is a kinematic term. We shall consider an Einstein-Hilbert action with a scalar field $\phi$ coupled to a non-relativistic barotropic fluid (dark matter) with a coupling $Q(\phi)$. This coupled quintessence scenario was proposed in Ref. as an extension of scalar-tensor theories with a nonminimally coupled scalar field. In fact the scalar-tensor action in Jordan frame can be transformed to the action in Einstein frame with an explicit coupling between the scalar field and the non-relativistic fluid.

The presence of the coupling $Q(\phi)$ means that the parametrisation of the Hubble rate $H(z)$ is not sufficient to determine the structure of theory. We shall use the equation of sub-Hubble matter perturbations recently derived in Ref. for the Lagrangian density $p(\phi, X)$ (see also Refs. and ). The coupling $Q(\phi)$ is determined once we know $H(z)$ and $\delta_m(z)$ observationally. We note that this places constraints on the strength of nonminimal couplings when the coupling $Q$ originates from scalar-tensor theories.

The observations of Sloan Digital Sky Survey (SDSS) and Two degree Field (2dF) galaxy clustering provide the information of matter perturbations. Galaxy clustering is proportional to matter clustering on large scales with a constant of proportionality called bias. The analysis of galaxy clustering itself does not determine the value of bias. Hence the evolution of matter perturbations is difficult to be known unless bias is determined observationally. However recent observations using the luminosity dependence of galaxy clustering together with the halo mass distribution began to provide good data for the determination of bias. It is expected that we will be able to determine the evolution of matter perturbations, $\delta_m(z)$, from upcoming high-precision observations.

The recent SN analysis using the Gold dataset implies that the parametrisation of $H(z)$ which crosses the cosmological-constant boundary ($w = -1$) shows a good
fit to data. This crossing to the phantom region \((w < -1)\) is not possible for an ordinary minimally coupled scalar field \([p = X - V(\phi)]\). This transition can occur for the Lagrangian density \(p(\phi, X)\) in which \(\partial p/\partial X\) changes sign from positive to negative, but we require nonlinear terms in \(X\) to realise the \(w = -1\) crossing\(^1\). It was shown in Ref. 33 that such a crossing is possible if a (phantom) dark energy fluid is coupled to a non-relativistic fluid with a specific coupling. This can be also realised in a multi-field system with a phantom and an ordinary scalar \(19, 21\). In this paper we shall show a simple one-field model \((p = -X + h(\phi)X^2)\) crossing the cosmological-constant boundary and perform the reconstruction of such a theory.

**II. RECONSTRUCTION PROGRAM**

We start with a general action given by

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + p(\phi, X) \right] + S_m(\phi, g_{\mu\nu}) ,
\]

where \(R\) is a scalar curvature, \(p(\phi, X)\) is a general function of \(\phi\) and \(X = -(1/2)(\nabla\phi)^2\). The gravitational constant, \(\kappa^2 \equiv 8\pi G\), is set to be unity. We assume that the field \(\phi\) is coupled to a barotropic perfect fluid with a coupling \(Q(\phi) \equiv -1/(\rho_m\sqrt{-g})\delta S_m/\delta \phi\), where \(\rho_m\) is the energy density of the fluid. We shall use a sign notation \((- + , + , + , +)\).

In a flat Friedmann–Robertson–Walker (FRW) spacetime with a scale factor \(a\), the field equations for the action \(11\) are

\[
\begin{align*}
3H^2 &= \rho_m + 2Xp_X - p, \\
2\dot{H} &= -\rho_m - p_m - 2Xp_X, \\
\dot{\rho}_m + 3H(\rho_m + p_m) &= Q(\phi)\rho_m\dot{\phi},
\end{align*}
\]

where \(H = \dot{a}/a\), \(X = \dot{\phi}^2/2\), \(p_x = \partial p/\partial X\) and \(p_m\) is the pressure density of the fluid. A dot denotes a derivative with respect to cosmic time \(t\). In what follows we shall consider the case of a non-relativistic barotropic fluid, i.e., \(p_m = 0\). Eq. (4) is written in an integrated form

\[
\rho_m = \rho_m^{(0)} \left( \frac{a_0}{a} \right)^3 I(\phi),
\]

where

\[
I(\phi) \equiv \exp \left( \int_{\phi_0}^{\phi} Q(\phi) d\phi \right).
\]

