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LORENTZ CONTRACTION AND DIFFRACTIVE EXCITATION

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A B S T R A C T

The formalism of light-plane wave functions is applied to the problem of diffractive excitation. Suggestions in the literature, for removing an experimentally embarrassing forward zero in the single scattering approximation to Glauber theory, are not borne out.

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1. - INTRODUCTION

It has been proposed by Parry ¹⁾ that Lorentz contraction be introduced into the non-relativistic Glauber theory of diffractive excitation. Specifically, he finds that, by differently contracting initial and final projectile states, an experimentally embarrassing zero is removed from the forward excitation amplitude. Czyz and Maximon ²⁾ have suggested that this effect may be important in justifying Van Hove's treatment ³⁾ of the propagation of complex hadronic systems in nuclear matter ⁴⁾. Does some such effect arise in a fully relativistic account? Such an account has been initiated ⁵⁾ with the Bethe-Salpeter approach. Here the problem is considered in the light-plane formalism ^{6),*)}. The effect suggested by Parry is found to be absent. The formulae can indeed be made to resemble his, or to resemble alternative formulae adopted by Le Yaouanc, Oliver, Pène and Raynal ⁷⁾, but in each case relativistic corrections to orthogonality relations change the result and restore the unwelcome zero.

In Sections 2 and 3 are set out in parallel the non-relativistic and covariant light-plane developments for the problem of Born approximation scattering of a composite system on an external potential. Given a suitable energy dependent pseudo-potential, diffractive scattering can be simulated in this way. A more direct approach is sketched in Section 4. It is hoped that the light-plane formalism is introduced sufficiently gently to be intelligible, even if not quite convincing, to some who have not already studied it.

*) The paper of Dirac seems to be the earliest reference on this subject. The more recent development is described and documented in the two review papers listed in Ref. 6).

2. - NON-RELATIVISTIC

The question at issue arises in the single scattering approximation to Glauber theory, and also in Born approximation. To minimize inessential complications consider then Born approximation for scattering in an external field. Suppose we have a system of several particles incident on a potential which, for simplicity, acts only on particle 1. It will be convenient to regard this potential as a superposition of instantaneous pseudopotentials of short range :

$$\int d\vec{x}_0 dt_0 V(\vec{x}_0, t_0) \delta(\vec{x}_1 - \vec{x}_0) \delta(t - t_0)$$

Consider scattering from a state of total energy and momentum $(K_0, \vec{K})$ into one with $(K'_0, \vec{K}')$. By the space-time displacement properties of these states the (x_0, t_0) integration just gives a factor

$$\int d\vec{x} dt V(\vec{x}, t) e^{i(\vec{K} - \vec{K}') \cdot \vec{x} - i(K_0 - K'_0)t} \quad (1)$$

So the problem reduces to that of the pseudopotential

$$\delta(\vec{x}_1) \delta(t) \quad (2)$$

The $\delta(t)$ simply means that no time integration is to be performed. We will work in momentum space, where $\delta(\vec{x}_1)$ has matrix elements

$$P = (2\pi)^3 \delta(\vec{k}_2 - \vec{k}'_2) (2\pi)^3 \delta(\vec{k}_3 - \vec{k}'_3) \dots \quad (3)$$

- i.e., the unit matrix with respect to all particles other than 1.

Let the initial state be written, in momentum space

$$\Psi_{n\vec{k}}(t, \vec{k}_1, \vec{k}_2, \dots) = e^{-iK_0 t} (2\pi)^3 \delta(\vec{K} - \vec{k}_1 - \vec{k}_2 \dots) \Psi_n(\vec{k}_1, \vec{k}_2, \dots) \quad (4)$$

and likewise the final state with

$$K_0 \rightarrow K'_0 \quad \vec{K} \rightarrow \vec{K}' \quad n \rightarrow m$$

Then the desired matrix element of (3) is

$$\int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}'_2}{(2\pi)^3} \dots \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \dots I \quad (5)$$

with

$$I = \Psi_{m\vec{k}'}^*(0, \vec{k}_1, \vec{k}'_2 \dots) P \Psi_{n\vec{k}}(0, \vec{k}_1, \vec{k}_2 \dots) \quad (6)$$

which gives

$$\int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \dots (2\pi)^3 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 \dots) \Psi_m^*(\vec{k}_1 + \vec{k}'_2 - \vec{k}, k_2 \dots) \Psi_n(\vec{k}_1, \vec{k}_2 \dots) \quad (7)$$

