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TOWARDS SATISFACTORY SCATTERING AMPLITUDES FOR DUAL FERMIONS

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A B S T R A C T

We find the modified Lorentz invariant propagator which, when used on Neveu-Schwarz meson lines joining fermion lines, guarantees the coupling of just the ghost-free transverse spectrum of meson states. The modification is necessary to take account of the new form of "reflected" Ward identity valid for such lines. Thus we can write down dual amplitudes for processes involving four external fermions which possess a high degree of inner consistency - no-ghost theorems and Lorentz invariance - but we have not yet been able to evaluate them.

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1. - INTRODUCTION

One of the outstanding problems in dual resonance theory is the construction of consistent amplitudes involving mesons and at least four external fermions. By consistent we mean that the spectrum in all channels should be ghost free, and that the meson spectrum should be the same whether it couples to purely meson states or states with additional fermion-anti-fermion combinations. The problem is difficult mathematically and progress has been slow but nevertheless definite, and in this paper we wish to describe another step towards the above-mentioned goal, the desirability and interest of which has been particularly emphasized by Schwarz ¹⁾.

The successive steps in the development of the dual fermion theory have been :

- 1) the construction of a generalized Dirac equation for the propagation of free dual fermions ²⁾ ;
- 2) the incorporation of interaction via the construction of a vertex for the emission of pseudoscalar mesons which respects the gauge symmetries of the free fermion propagation ^{3),4)} ;
- 3) the construction of a "good" vertex for fermion emission which is likely to transmute fermionic and mesonic gauges ^{4),5)} ;
- 4) the establishment of all the Ward identities for the vertex ^{5),6),7)}.

On the basis of these Ward identities, it was shown that at least in certain channels the spectrum of amplitudes involving two external fermions are indeed consistent in the sense mentioned above, providing the ground state fermion is massless ⁷⁾.

Now the way is open to the construction of amplitudes with two fermion lines, for example that illustrated in Fig. 1 (wavy lines are mesons, solid lines fermions)

- Figure 1 -

This would enable us to study new sorts of duality which would hopefully tell us what sort of fermion we have - a baryon, a quark or whatever.

We know all the vertices. The only unknown is the propagator on the internal meson line indicated. Experience with the Pomeron ⁸⁾ has taught us that it would be wrong to assume that it is the usual mesonic propagator. What should be the same is the spectrum of physical states which actually couple in the residues and this is determined by the combination of the form of the propagator and the form of the Ward identities valid on the line concerned. Since the Ward identities differ on a mesonic line coupled to a fermion-antifermion pair because of the phenomenon whereby the G gauges are reflected ⁷⁾, we can anticipate that the propagator will also differ.

We shall assume that the residues in the indicated meson of the amplitude corresponding to Fig. 1 are saturated by the same transverse states that saturate mesonic residues in amplitudes with one or less fermionic line. Of course this is a physical assumption, justified by economy and the fact that it will guarantee the continued validity of the no-ghost theorem for mesons. The difficulty is that it is necessary to show that the residues constructed in this way are Lorentz invariant - which is not obvious since the transverse states are defined in some special Lorentz frame of reference. Usually the Lorentz invariance is restored by the use of the Ward identities and this is what we shall once again check. However, owing to the reflection behaviour of the G gauges, these G 's are reflected to and fro along the internal meson line indicated an infinite number of times and this new phenomenon gives rise to a Lorentz invariant correction factor in the propagator. This correction factor has the form of a determinant of an infinite dimensional matrix and it is intriguing that it is closely related to another factor already occurring in the above amplitudes. The final obstacle, which we have not overcome, is to evaluate these determinants, or at least understand certain of their properties. This is a well-defined mathematical problem which, we hope, will find a solution.

In the next section we supply the necessary definitions and mathematical statements of our results. Technically the residues are defined by the insertion of the transverse state projection operator ⁹⁾, expressed in terms of L_n 's and G_r 's. To eliminate (or simplify) it we need the Ward identity written with any one single incident G_r - rather than with certain finite linear combinations as in Ref. 7). This is found in Section 3. The result is interesting because the matrix α specifying the reflection

pattern has already been met in the fermion emission vertex $4), 1), 5)$. This matrix is rather complicated and we do not understand why it should turn up twice : we can only ascribe the coincidence to one of those dual theory miracles which probably have a deep significance. Our new form of gauge condition enables us to rederive the saturation of transverse meson states $7)$ when coupled between mesons on one side and mesons plus a fermion-antifermion on the other side.