Here \(\rho_m^{(0)}\), \(a_0\) and \(\phi_0\) are the present values of the energy density \(\rho_m\), the scale factor \(a\) and the scalar field \(\phi\), respectively. By using the present ratio \(\Omega_m\) of the matter fluid and the Hubble parameter \(H_0\), \(\rho_m^{(0)}\) is given by \(\rho_m^{(0)} = 3H_0^2\Omega_m\). We also define the redshift parameter \(z\), as \(1 + z \equiv a_0/a\). Then the energy density \(\rho_m\) is written as

\[
\rho_m = 3\Omega_m H_0^2 (1 + z)^3 I(\phi).
\]

One has \(I(\phi) = 1\) in the absence of the coupling \(Q(\phi)\).

In this case one can reconstruct the structure of theory by using Eqs. (2), (3) and (7) if the evolution of the Hubble parameter is known from observations. This was actually carried out for an ordinary scalar field with a Lagrangian density: \(p = X - V(\phi)\) \(14, 12, 10\). When the field \(\phi\) is coupled to dark matter, we require additional information to determine the strength of the coupling. We shall use the equation of matter density perturbations for this purpose as in the case of scalar-tensor theories \(19, 21\).

The evolution equation for the matter density contrast, \(\delta_m \equiv \delta \rho_m/\rho_m\), was recently derived in Ref. 32. On sub-Hubble scales this is given by

\[
\ddot{\delta}_m + \left[ 2H + Q(\phi) \right] \dot{\delta}_m - \frac{1}{2} \left[ 1 + \frac{2Q^2(\phi)}{p_X} \right] \rho_m \delta_m = 0.
\]

This is a very general equation which holds for any scalar-field Lagrangian density \(p(\phi, X)\) and also for the case in which the coupling \(Q\) depends on the field \(\phi\). Eq. (8) can be solved analytically for scaling solutions \(32\). In fact it was shown that matter perturbations are suppressed for a phantom \((p_X < 0)\) whereas they are not for an ordinary field \((p_X > 0)\). Generally we need to specify the Lagrangian density \(p(\phi, X)\) in order to solve Eq. (8).

We note that the gravitational potential \(\Phi\) is related with the matter perturbation \(\delta_m\) through the relation \(\Phi \simeq -(3a^2H^2/2k^2)\Omega_m\delta_m\) on sub-Hubble scales \(32\) \((k\) is a comoving wavenumber).

One can rewrite the equations \(22, 23\) and \(8\) by using a dimensionless quantity

\[
r \equiv H^2 / H_0^2.
\]

Making use of the relation \(7\), we find that

\[
p = \left[ (1 + z)r' - 3r \right] H_0^2 ,
\]

\[
\phi'^2 p_X = \frac{r' - 3\Omega_m(1 + z)^2 I}{r(1 + z)} ,
\]

\[
\delta_m'' + \left( 2r' - \frac{1}{1 + z} + \frac{I'}{I} \right) \delta_m' = \frac{-3}{2} \Omega_m \left( 1 + \frac{2I'^2}{\phi'^2 p_X I^2} \right) \left( 1 + z \right) I \delta_m = 0 ,
\]

where a prime represents a derivative with respect to \(z\). Eliminating the \(\phi'^2 p_X\) term from Eqs. (11) and (12), we obtain

\[
I' = \frac{I}{4r(1 + z)A} \left[ \delta_m' \pm \sqrt{\delta_m'^2 - 8r(1 + z)AB} \right] ,
\]