Now if the potential V is static, the integral (1) contains an energy conserving δ function. Then we have

$$\vec{k}^2 - \vec{k}'^2 = M_m^2 - M_n^2 \quad (8)$$

with relativistic kinematics, or (more consistently) *)

$$\vec{k}^2 - \vec{k}'^2 = 2(M_1 + M_2 + \dots)(M_m - M_n) \quad (9)$$

with non-relativistic kinematics. Either way

$$|\vec{k}| - |\vec{k}'| \rightarrow 0 \quad (10)$$

at large energy. Then in the case of forward scattering (7) becomes

$$\int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \dots (2\pi)^3 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 \dots) \Psi_m^*(\vec{k}_1, \vec{k}_2 \dots) \Psi_n(\vec{k}_1, \vec{k}_2 \dots) \quad (11)$$

*) In this formula the M 's are internal energies in the non-relativistic sense.

The internal wave functions are Galilei-invariant :

$$\psi_n(\vec{k}_1, \vec{k}_2 \dots) = \psi_n(\vec{k}_1 + m_1 \vec{v}, \vec{k}_2 + m_2 \vec{v} \dots) \quad (12)$$

Then the integral (11) is in fact $\vec{K}$ independent, and we may put $\vec{K} = 0$. However, it is then seen to be zero, for $n \neq m$, by the orthogonality property of internal wave functions :

$$\delta_{nm} = \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \dots (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 \dots) \psi_m^*(\vec{k}_1, \vec{k}_2 \dots) \psi_n(\vec{k}_1, \vec{k}_2 \dots) \quad (13)$$

This is the unwanted zero.

To adapt these non-relativistic formulae to a relativistic situation, Parry tentatively proposed the incorporation of Lorentz contractions in the configuration space wave functions, of which the above ψ 's are the Fourier transforms. The result is to substitute for all longitudinal components of the momenta appearing in ψ_n and ψ_m^* in (13)

$$\left. \begin{aligned} k_z &\rightarrow \frac{M_n}{E} k_z && \text{in } \psi_n \\ k_z &\rightarrow \frac{M_m}{E} k_z && \text{in } \psi_m^* \end{aligned} \right\} \quad (14)$$

with appropriate over-all renormalization factors. Parry's point is that the masses of initial and final systems being different, and their energies equal, the Lorentz contractions are different - and the orthogonality lifted.

Another direction is taken by Byers and Frautschi⁶⁾ and by Le Yaouanc, Oliver, Pène and Raynal⁷⁾. They argue that a frame of reference should be used in which the composite system moves slowly, both initially and finally, so that non-relativistic description may be appropriate. Such systems can be found if mass change and scattering angle are small. The target potential V if static in the lab. system, will not be static in these projectile systems. Rather it will depend only on x, y and $z + vt$, where $-V$ is the velocity of the target in the chosen system. Then (1) will give conservation, not of energy, but of a certain combination of energy and momentum :

$$K_0 + v K_z = K_0' + v K_z' \quad (15)$$

or approximately

$$K_z - K'_z = (M_m - M_n)/v \quad (16)$$

This does not vanish for $v \rightarrow 1$. In this approach it is this non-vanishing momentum transfer which prevents (7) from vanishing in the forward direction.

3. - LIGHT-PLANE

In (4) the state of a system extended in space is specified at given time t . The basic problem of Lorentz transformation in such a formalism is that space time surfaces of constant t transform into surfaces on which t is not constant. In the light plane formalism one takes instead surfaces of constant

$$\tau = (t + z)/\sqrt{2} \quad (17)$$

with varying

$$x, y, \zeta = (z - t)/\sqrt{2} \quad (18)$$

The Lorentz transformation

$$x \rightarrow x, \quad y \rightarrow y, \quad z \rightarrow \frac{z + vt}{\sqrt{1-v^2}}, \quad t \rightarrow \frac{t + vz}{\sqrt{1-v^2}} \quad (19)$$

becomes

$$x \rightarrow x, \quad y \rightarrow y, \quad \zeta \rightarrow \lambda^{-1} \zeta, \quad \tau \rightarrow \lambda \tau \quad (20)$$

with

$$\lambda = \sqrt{1+v} / \sqrt{1-v} \quad (21)$$

The surfaces $\tau = \text{constant}$ map into one another ; as a result the description of these particular Lorentz transformations becomes rather simple in the subsequent formalism. Conversely the description of some ordinary rotations and reflections, because they mix τ with other variables, is greatly obscured.