This insight helps us in Section 4 to compute the effect of inserting the transverse state projection operator $9)$ on the meson line indicated in Fig. 1, and to show that it effectively modifies the propagator in a Lorentz covariant way, as already explained.

In the course of the work we need certain specific properties of the matrix α mentioned above. The most elegant proofs we have found involve contour integral methods applied to the generating function of this matrix, and are described in an Appendix.

2. - PRELIMINARIES

Guided by economy and a desire to maintain the no-ghost theorem, we take as our fundamental assumption that the physical spectrum of Neveu-Schwarz mesons always be the subspace of transverse (ghost-free) states. This space is most conveniently specified by its projection operator $9)$

$$\boxed{\square} = \oint \frac{dy}{4\pi i y} y^E \quad (2.1)$$

$$E = (L_0 - \frac{1}{2})(D_0 - 1) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n) + \sum_{r=\frac{1}{2}}^{\infty} (E_{-r} G_r - G_{-r} E_r) \quad (2.2)$$

where we shall say more presently about the quantities L_n, G_r, D_n, E_r [which are defined in Ref. 9]. The double circle in (2.1) means that the origin is encircled twice. Let $|M\rangle$ and $|F\bar{F}\rangle$ denote respectively the trees of Fig. 2

$$\begin{aligned}
 |M\rangle &= \text{diagram with a horizontal wavy line and three vertical wavy lines above it} \\
 |F\bar{F}\rangle &= \text{diagram with a horizontal wavy line and four vertical wavy lines above it, with an ellipsis between the second and third lines}
 \end{aligned}$$

- Figure 2 -

These describe the coupling of a meson Reggeon to a purely mesonic state in the case of $|M\rangle$ and a state including an additional fermion-antifermion pair in the case of $|F\bar{F}\rangle$. (For technical reasons we have had to exclude twists in the case of the $|F\bar{F}\rangle$ tree.) These states are written down in the "F₂" formalism since this is the only well-defined formalism if the ground state fermion has zero mass ⁷⁾.

The mathematical statement that transverse states factorize the residues of purely mesonic amplitude is

$$\langle M | \mathcal{J} \mathcal{M} | M \rangle = \langle M | \mathcal{m} | M \rangle \quad (2.3)$$

and was proved in Ref. 9).

$$\mathcal{m} = \oint \frac{dx}{4\pi i x} x^{L_0 - \frac{1}{2}} \quad (2.4)$$

is the mass-shell projection operator and also the residue of the usual mesonic propagator

$$\int_0^1 \frac{dx}{x} x^{L_0 - \frac{1}{2}}$$

The same states also factorize certain residues of amplitudes involving a single fermion line :

$$\langle F\bar{F} | \mathcal{J} \mathcal{M} | M \rangle = \langle F\bar{F} | \mathcal{m} | M \rangle \quad (2.5)$$

as was proved in Ref. 7) if the ground state fermion has zero mass. We shall give an alternative proof of this in the next section. The main result of this paper is that when we consider a mesonic line joining two fermion lines, we find

$$\langle F \bar{F} | \mathcal{J} \mathcal{M} | F \bar{F} \rangle = \langle F \bar{F} | \oint \frac{dx}{4\pi i x} \frac{x^{L_0 - \frac{1}{2}}}{\Delta(x)} | F \bar{F} \rangle \quad (2.6)$$

where $1/\Delta(x)$ is a correction factor function to be defined later.

This means :

- 1) the residues of amplitudes involving two fermion lines defined by transverse mesonic states are Lorentz invariant ; since the space of transverse state is not Lorentz invariant this is not obvious, and will be seen to be an extremely non-trivial consequence of the gauge conditions ;
- 2) if we take for the meson propagator joining two fermion lines the Lorentz invariant quantity :

$$\int_0^1 \frac{dx}{x} \frac{x^{L_0 - \frac{1}{2}}}{\Delta(x)} \quad (2.7)$$

it has as residue the integral in (2.6) and hence yields the ghost-free spectrum of meson states specified by $\mathcal{J}$ [Eq. (2.1)].