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1 I thank Alexander Vikman for pointing out that the \(w = -1\) crossing is hard to be realised only in the presence of the terms linear in \(X\) \(35\).
where

\[ A = \frac{3\Omega_m(1+z)\delta_m I}{2r[r^2 - 3\Omega_m(1+z)^2]}. \]  
\[ B \equiv [r' - 3\Omega_m(1+z)^2 I]A \]
\[ - \delta_m'' \left( \frac{r'}{2r} \right) \frac{1}{1+z} \delta_m'. \]  

(15)

From Eq. (13) we require the following condition

\[ \delta_m^2 > 8r(1+z)AB. \]  

(16)

Once we know \( r \) and \( \delta_m \) in terms of \( z \) observationally, Eq. (13) is integrated to give the functional form of \( I(z) \). It is worth mentioning that the function \( I(z) \) is determined without specifying the Lagrangian density \( p(\phi, X) \). By using Eqs. (10) and (11), we obtain \( p \) and \( \phi^2 p_X \) as the functions of \( z \). The energy density of the scalar field, \( p = \phi^2 p_X - p \), is also known. From Eq. (16) we find

\[ Q = \frac{(\ln I)'}{\phi'}. \]  

(17)

Hence the coupling \( Q \) is determined once \( I \) and \( \phi' \) are known. We need to specify the Lagrangian density \( p(\phi, X) \) to find the evolution of \( \phi' \) and \( Q \).

The equation of state for dark energy, \( w \equiv p/\rho \), is given by

\[ w = \frac{p}{\phi^2 p_X - p} = \frac{(1+z)r' - 3r}{3r - 3\Omega_m(1+z)^2 I}. \]  

(18)

(19)

A normal scalar field corresponds to \( w > -1 \), which translates into the condition \( p_X > 0 \) by Eq. (18). From Eq. (11) we find that this condition corresponds to \( r' > 3\Omega_m(1+z)^2 I \), which can be also checked by Eq. (13).

Meanwhile a phantom field corresponds to \( p_X < 0 \) and \( r' < 3\Omega_m(1+z)^2 I \). As we already mentioned, the evolution of \( I(z) \) is determined if \( r \) and \( \delta_m \) are known observationally. Then the equation of state of dark energy is obtained by Eq. (15) without specifying the Lagrangian density \( p(\phi, X) \). In the next section we shall apply our formula to several different forms of scalar-field Lagrangian.

### III. APPLICATION TO SPECIFIC CASES

Most of scalar-field dark energy models can be classified into two classes: (A) \( p = f(X) - V(\phi) \) and (B) \( p = f(X) V(\phi) \). There are special cases in which cosmological scaling solutions exist. This corresponds to the Lagrangian density (C) \( p = g(X e^{\lambda \phi}) \), where \( \lambda \) is a constant and \( g \) is an arbitrary function. In what follows we shall consider these classes of models separately.

#### A. Case of \( p = f(X) - V(\phi) \)

This case includes quintessence \( [f(X) = X] \) and a phantom field \( [f(X) = -X] \). From Eq. (11) we find

\[ \phi^2 f_X = \frac{r' - 3\Omega_m(1+z)^2 I}{r(1+z)}. \]  

(20)

If we specify the function \( f(X) \), the evolution of \( \phi' \) and \( \phi(z) \) is known from \( r(z) \) and \( f(z) \). By Eq. (17) we find the coupling \( Q \) in terms of \( z \) and \( \phi \). Finally Eq. (10) gives

\[ V = f + [3r - (1+z)r']H_0^2. \]  

(21)

The r.h.s. is determined as a function of \( z \). Since \( z \) is expressed by the field \( \phi \), one can find the potential \( V(\phi) \) in terms of \( \phi \). In the case of quintessence without a coupling \( Q \), this was carried out by a number of authors [10, 11, 12, 13]. We have generalized this to a more general Lagrangian density \( p = f(X) - V(\phi) \) in the presence of the coupling \( Q \).