The free particle plane wave

$$e^{i\vec{k} \cdot \vec{x} - i k_0 t} \quad (22)$$

with

$$k_0 = \sqrt{m^2 + \vec{k}^2}$$

can be rewritten

$$e^{i\vec{h} \cdot \vec{\zeta} - i\omega\tau} \quad (23)$$

with

$$\begin{aligned} \vec{\zeta} &\equiv (x, y, z) \\ \vec{h} &\equiv (k_x, k_y, h) \quad h = (k_x + k_0)/\sqrt{2} \\ \omega &= \frac{k_0 - k_z}{\sqrt{2}} = \frac{m^2 + k_x^2 + k_y^2}{2h} \end{aligned} \quad (24)$$

Note that ω and h are both positive. Boosting in the z direction gives

$$h \rightarrow \lambda h \quad \omega \rightarrow \lambda' \omega \quad (25)$$

The wave packet [with $\psi(t, \vec{k}) = \psi(0, \vec{k}) \exp(-ik_0 t)$]

$$\int \frac{d\vec{k}}{(2\pi)^3 2k_0} \psi(t, \vec{k}) e^{i\vec{k} \cdot \vec{x}} \quad (26)$$

[interpreting each plane wave (22) as meaning particle probability $2k_0$ per unit volume at given time] has norm

$$\int \frac{d\vec{k}}{(2\pi)^3 2k_0} |\psi(t, \vec{k})|^2 \quad (27)$$

Writing the same wave packet as [with $\phi(\tau, h) = \phi(0, h) \exp(-i \omega \tau)$]

$$\int \frac{d\vec{h}}{(2\pi)^3 2h} \phi(\tau, \vec{h}) e^{i\vec{h} \cdot \vec{\xi}} \quad (28)$$

the same norm is

$$\int \frac{d\vec{h}}{(2\pi)^3 2h} |\phi(\tau, \vec{h})|^2 \quad (29)$$

Note that under z boosting

$$\phi(h_x, h_y, h) \rightarrow \phi(h_x, h_y, \lambda' h) \quad (30)$$

By considering a product of such free particle wave packets, at equal τ , we can then guess the following about the description of many particle interacting systems. More serious developments can be found in and through excellent reviews ⁶⁾.

The many-body system is described by a wave function

$$\Phi(\tau, \vec{h}_1, \vec{h}_2, \vec{h}_3, \dots) \quad (31)$$

which can be boosted in the z direction simply by making the substitution

$$\tau \rightarrow \lambda' \tau \quad h_n \rightarrow \lambda' h_n \quad (32)$$

(where h_n is the third component of $\vec{h}_n$). The norm is

$$\int \frac{d\vec{h}_1}{(2\pi)^3 2h_1} \frac{d\vec{h}_2}{(2\pi)^3 2h_2} \dots |\Phi(\tau, \vec{h}_1, \vec{h}_2, \dots)|^2 \quad (33)$$

which is manifestly invariant under z boosting. In analogy with (4) there are eigenstates of $\vec{\xi}$ and τ displacements

$$\Phi_{n\vec{h}} = e^{i\omega\tau} (2\pi)^3 2h \delta(\vec{h} - \vec{h}_1 - \vec{h}_2 - \dots) \phi_n(\vec{h}_1, \vec{h}_2, \dots) \quad (34)$$

with internal states invariant under z boosting

$$\phi_n(\vec{h}_1, \vec{h}_2, \dots) = \phi_n(\vec{h}'_1, \vec{h}'_2, \dots) \quad \text{with} \quad \vec{h}'_n = (h_{nx}, h_{ny}, \lambda' h_n) \quad (35)$$

This replaces (12). The requirement that arbitrary wave packets of states (34) have constant norm yields the orthogonality relation

$$\int \frac{d\vec{h}_1}{(2\pi)^3 2h_1} \frac{d\vec{h}_2}{(2\pi)^3 2h_2} \dots (2\pi)^3 2h \delta(\vec{h} - \vec{h}_1 - \vec{h}_2 \dots) \Phi_m^* \Phi_n = \delta_{mn} \quad (36)$$

This replaces (13).