These saturation equations (2.3), (2.5) and (2.6) are consequences of the structure of $\mathcal{J}$ in terms of L_n and G_r [Eqs. (2.1) and (2.2)] and the different Ward identities satisfied by the states $|M\rangle$ and $|F\bar{F}\rangle$

$$L_n \mathcal{M} |M\rangle = 0 \quad n \geq 1 \quad (2.8)$$

$$G_r \mathcal{M} |M\rangle = 0 \quad r \geq \frac{1}{2} \quad (2.9)$$

$$\langle F \bar{F} | L_{-n} = \langle F \bar{F} | (L_0 + \frac{5(n-1)}{8}) \quad , n \geq 1 \quad (2.10)$$

$$\langle F \bar{F} | \sum_{s=-r}^{\infty} (-1)^{s+r} \binom{-K}{s+r} G_s = 0 \quad r \geq \frac{1}{2} \quad (2.11)$$

The last two equations were proved in Ref. 7) and are valid in this form only if space time has 10 dimensions and the ground state fermions have zero mass - as we shall henceforth assume.

Equations (2.10) and (2.11) are intriguing because they are so different from Eqs. (2.8) and (2.9) - in (2.11) we see a finite linear combination of incoming G gauges (with negative index) being reflected into an infinite linear combination of G 's with positive index, and in (2.10) we see a strange n dependent c number.

Unfortunately (2.11) is not in a convenient form for our present purposes - we should like to have a single incident G gauge in order to simplify expressions (2.5) and (2.6). In the next section we shall achieve this and give a second proof of Eq. (2.5).

3. - MODIFIED FORM OF WARD IDENTITY ¹⁰⁾

We want to rewrite a linear combination of the gauge conditions (2.11) in the form

$$\langle F\bar{F} | G_{-r} = \langle F\bar{F} | \left(\sum_{s=\frac{1}{2}}^{\infty} \alpha_{rs} G_s \right) \quad r \gg \frac{1}{2} \quad (3.1)$$

so that we can deal with each G_{-r} independently.

This would be achieved if we could write

$$G_{-r} = \sum_{s=\frac{1}{2}}^{\infty} \alpha_{rs} G_s + \sum_{s=\frac{1}{2}}^r \beta_{rs} G(x^{-s+\frac{1}{2}} \sqrt{1-\lambda}) \quad (3.2)$$

where

$$G(x^{-s+\frac{1}{2}} \sqrt{1-\lambda}) = i \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} G_{n-s} \quad (3.3)$$

since (3.3) annihilates the state $\langle F\bar{F} |$ on the left by (2.11).

It is obvious that an equation such as (3.2) exists : the problem is to find simple and useful forms for the matrices α and β . Our first result will be that the matrix α has the generating function

$$\sum_{r,s=\frac{1}{2}}^{\infty} x^r \alpha_{rs} y^s = \frac{\sqrt{xy}}{y-x} \left\{ 1 - \sqrt{\frac{1-y}{1-x}} \right\} \quad (3.4)$$

$$= \frac{1}{2} \sqrt{xy} \int_1^{\infty} d\xi (\xi-y)^{-1/2} (\xi-x)^{-3/2} \quad (3.5)$$

This is nice for two reasons :

- (i) it is simple ;
- (ii) the same matrix α enters the fermion emission vertex $(4), (1), (5)$ via the factor $^{11)}$

$$e^{-\frac{1}{2} \sum_{r,s=\frac{1}{2}}^{\infty} b_r \alpha_{rs} b_s} \quad (3.6)$$

At our present level of understanding this is just a coincidence but it probably has a deep significance.