#### B. Case of \( p = f(X) V(\phi) \)

This case includes k-essence [8] and tachyon [9]. The pressure density of the form

\[ p(\bar{X}, \varphi) = K(\varphi)\bar{X} + L(\varphi)\bar{X}^2, \quad \bar{X} = -\langle \nabla \varphi \rangle^2/2, \]  

(22)

is transformed to the Lagrangian density \( p = f(X) V(\phi) \) with \( f(X) = -X + X^2 \) and \( V(\phi) = K^2/L \) by field redefinitions:

\[ \phi = \int^\varphi d\varphi \sqrt{\frac{L}{|K|}}, \quad X = \frac{L}{|K|}\bar{X}. \]  

(23)

We note that dilatonic ghost condensate model [12] corresponds to a choice \( K(\varphi) = -1 \) and \( L(\varphi) = e^{\lambda \varphi} \). In this case the potential of the scalar field has a dependence \( V(\phi) \propto \phi^{-2} \) by the field redefinitions [24].

For the Lagrangian density \( p = f(X) V(\phi) \) we obtain the following relation from Eqs. (10) and (11):

\[ \phi^2 f_X \frac{f_X}{f} = \frac{r' - 3\Omega_m(1+z)^2 I}{r(1+z)[(1+z)r' - 3r]H_0^2}; \]  

(24)

\[ V = \frac{[(1+z)r' - 3r]H_0^2}{f}. \]  

(25)

Once we specify the form of \( f(X) \), one can determine the functions \( \phi' \) and \( \phi(z) \) from Eq. (23). Then we obtain the potential \( V(\phi) \) from Eq. (20).

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2 We note that a coupled dark energy scenario between k-essence field and dark matter was recently studied in Ref. [11].
C. Case of $p = X g(X e^{\lambda \phi})$

When we construct realistic dark energy models, scaling solutions may play an important role to solve coincident problems. In this case the energy density of the scalar field is proportional to that of a barotropic fluid ($\rho \propto \rho_m$). It was shown in Refs. [12, 14] that the existence of scaling solutions restricts the form of the scalar-field Lagrangian to be

$$p = X g(X e^{\lambda \phi}).$$  \hspace{1cm} (26)

We note that this property holds both for coupled and uncoupled models of dark energy.

For the Lagrangian density \cite{20} Eqs. \cite{10} and \cite{11} give

$$Y = \frac{6r - (1 + z)r' - 3\Omega_m(1 + z)^3I}{2[(1 + z)r' - 3r]}$$

$$\phi^2 = \frac{2[(1 + z)r' - 3r]}{r(1 + z)^2g},$$

where $Y = X e^{\lambda \phi}$. If we specify the functional form of $g(Y)$, one can determine the function $Y = Y(z)$ from Eq. (27). Then we find $\phi'(z)$ and $\phi(z)$ by Eq. (28). The parameter $\lambda$ is known by the relation $Y = (1/2)\phi^2 e^{\lambda \phi}$.

It was shown in Refs. [12, 14] that the quantity $Y$ is constant along scaling solutions, in which case the l.h.s. of Eq. (27) is constant. In the absence of the coupling $Q$, i.e., $I = 1$, accelerated expansion is not realized for scaling solutions. This means that the r.h.s. of Eq. (27) should not be constant for $Q = 0$ when we use observational data. If the coupling $Q$ is present, there is a possibility to find a situation in which the r.h.s. of Eq. (27) does not vary. This corresponds to the case where accelerated expansion is realized by scaling attractors. Thus one can directly check whether the present acceleration originates from scaling solutions with a non-zero coupling $Q$ once we obtain accurate observational data of $r(z)$ and $\delta_m(z)$.