The Schrödinger equation of the theory is

$$\frac{\partial \Phi}{\partial \tau} = -i (\Omega_0 + \Omega_I) \Phi \quad (37)$$

where

$$\Omega_0 = \sum_n \frac{m_n^2 + h_{nx}^2 + h_{ny}^2}{2h_n} \quad (38)$$

and Ω_I describes interparticle interactions (which must transform like Ω_0 under z boosts) and interactions with external fields. Scattering theory can be developed from the Schrödinger equation following closely the non-relativistic pattern.

Consider again the scattering of a bound system on an external field which acts only on particle 1. In Born approximation, the displacement properties of states (34) again allow the separation of a factor (1). It remains only to deal with a short-range pseudopotential. We replace (3) by

$$P = (h_1 + h'_1) (2\pi)^3 2h_2 \delta(\vec{h}_2 - \vec{h}'_2) (2\pi)^3 2h_3 \delta(\vec{h}_3 - \vec{h}'_3) \dots \quad (39)$$

The factors referring to unscattered particles 2, 3, etc., are just the appropriate forms of the unit operator; the appearance of factors h_n is associated with the use below of the integration element $(d\vec{h}_n/2h_n)$. The extra factor for particle 1 is inserted to match, in the case of rapid motion in the positive z direction, the energy dependence of diffraction scattering. In the case of forward scattering of a free single particle it gives a factor

$$2h = 2(k_0 + k_z) \approx 4k_0 \quad (40)$$

for large positive k_z . For the composite system we then have the remaining factor in the scattering amplitude, in analogy with (5), (6) :

$$\int \frac{d\vec{h}'_1}{(2\pi)^3 2h'_1} \frac{d\vec{h}'_2}{(2\pi)^3 2h'_2} \dots \int \frac{d\vec{h}_1}{(2\pi)^3 2h_1} \frac{d\vec{h}_2}{(2\pi)^3 2h_2} \dots I \quad (41)$$

$$I = \Phi_{m\vec{h}}^* (0, \vec{h}'_1, \vec{h}'_2 \dots) P \Phi_{n\vec{h}} (0, \vec{h}_1, \vec{h}_2 \dots) \quad (42)$$

which gives on evaluation

$$\int \frac{d\vec{h}_1}{(2\pi)^3 2h_1} \frac{d\vec{h}_2}{(2\pi)^3 2h_2} \dots (2\pi)^3 2h \delta(\vec{h} - \vec{h}_1 - \vec{h}_2 \dots) \\ (2h'/2h'_1)(h_1 + h'_1) \Phi_m^* (\vec{h}'_1, \vec{h}_2 \dots) \Phi_n (\vec{h}_1, \vec{h}_2 \dots) \quad (43)$$

[which replaces (7)] where

$$\vec{h}'_1 = \vec{h}_1 + \vec{h}' - \vec{h} \quad (44)$$

When the target potential is static we again have projectile energy conservation, and at high energy also momentum conservation (10). So in the case of forward scattering

$$h' - h = (K'_z + K'_0) - (K_z + K_0) \rightarrow 0 \quad (45)$$

asymptotically. Then (43) reduces to

$$2h \int \frac{d\vec{h}_1}{(2\pi)^3 2h_1} \frac{d\vec{h}_2}{(2\pi)^3 2h_2} \dots (2\pi)^3 2h \delta(\vec{h} - \vec{h}_1 - \vec{h}_2 \dots) \Phi_m^* \Phi_n \quad (46)$$

which is just a multiple of the orthogonality integral (36). Again we have zero for $n \neq m$.