Notice that (3.4) is analytic in x and y for $|x| < 1, |y| < 1$ despite the apparent singularity at $x = y$. If we equate coefficients of x^r on both sides of Eq. (3.4), we see that

$$\sum_{s=\frac{1}{2}}^{\infty} \alpha_{rs} y^s = y^{-r} + i \sqrt{1-y} \sum_{s=\frac{1}{2}}^{\infty} y^{-s+\frac{1}{2}} \binom{-\frac{1}{2}}{r-s} (-1)^{r-s}$$

So multiplying by $\oint (dy/2\pi i y) P(y) \cdot H(y)$ we obtain (3.2) with the additional information that

$$\beta_{rs} = i \binom{-\frac{1}{2}}{r-s} (-1)^{r-s} \quad (3.7)$$

Also expanding (3.5) in powers of x and y and then integrating we find that

$$\alpha_{rs} = \frac{1}{2} \frac{(-1)^{r+s+1}}{r+s} \binom{-3/2}{r-\frac{1}{2}} \binom{-1/2}{s-\frac{1}{2}} = \frac{(-1)^{r+s+1}}{r+s} \binom{-1/2}{r-\frac{1}{2}} \binom{-1/2}{s-\frac{1}{2}} \quad (3.8)$$

We now wish to show how (3.1) and (3.4) can be used to prove Eq. (2.5) which states that transverse meson states saturate certain residues of single fermion amplitudes.

As usual ⁹⁾ we consider

$$\langle F \bar{F} | (y^{E-1}) \mathcal{M} | M \rangle = \int_1^y dy' \langle F \bar{F} | E y'^{E-1} \mathcal{M} | M \rangle \quad (3.9)$$

By (2.2), (2.8) and (2.9) this immediately simplifies to

$$\int_1^y dy' \langle F \bar{F} | \left[\sum_{n=1}^{\infty} L_{-n} D_n - \sum_{r,s=\frac{1}{2}}^{\infty} G_{-r} E_r \right] y'^{E-1} \mathcal{M} | M \rangle \quad (3.10)$$

We shall now show

$$\langle F \bar{F} | \left[\sum_{n=1}^{\infty} L_{-n} D_n - \sum_{r,s=\frac{1}{2}}^{\infty} G_{-r} E_r \right] = \langle F \bar{F} | \left\{ \left(\sum_{n=1}^{\infty} D_n \right) (L_0 - \frac{1}{2}) + \sum_{r,s=\frac{1}{2}}^{\infty} \alpha_{rs} E_r G_s \right\} \quad (3.11)$$

This result and the Ward identities (2.8) and (2.9) imply that (2.10) and hence (3.9) vanish. The desired result (2.5) will then follow by integration. Notice how the left facing G gauges have to be reflected once before they annihilate on the right on the meson tree. The proof of (3.11) follows simply from the Ward identities (2.10), (3.1) and the commutators ⁹⁾

$$[L_n, D_m] = -(2n+m) D_{n+m} \quad (3.12)$$

$$\{G_r, E_s\} = -\frac{(3r+s)}{2} D_{r+s} \quad (3.13)$$

except for an additional term

$$\langle F \bar{F} | \sum_{n=1}^{\infty} \left\{ \sum_{\substack{r,s=\frac{1}{2} \\ r+s=n}}^{\infty} \left(\frac{3s+r}{2} \right) \alpha_{rs} - \frac{3n+1}{8} \right\} D_n$$

which vanishes term by term by virtue of the following identity, proved in the Appendix

$$\sum_{r+s=n} \left(s + \frac{n}{2}\right) \alpha_{rs} = \frac{3n+1}{8} \quad (3.14)$$

4. - DERIVATION OF THE MODIFIED PROPAGATOR

We are now in a position to specify that the $\Delta(x)$ appearing in Eq. (2.6) is the determinant

$$\Delta(x) = \det (1 - A(x)^2) \quad (4.1)$$

where $A(x)$ is the infinite dimensional matrix

$$A(x)_{rs} = x^r \alpha_{rs} \quad (4.2)$$

with α_{rs} defined by (3.4) or (3.8).

The procedure of eliminating the transverse state projection operator in favour of a Lorentz invariant correction factor [in this case $1/\Delta(x)$] is now fairly standard : it follows the loop calculation ¹²⁾ and the analogous Pomeron calculation ⁸⁾. The difference is that the technical details are now more complicated because of the occurrence of the matrix α_{rs} specifying how the G 's are reflected.