IV. EXAMPLES OF RECONSTRUCTION

In this section we show concrete examples of reconstruction for several dark energy models. We shall use the following parametrization for the Hubble parameter \cite{45, 46}

$$r(x) = \Omega_m x^3 + A_0 + A_1 x + A_2 x^2,$$  \hspace{1cm} (29)

where $x = 1 + z$ and $A_0 = 1 - A_1 - A_2 - \Omega_m$. This corresponds to the following expansion for dark energy

$$\rho = \rho_0c \left( A_0 + A_1 x + A_2 x^2 \right),$$  \hspace{1cm} (30)

where $\rho_0c = 3H_0^2$.

For a prior $\Omega_m = 0.3$, the Gold dataset of SN observations gives $A_1 = -4.16 \pm 2.53$ and $A_2 = 1.67 \pm 1.03$.

We note that the weak energy condition for dark energy, $\rho \geq 0$ and $w = p/\rho \geq -1$, corresponds to \cite{44}

$$A_0 + A_1 x + A_2 x^2 \geq 0, \quad A_1 + 2A_2 x \geq 0.$$  \hspace{1cm} (31)

If we use the best-fit values $A_1 = -4.16$ and $A_2 = 1.67$, for example, we find that the second condition in Eq. (31) is violated around present ($x \sim 1$). This means that the field behaves as a phantom ($w < -1$). In the case of an ordinary scalar field such as quintessence, we need to put a prior $A_1 + 2A_2 x \geq 0$.

The reconstruction program of a quintessence-type scalar field has been already carried out in Ref. \cite{15}, so we do not repeat it here. We shall study the cases of tachyon and generalised ghost condensate when the coupling $Q$ is zero. We have not yet obtained good dataset about $\delta_m(z)$, so the coupling $Q$ is not well determined by current observations.

A. Tachyon

Tachyon corresponds to the case $f = -(1 - 2X)^{1/2}$ in Eqs. \cite{20} and \cite{26}. Then we have

$$\phi^2 = \frac{r' - 3\Omega_m x^2 I}{H_0^2 r x (r' - 3\Omega_m x^2 I) + 3r - rx'},$$  \hspace{1cm} (32)

$$V = \frac{(3r - x r')H_0^2}{\sqrt{1 - r^2 x^2 H_0^2 \phi^2}}.$$  \hspace{1cm} (33)

We note that the reconstruction equations were derived in Ref. \cite{15} for $Q = 0$. The equation of state for tachyon is $w = \phi^2 - 1$, which means that $w \geq -1$. Hence we should impose the prior given by Eq. (31).

In Fig. 11 we show one example of the reconstruction of the tachyon model for $Q = 0$ ($I = 1$). The field value is chosen to be $\phi = 0$ at present ($z = 0$). The tachyon potential needs to be flat around 0 $< z < 1$ to give rise to an accelerated expansion. The potential has a minimum with a non-zero energy density for the parametrisation given in this figure ($A_1 = -3.80$ and $A_2 = 1.95$). This implies that the rolling massive scalar field model with potential $V(\phi) = V_0 \exp(m^2\phi^2/2)$ \cite{15} can be a viable dark energy model.

B. Generalised ghost condensate

When the field satisfies the condition $p_X > 0$, we need to put the prior \cite{47} for consistency. Let us consider a situation in which crossing of the cosmological-constant boundary is possible. This can be realised for the following type of Lagrangian density:

$$p = -X + h(\phi) X^2,$$  \hspace{1cm} (34)

where $h(\phi)$ is a function in terms of $\phi$. Dilatonic ghost condensate model \cite{12} corresponds to a choice $h(\phi) = \frac{1}{2} \rho_0c g(X e^{\lambda \phi}) \left( A_0 + A_1 x + A_2 x^2 \right)$.
though some difference appears for $z > 0$, the current observational dataset is not still sufficient to future, after which the perturbations become stable.