To bring out the difference between (46) and non-relativistic formulae, let us reduce it to essentials. As well as leaving out trivial factors, omit now the transverse degrees of freedom. Consider moreover just two constituents, and substitute

$$\phi_n(h_1, h_2) \Big|_{h_1+h_2=h} \rightarrow \phi_n\left(\frac{h_1}{h}\right) \quad (47)$$

where the form (h_1/h) of the argument is consistent with boost invariance (35). Then (46) reduces to, after performing the h_2 integration,

$$\int \frac{dh_1}{h_1} \left(1 - \frac{h_1}{h}\right)^{-1} \phi_m^*\left(\frac{h_1}{h}\right) \phi_n\left(\frac{h_1}{h}\right) \quad (48)$$

Something like Parry's effect can be seen here by writing, for large h ,

$$h = (K_0 + K_z) \approx 2K_0 = 2M_n \gamma_n = 2M_m \gamma_m \quad (49)$$

where

$$\gamma_n = K_0/M_n \quad \gamma_m = K_0/M_m = K'_0/M_m \quad (50)$$

are Lorentz contraction factors. The wave functions appearing in (48) are

$$\phi_m\left(\frac{h_1}{2M_m \gamma_m}\right) \quad \phi_n\left(\frac{h_1}{2M_n \gamma_n}\right) \quad (51)$$

If the wave functions

$$\phi_m\left(\frac{h_1}{2M_m}\right) \quad \phi_n\left(\frac{h_1}{2M_n}\right) \quad (52)$$

are thought of as "uncontracted", then those in (51) have suffered differential Lorentz contraction. This would destroy orthogonality - if the "uncontracted" wave functions were indeed orthogonal. But the relativistically correct orthogonality integral is (48), and involves precisely the "differentially contracted" wave functions.

To rewrite (48) to resemble the prescription of Le Yaouanc et al., introduce "rapidity" variables :

$$\left. \begin{aligned} h_i &= m_i e^{\omega} \\ h &= M_n e^{\Omega_n} = M_m e^{\Omega_m} \end{aligned} \right\} \quad (53)$$

$$\Omega_m - \Omega_n = \log \frac{M_n}{M_m} \approx \frac{M_n - M_m}{M_n} \quad (54)$$

With

$$\phi_n \left(\frac{m_i}{M_n} e^{\omega - \Omega_n} \right) = \psi_n(\omega - \Omega_n) \quad (55)$$

and likewise with $n \rightarrow m$, (48) becomes

$$\int d\omega \left(1 - \frac{m_i}{M_n} e^{\omega - \Omega_n} \right)^{-1} \psi_m^*(\omega - \Omega_m) \psi_n(\omega - \Omega_n) \quad (56)$$

The first factor in the integrand would be essentially constant for a weakly bound system with small relative rapidities : let us ignore it for a moment. Then thinking of the ψ 's as non-relativistic wave functions and of ω and Ω as velocities (not momenta) (56) is indeed essentially the prescription of Le Yaouanc et al. The orthogonality of the wave functions would be lifted by the differential displacement of the arguments - if indeed they were otherwise orthogonal. Actually, however, when

$$\Omega_n - \Omega_m \approx \frac{M_n - M_m}{M_n}$$

is regarded as significantly different from zero, one must both restore the initial factor in the integrand of (56) and note that it is precisely the differentially displaced wave functions which appear in this relativistically correct orthogonality integral.

4. - DIFFRACTION

Note finally an alternative derivation of (46) in which the pseudopotential P appears in a less ad hoc way. Suppose the composite system moves rapidly in the positive z direction, and that constituent 1 encounters a static energy independent refractor-absorber. In a wave packet description, suppose that at high energy passage through the target changes the initial wave packet simply by a factor

$$1 - \Gamma(\vec{x}_{1\perp}) \quad (57)$$

where $\vec{x}_1$ is the transverse position (conjugate to $\vec{h}_1$) of particle 1. The transition amplitude is then

$$(\Phi_{m\vec{h}}, \Gamma(\vec{x}_{1\perp}) \Phi_{n\vec{h}}) \quad (58)$$

In $\vec{h}$ space Γ is equivalent to the integration kernel

$$\tilde{\Gamma}(\vec{h}_{1\perp} - \vec{h}'_{1\perp}) 2\pi 2h_1 \delta(h_1 - h'_1) (2\pi)^3 2h_2 \delta(\vec{h}_2 - \vec{h}'_2) \dots \quad (59)$$