We shall set up a differential equation for the quantity

$$X(x, y) = \langle F \bar{F} | x^{L_0 - \frac{1}{2}} y^E | F \bar{F} \rangle \quad (4.3)$$

and define the auxiliary quantities :

$$\left. \begin{aligned} Z_0(x, y) &= \langle F\bar{F} | (D_0 - 1) x^{L_0 - \frac{1}{2}} y^E | F\bar{F} \rangle \\ Z_N(x, y) &= \langle F\bar{F} | D_N x^{L_0 - \frac{1}{2}} y^E | F\bar{F} \rangle \\ Z_{-N}(x, y) &= \langle F\bar{F} | x^{L_0 - \frac{1}{2}} y^E D_{-N} | F\bar{F} \rangle \end{aligned} \right\} \begin{matrix} N \geq 1 \end{matrix} \quad (4.4)$$

Notice how we have put D_N to the left or right according as N is positive or negative.

Remembering that $D_{-n} L_n$ and $E_{-r} G_r$ each commute with L_0 and E we find, on differentiating $X(x, y)$ with respect to y

$$\begin{aligned} y \frac{\partial}{\partial y} X(x, y) &= x \frac{\partial}{\partial x} Z_0(x, y) + \langle F\bar{F} | \left[\sum_{n=1}^{\infty} L_{-n} D_n - \sum_{r=1/2}^{\infty} G_{-r} E_r \right] x^{L_0 - \frac{1}{2}} y^E | F\bar{F} \rangle \\ &\quad + \langle F\bar{F} | x^{L_0 - \frac{1}{2}} y^E \left[\sum_{n=1}^{\infty} D_{-n} L_n + \sum_{r=1/2}^{\infty} E_{-r} G_r \right] | F\bar{F} \rangle \end{aligned}$$

We have also replaced an $L_0 - \frac{1}{2}$ by $x(\partial/\partial x)$, a manoeuvre we shall immediately repeat when we use the Ward identity (3.11) and its conjugate to simplify further to

$$y \frac{\partial}{\partial y} X(x, y) = x \frac{\partial}{\partial x} \sum_{n=-\infty}^{\infty} Z_n(x, y) + \sum_{r, s = \frac{1}{2}}^{\infty} \alpha_{rs} \langle F\bar{F} | [E_r G_s x^{L_0 - \frac{1}{2}} y^E - x^{L_0 - \frac{1}{2}} y^E G_s E_{-r}] | F\bar{F} \rangle \quad (4.5)$$

We now obtain an equation for $\langle F\bar{F} | E_r G_s x^{L_0 - \frac{1}{2}} y^E | F\bar{F} \rangle$ by moving G_s first to the right then to the left and finally to the right into its original position. The behaviour at each of the two reflections is given by (3.1). Each time a G_r passes through $x^{L_0 - \frac{1}{2}} y^E$ it picks up a factor $(x/y)^r$ and each time it passes through an E it gives rise to a D via the anticommutator (3.13). The result is

$$\begin{aligned}
 & \left[\delta_{sv} - A^2(x/y)_{sv} \right] \langle F \bar{F} | E_r G_v x^{L_0 - \frac{1}{2}} y^E | F \bar{F} \rangle \\
 &= \sum_{N=-\infty}^{\infty} \langle F \bar{F} | D_N x^{L_0 - \frac{1}{2}} y^E | F \bar{F} \rangle \sum_{\substack{t=y \\ \tau-t=N}}^{\infty} (x/y)^s \alpha_{st} (x/y)^t (t - \frac{N}{2}) \\
 &+ \sum_{N=1}^{\infty} Z_N(x, y) \sum_{\substack{u=y \\ u+r=N}}^{\infty} A^2(x/y)_{su} (u + \frac{N}{2})
 \end{aligned}$$

So, formally inverting the matrix $1-A^2$, we find

$$\begin{aligned}
 & \sum_{r,s=\frac{1}{2}}^{\infty} \alpha_{rs} \langle F \bar{F} | E_r G_s x^{L_0 - \frac{1}{2}} y^E | F \bar{F} \rangle \\
 &= \sum_{N=-\infty}^{\infty} \langle F \bar{F} | D_N x^{L_0 - \frac{1}{2}} y^E | F \bar{F} \rangle \sum_{r-t=N} \alpha_{rs} \left[(1-A^2(x/y))^{-1} A(x/y) \right]_{st} (x/y)^t (t - \frac{N}{2}) \\
 &+ \sum_{N=1}^{\infty} Z_N(x, y) \sum_{u+r=N} \alpha_{rs} \left[(1-A^2(x/y))^{-1} A^2(x/y) \right]_{su} (u + \frac{N}{2}) \quad (4.6)
 \end{aligned}$$