Cosmological-constant boundary crossing occurs again in the case of dilatonic ghost condensate model (see, e.g., Fig. 4 in Ref. [12]). In this case the behaviour was found in the case of dilatonic ghost condensate model. We note that the potential is normalised by $H_0^2 m_{pl}^2$, where $m_{pl}$ is the Planck mass.

c$e^{\lambda \phi}$. From Eqs. (10) and (11) we obtain

$$\phi^2 = \frac{12r - 3x'r - 3\Omega_{0m}x^3 I}{r x^2}, \quad (35)$$

$$h(\phi) = \frac{2(2x^2 - 6r + r x^2 \phi^2)}{H_0^2 r^2 x^4 y^4}. \quad (36)$$

In Fig. 2 we plot $h(\phi)$ in terms of the function $\phi$ when we use the best-fit values of $A_1$ and $A_2$. The crossing of the cosmological-constant boundary corresponds to $hX = 1/2$, which occurs around the redshift $z = 0.24$ for the best-fit parametrisation. The system can enter the phantom region ($hX < 1/2$) without discontinuous behaviour of $h$ and $X$.

However we have to caution that the perturbation of the field $\phi$ is plagued by a quantum instability whenever it behaves as a phantom [12]. Even at the classical level the perturbation is unstable for $1/6 < hX < 1/2$, since the speed of sound, $c_s^2 = p_X/(p_X + 2X p_{XX})$, becomes negative. One may avoid this instability if the phantom behaviour is just transient. In fact transient phantom behavior was found in the case of dilatonic ghost condensate model (see, e.g., Fig. 4 in Ref. [12]). In this case the cosmological-constant boundary crossing occurs again in future, after which the perturbations become stable.

We found that the function $h(\phi)$ can be approximated by an exponential function $e^{\lambda \phi}$ near to the present, although some difference appears for $z \gtrsim 0.2$. However the current observational dataset is not still sufficient to rule out the dilatonic ghost condensate model. We hope that future high-precision observations will determine the functional form of $h(\phi)$ more accurately.

**C. Scaling solutions**

As we already mentioned, the existence of scaling solutions can be found by evaluating the r.h.s. of Eq. (27). When $Q = 0$ we checked that the r.h.s. of Eq. (27) is not constant when we use the Gold dataset with the parametrisation [23], which is consistent with the fact that scaling solutions do not lead to an accelerated expansion for $Q = 0$.

In the presence of the coupling $Q$ if the solution around $0 < z < 1$ corresponds to a scaling solution, the r.h.s. of Eq. (27) is constant. This gives the constraint for the evolution of $I$:

$$I(z) = \frac{r(r_0' - 3\Omega_{0m}) - x r'(1 - \Omega_{0m})}{\Omega_{0m}(r_0' - 3)x^3}, \quad (37)$$

where $I(0) = 1$. The r.h.s. of this equation is independent of the scalar-field Lagrangian.

If both the evolution of $r(z)$ and $\delta_m(z)$ are known, one can determine $I(z)$ by Eq. (18). Then by comparing this with Eq. (37), one can check the existence of scaling solutions. It was shown in Ref. [50] that in the case of a phantom field the final attractor does not correspond to scaling solutions but to scalar-field dominant solutions.
with $\Omega_\phi = 1$. Hence we have to caution that Eq. \ref{eq:37} can be used for the region characterised by Eq. \ref{eq:31}.

V. RELATIONSHIP WITH GENERALISED EINSTEIN THEORIES

Having considered the coupling $Q$ between dark energy and dark matter, we would like to relate this scenario with theories giving rise to such a coupling. Let us consider the following 4-dimensional Lagrangian density with a scalar field $\phi$ and a barotropic perfect fluid:

$$\mathcal{L} = \frac{1}{2} F(\phi) \dot{R} - \frac{1}{2} \zeta(\phi) (\nabla \phi)^2 - U(\phi) - \tilde{\mathcal{L}}_m, \quad (38)$$

where $F(\phi)$, $\zeta(\phi)$ and $U(\phi)$ are the functions in terms of $\phi$. This includes a wide variety of gravity models—such as Brans-Dicke theories, non-minimally coupled scalar fields and dilaton gravity\(^3\).