The first factor is the transverse Fourier transform of Γ . The second represents the longitudinal unit operator for particle 1; Γ does not affect longitudinal motion. Then come unit operators for other particles. Taking account of the delta functions in both (59) and (34), (59) is equivalent to

$$\{ \tilde{\Gamma}(\vec{h}'_1 - \vec{h}_1) 2\pi \delta(h - h') \} (h_1 + h'_1) (2\pi)^3 2h_2 \delta(\vec{h}'_2 - \vec{h}_2) \dots \quad (60)$$

The first two factors here correspond to those separated out in (1). That we get longitudinal h conservation rather than energy conservation shows that the present argument can be valid only for small angles and high energy of incidence in the positive z direction. The remaining factors in (60) are just the P of (39), and the calculation proceeds as before.

In this approach it is almost trivial that there is no excitation of the forward scattered system. For the forward scattering is insensitive to the detailed form of Γ , depending only on its integral $\tilde{\Gamma}(0)$. So we can consider Γ to be thinly and widely distributed, so that

$$\tilde{\Gamma}(\vec{h}_{1\perp} - \vec{h}'_{1\perp}) \rightarrow \delta(\vec{h}_{1\perp} - \vec{h}'_{1\perp})$$

in (59). But then (59) is just the unit operator in all variables, so that the scattered state is just a multiple of the initial one.

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APPENDIX

In passing from (43) to (46) it has been casually assumed that the limit of the integral is the integral of the limit. To illustrate the problem consider again the simple case of two constituents without transverse motion. Using (47), (43) becomes

$$(h+h') \int \frac{dh_1}{(h-h_1)} \frac{hh'}{h_1h'_1} \frac{h_1+h'_1}{h+h'} \phi_m^*\left(\frac{h'_1}{h'_1}\right) \phi_n\left(\frac{h_1}{h}\right) \quad (\text{A.1})$$

with

$$h-h' = h_1-h'_1 \approx \frac{M_m^2 - M_n^2}{2h}$$

We are interested in the limit of the coefficient of $(h+h')$ in (A.1) as $h \rightarrow \infty$. With

$$h_1 = \beta h$$

this coefficient may be written

$$\int_0^1 \frac{d\beta_1}{1-\beta_1} \frac{1}{\beta} \frac{1-\delta}{\beta-\delta} \frac{2\beta-\delta}{2-\delta} \phi_m^*\left(\frac{\beta-\delta}{1-\delta}\right) \phi_n(\beta) \quad (\text{A.2})$$

with

$$\delta \approx \frac{M_m^2 - M_n^2}{2h^2}$$

The integration limits are dictated by the requirement that the arguments of the wave functions lie between zero and unity. Now in (A.2), one hesitates to take the limit simply by setting $\delta = 0$ mainly because of the factors which are singular on the boundaries. But such singular factors occur also in normalization integrals

$$\int_0^1 \frac{d\beta}{1-\beta} \frac{1}{\beta} |\phi(\beta)|^2$$

So we expect $|\phi|$ to vanish on the boundaries, at least for the discrete states. Assume that $|\phi|$ vanishes as some positive power

$$\begin{aligned} |\phi| &< c \beta^\epsilon \quad \text{near } \beta = 0 \\ &< c' (1-\beta)^\epsilon \quad \text{near } \beta = 1 \end{aligned}$$

Then if I is the integrand in (A.2)

$$\left| \int_{\delta}^{\Delta} d\beta I(\beta, \delta) \right| < C'' \Delta^{2\epsilon}$$

independently of δ , for small δ . This can be made arbitrary small by reducing Δ . Similar reasoning applies near $\beta = 1$. Then if we assume for the restricted integral

$$\lim_{\delta \rightarrow 0} \int_{\Delta}^{1-\Delta} d\beta I(\beta, \delta) = \int_{\Delta}^{1-\Delta} d\beta I(\beta, 0)$$

we have also, finally letting $\Delta \rightarrow 0$,

$$\lim_{\delta \rightarrow 0} \int_{\delta}^1 d\beta I(\beta, \delta) = \int_0^1 d\beta I(\beta, 0)$$

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