In a similar way we find

$$\begin{aligned}
 & - \sum_{r,s=\frac{1}{2}}^{\infty} \alpha_{rs} \langle F \bar{F} | x^{L_0 - \frac{1}{2}} y^E G_{-s} E_{-r} | F \bar{F} \rangle \\
 &= \sum_{N=-\infty}^{\infty} \langle F \bar{F} | x^{L_0 - \frac{1}{2}} y^E D_{-N} | F \bar{F} \rangle \sum_{\tau-t=N} \alpha_{ru} \left[(1-A^2(x/y))^{-1} A(x/y) \right]_{ut} (x/y)^t (t - \frac{N}{2}) \\
 &+ \sum_{N=1}^{\infty} Z_{-N}(x, y) \sum_{v+r=N} \alpha_{ru} \left[(1-A^2(x/y))^{-1} A^2(x/y) \right]_{uv} (v + \frac{N}{2}) \quad (4.7)
 \end{aligned}$$

Let us pause to examine the structure of the result we are finding :

$$\sum_{r,s=\frac{1}{2}}^{\infty} \alpha_{rs} \langle F\bar{F} | [E_r G_s x^{L_0 - \frac{1}{2}} y^E - x^{L_0 - \frac{1}{2}} y^E G_{-s} E_r] | F\bar{F} \rangle$$

$$= \lambda_0(x/y) X(x,y) + \sum_{N=-\infty}^{\infty} \lambda_N(x/y) Z_N(x,y) \quad (4.8)$$

where the functions $\lambda_N(x/y)$ can be read off Eqs. (4.6) and (4.7) and will be specified later. Inserting (4.8) into (4.5) yields the differential equation

$$\left[y \frac{\partial}{\partial y} - \lambda_0(x/y) \right] X(x,y) = \sum_{N=-\infty}^{\infty} \left[x \frac{\partial}{\partial x} + \lambda_N(x/y) \right] Z_N(x,y) \quad (4.9)$$

We seek a solution of this equation for $X(x,y)$ in which the $Z_N(x,y)$ have disappeared. The $Z_N(x,y)$ could be eliminated by the fact that

$$\oint \frac{dy}{4\pi i y} Z_N(x,y) = 0 \quad \text{all } N \quad (4.10)$$

as a consequence of the equations $D_N \mathcal{J} = 0$, $(D_0 - 1) \mathcal{J} = 0$ which express the fact that we are working in a certain Lorentz non-covariant transverse gauge. The amazing facts that will enable us to utilize the above conditions and solve the differential equation are the following equalities (which will be proved later)

$$\lambda_N(x/y) = -\Delta'(x/y) / \Delta(x/y) \quad \text{all } N \quad (4.11)$$

where $\Delta(x/y)$ is the determinant defined in Eq. (4.1). The dash denotes the following differentiation of any function of the ratio x/y :

$$f'(x/y) = \frac{x}{y} \frac{d}{d(x/y)} f(x/y) = x \frac{\partial}{\partial x} f(x/y) = -y \frac{\partial}{\partial y} f(x/y) \quad (4.12)$$

Let us see how the result (4.11) will help us. It means that $1/\Delta(x/y)$ is an integrating factor and that (4.9) can be rewritten as

$$y \frac{\partial}{\partial y} \left[\frac{\chi(x,y)}{\Delta(x/y)} \right] = x \frac{\partial}{\partial x} \left[\frac{\sum_{N=-\infty}^{\infty} Z_N(x,y)}{\Delta(x/y)} \right] \quad (4.13)$$

This we integrate $\int_1^y (dy'/y')$ to get

$$\chi(x,y) = \Delta(x/y) \frac{\chi(x,1)}{\Delta(x)} + \Delta(x/y) x \frac{\partial}{\partial x} \int_1^y \frac{dy'}{y'} \frac{\sum_{N=-\infty}^{\infty} Z_N(x,y')}{\Delta(x/y')} \quad (4.14)$$