Making a conformal transformation $g_{\mu\nu} = F(\phi) \tilde{g}_{\mu\nu}$, the above action reduces to that of the Einstein frame\(^5\):

$$\mathcal{L} = \frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) - \mathcal{L}_m(\phi), \quad (39)$$

where

$$\phi \equiv \int G(\phi) d\varphi, \quad G(\phi) \equiv \sqrt{\frac{3}{2} \left( \frac{F_\phi}{F} \right)^2 + \zeta}. \quad (40)$$

We note that several quantities in Einstein frame are related with those in string frame via $a = \sqrt{F} \tilde{a}$, $dt = \sqrt{F} \tilde{a} \tilde{d}t$, $\rho_m = \tilde{\rho}_m/F^2$, $p_m = \tilde{\rho}_m/F^2$ and $V = U/F^2$. In Einstein frame the background equations are given by Eqs. \ref{eq:2} and \ref{eq:4} with $p = X - V(\phi)$ and

$$\dot{\tilde{\rho}}_m + 3H(\rho_m + p_m) = - \frac{F_\phi}{2FG}(\rho_m - 3p_m) \dot{\phi}. \quad (41)$$

In the case of non-relativistic dark matter ($p_m = 0$) this corresponds to Eq. \ref{eq:41} with a coupling

$$Q(\phi) = - \frac{F_\phi}{2F} \left[ \frac{3}{2} \left( \frac{F_\phi}{F} \right)^2 + \frac{\zeta}{F} \right]^{-1/2}. \quad (42)$$

For example a nonminimally coupled scalar field with a coupling $\xi$ corresponds to $F(\phi) = 1 - \xi \phi^2$ and $\zeta(\phi) = 1$. In this case we find

$$Q(\phi) = \frac{\xi \phi}{(1 - \xi \phi^2(1 - 6\xi))^{1/2}}. \quad (43)$$

Then $\xi$ is expressed in terms of $Q$:

$$\xi = \frac{Q^2}{2(1 - 6Q^2)} \left[-1 \pm \sqrt{1 + \frac{4(1 - 6Q^2)}{Q^2 \phi^2}} \right]. \quad (44)$$

Once we know $r(z)$ and $\delta_m(z)$, the coupling $\xi$ is evaluated by using Eqs. \ref{eq:14} and \ref{eq:41} together with the relation \ref{eq:10} between $\phi$ and $\varphi$. Hence it is possible to determine the strength of the nonminimal coupling from observations. We note that $Q \simeq \xi \varphi$ for $|\xi| \ll 1$ and $Q \simeq \pm 1/6$ in the large-coupling limit ($|\xi| \gg 1$). The large-coupling case is excluded from solar system experiments provided that the scalar field is universally coupled to all matter\(^24\),\(^25\). It is certainly of interest to place constraints on the strength of $\xi$ by using our reconstruction formula together with other experiments about the time-variation of a gravitational “constant” $G$.

String theory also gives rise to the coupling $Q$ after a conformal transformation from string frame to Einstein frame. The tree-level dilaton gravity\(^53\) corresponds to $F(\varphi) \sim e^{-\varphi}$ and $\zeta(\varphi) \sim -e^{-\varphi}$, which gives a constant value of $Q$ by Eq. \ref{eq:12}. It is typically assumed that non-perturbative effects would stabilize the dilaton with a potential so that it does not contradict with solar system experiments. An alternative possibility is the runaway dilaton scenario\(^54\) in which the dilaton is effectively decoupled from gravity in the limit $\varphi \rightarrow \infty$ with the dependence $F(\varphi) \sim B_1 + C_1 e^{-\varphi}$ and $\zeta(\varphi) \sim B_2 + C_2 e^{-\varphi}$. One can also check the viability of this scenario by comparing Eq. \ref{eq:12} with the coupling $Q$ obtained from observations.

There is another interesting cosmological scenario in which neutrinos are coupled to dark energy\(^55\). In this model the neutrino mass, $m_\nu$, is a function of a scalar field, $\phi$. In a situation where neutrinos are collisionless, the neutrino energy density, $\rho_\nu$, satisfies the equation of motion\(^56\)

$$\dot{\rho}_\nu + 3H (\rho_\nu + p_\nu) = - \frac{\partial \ln m_\nu}{\partial \phi} (p_\nu - 3p_\nu) \dot{\phi}, \quad (45)$$

where $p_\nu$ is the pressure density of neutrinos. When the neutrinos become non-relativistic, Eq. \ref{eq:15} shows that the coupling between neutrinos and dark energy is given by $Q(\phi) = \partial \ln m_\nu/\partial \phi$. Hence if we know the coupling $Q(\phi)$ observationally, the evolution of the neutrino mass is found as a function of $\phi$ (and $z$).

VI. CONCLUSIONS

In this paper we have provided a method to reconstruct scalar-field Lagrangian in an accelerating universe from observations. Our starting point is the general Lagrangian\(^1\) which is the function of a scalar-field $\phi$ and a kinematic term $X$. We have also taken into account the coupling $Q(\phi)$ between dark energy and a non-relativistic perfect fluid in order to include the coupled quintessence scenario.
In the absence of the coupling $Q$, one can reconstruct the structure of theory by parametrising the Hubble rate $H$ in terms of the redshift $z$ from the luminosity distance of supernovae observational data. We need to know additional information in order to determine the coupling $Q$ from observations. We have made use of the equation for matter density perturbations $\delta_m$ on sub-Hubble scales for this purpose. Our reconstruction formula is given by Eqs. (10), (14) and (15), from which the coupling $Q$ is also determined with the use of Eq. (17). We note that the equation of state (19) for dark energy is known from observables only without specifying any Lagrangian.

In Sec. III we have applied our reconstruction formula for several forms of Lagrangian density: (i) $p = f(X) - V(\phi)$, (ii) $p = f(X)V(\phi)$ and (iii) $p = X g(Xe^{\lambda\phi})$ where $g$ is an arbitrary function. In the cases (i) and (ii) reconstruction equations can be decomposed into two contributions coming from a kinematic term and a potential term. Hence the potentials of such theories can be obtained together with the coupling $Q$ once the evolution of $H(z)$ and $\delta_m(z)$ is known. The case (iii) corresponds to the Lagrangian density for the existence of scaling solutions. One can check the existence of scaling solutions if we evaluate the r.h.s. of Eq. (27) using observational data.

In Sec. IV we have presented concrete examples of our reconstruction with the parametrization given by Eq. (29). We studied two dark energy scenarios in the absence of the coupling

$$w > -1,$$

which means that the prior (31) needs to be imposed. In Fig. 1 we plotted one example for the reconstruction of the tachyon potential. The model (a) allows a possibility to cross the cosmological-constant boundary: $w = -1$. We carried out the reconstruction of this model by using the best-fit values coming from the Gold dataset. The result is illustrated in Fig. 2 which shows that the field behaves as a phantom for the redshift $0 < z < 0.24$.

In Sec. V we presented a Lagrangian in generalised Einstein theories which gives rise to the coupling $Q$ by a conformal transformation to Einstein frame. For example, nonminimal coupling $\xi$ is directly related with $Q$ as given in Eq. (11). This allows a possibility to determine the strength of the nonminimal coupling from observational data of $H(z)$ and $\delta_m(z)$.

For the moment we have not yet obtained the accurate evolution of $\delta_m(z)$ from observations of clustering. We only know the total amount of growth between the decoupling epoch and present. This is associated with the fact that all probes of clustering are plagued by a bias problem. However upcoming galaxy surveys such as KAOS, LSST and PANSTARS will pin down the matter power spectrum to exquisite accuracy, allowing the ultimate measurement of the power spectrum. By that time we should have an excellent understanding of bias and will be able to obtain the time-evolution of $\delta_m$. We hope that this will provide us an exciting possibility to reveal the origin of dark energy.

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