This we now integrate $\oint (dx/4\pi ix)$ $\oint (dy/4\pi iy)$ and use

$$\oint \frac{dy}{4\pi iy} \Delta(x/y) = 0$$

since $\Delta(x)$ is analytic in $\sqrt{x}$ within the unit circle and $\Delta(0) = 1$. [This can be proved by a pinch analysis applied to the Fredholm formula for $\Delta(x)$.] The resulting second term on the right-hand side of (4.14) we integrate by parts with respect to x , use (4.12) to interchange x and y differentiations of Δ and finally integrate by parts with respect to y . The result is

$$\sum_{N=-\infty}^{\infty} \oint \frac{dx}{4\pi ix} \oint \frac{dy}{4\pi iy} Z_N(x,y)$$

which vanishes term by term by (4.10). The surviving terms yield the announced result, Eq. (2.6).

It remains to show that the coefficients λ_N defined by Eqs. (4.6), (4.7) and (4.8) actually satisfy Eq. (4.11).

Equal contributions to λ_0 come from the first terms of (4.6) and (4.7)

$$\begin{aligned} \lambda_0 &= 2 \sum_{t=1/2}^{\infty} \left[(1 - A^2(x/y))^{-1} A^2(x/y) \right]_{tt} t \\ &= 2 \text{Tr} \left[(1 - A^2(x/y))^{-1} A^2(x/y) A'(x/y) \right] \end{aligned} \quad (4.15)$$

where, according to (4.2) and (4.12) $A'(x/y)_{rs} = rA(x/y)_{rs}$ and we have used the trivial fact that A and $(1-A^2)^{-1}$ commute. Expanding the inverse as a power series and using the cyclic symmetry of the trace we find

$$\lambda_0 = 2 \text{Tr} \left(\sum_{n=0}^{\infty} A^{2n+1} A' \right) = \text{Tr} \left[\sum_{n=1}^{\infty} \frac{(A^2)^n}{n} \right]'$$

This is a logarithmic series which can be summed to give the Fredholm formula for the logarithm of the determinant

$$\begin{aligned} - \left\{ \text{Tr} [\ln(1-A^2)] \right\}' &= - \left\{ \ln [\det(1-A^2)] \right\}' \\ &= - \Delta'(x/y) / \Delta(x/y) \end{aligned}$$

which is in agreement with (4.11).

Contributions to λ_N ($N \geq 1$) come from both terms in (4.6) and from the first term in (4.7). In this last mentioned contribution we must move the D to the correct position by

$$x^{L_0 - \frac{1}{2}} y^E D_{-N} = (x/y)^N D_{-N} x^{L_0 - \frac{1}{2}} y^E$$

and then relabel N . We then find that for $N \geq 1$

$$\begin{aligned} \lambda_N = \sum_{s,t=\frac{1}{2}}^{\infty} \left[(1-A^2)^{-1} A \right]_{st} (x/y)^t &\left\{ \sum_{u+v=N} \left(\alpha_{tu}(u+\frac{N}{2}) \alpha_{vs} \right. \right. \\ &\left. \left. - (\frac{N}{2}-t) \alpha_{t+N,s} + \alpha_{t,s+N}(s+\frac{3N}{2}) \right) \right\} \end{aligned}$$

if we again remember that A and $(1-A^2)^{-1}$ commute. λ_{-N} is given by precisely the same expression. In the Appendix we shall show that the quantity in the curly brackets satisfies

$$\sum_{u+v=N} \alpha_{tu}(u+\frac{N}{2}) \alpha_{vs} = (\frac{N}{2}-t) \alpha_{t+N,s} - \alpha_{t,s+N}(s+\frac{3N}{2}) + (s+t) \alpha_{ts} \quad (4.16)$$

so λ_N is independent of N . Working further, we find

$$\begin{aligned}\lambda_N = \lambda_{-N} &= \sum_{s,t} [(1-A^2)^{-1}A]_{st} (s+t) A_{ts} \\ &= \text{Tr} \{ A'(1-A^2)^{-1}A + (1-A^2)AA' \} \\ &= 2 \text{Tr} ((1-A^2)^{-1}AA') = \lambda_0\end{aligned}$$

by (4.15) and this completes the proof of (4.11) and hence (2.6).

5. - DISCUSSION

It is convenient to define a new matrix

$$\beta_{rs}(x) = x^{\frac{r+s}{2}} \alpha_{rs} = x^{-r/2} A_{rs} x^{s/2} \quad (5.1)$$

with generating function

$$\sum_{r,s=1/2}^{\infty} y^r \beta_{rs}(x) z^s = \frac{\sqrt{zy}}{z-y} \left\{ 1 - \sqrt{\frac{1-z\sqrt{x}}{1-y\sqrt{x}}} \right\} \quad (5.2)$$

In terms of this our correction factor $\Delta(x)^{-1}$ is

$$\frac{1}{\det(1-\beta^2(x))} \quad (5.3)$$

In evaluating the four fermion amplitude we must eliminate the b operators on the meson line we have been considering. The factors (3.6) occurring in the vertices give rise to a factor

$$\left\{ \det \left(1 - \left[\frac{\beta(x) - \beta^T(x)}{2} \right]^2 \right) \right\}^5 \quad (5.4)$$

where the suffix T denotes transpose. It is intriguing that the factors (5.3) and (5.4) arising from different sources are so similar yet apparently do not cancel (as they do in the corresponding Pomeron case) [but notice that $\beta + \beta^T$ is a simple matrix of rank 1].

The most important physical question is to understand what happens in the t channel - are there singularities? - if poles, what is the spectrum? To answer this one must understand the behaviour of the product of factors (5.3) and (5.4) near $x = 1$. This is a self-contained mathematical problem. The problem of analyzing (5.4) itself ¹³⁾ has proved difficult. One must anticipate that if dual theory miracles continue to aid us, the analysis of (5.3) times (5.4) will be simpler.

APPENDIX

We shall show how identities (3.14) and (4.16) can be proved. The most elegant method we have found involves contour integral methods applied to the generating function (3.4).

Thus

$$\sum_{r+s=N} \alpha_{rs} = \oint \frac{dx}{2\pi i x^{N+1}} \alpha(x, x) \quad (\text{A.1})$$

Since

$$\alpha(x, x+\epsilon) = \frac{x}{2(1-x)} + \frac{(2-x)\epsilon}{8(1-x)^2} + O(\epsilon^2) \quad (\text{A.2})$$

equation (A.1) can be evaluated by distorting the contour, which originally encloses $x = 0$ and lies in $|x| \leq 1$ out to ∞ , picking up the residue at $x = 1$. The result is

$$\sum_{r+s=N} \alpha_{rs} = \frac{1}{2} \quad (\text{A.3})$$

Similarly

$$\begin{aligned} \sum_{r+s=N} s \alpha_{rs} &= \oint \frac{dx}{2\pi i x^{N+1}} y \frac{\partial}{\partial y} \alpha(x, y) \Big|_{x=y} \\ &= \oint \frac{dx}{2\pi i x^N} \frac{2-x}{8(1-x)^2} = \frac{N+1}{8} \end{aligned} \quad (\text{A.4})$$

again using (A.2) and the residue theorem. Equations (A.3) and (A.4) imply (3.14).

We shall illustrate the method of proving (4.16) by proving a similar but slightly simpler identity

$$\sum_{u+v=N} \alpha_{su} \alpha_{vt} = \alpha_{s+N,t} - \alpha_{s,t+N} \quad (\text{A.5})$$

The generating function of the left-hand side of this equation is

$$\sum_{s,t} \sum_{u+v=N} x^s y^t \alpha_{su} \alpha_{vt} = \oint \frac{dz}{2\pi i z^{N+1}} \alpha(x, z) \alpha(z, y) \quad (\text{A.6})$$

Inserting

$$\left(1 - \sqrt{\frac{z-1}{x-1}}\right) \left(1 - \sqrt{\frac{y-1}{z-1}}\right) = \left(1 - \sqrt{\frac{y-1}{z-1}}\right) + \left(1 - \sqrt{\frac{z-1}{x-1}}\right) - \left(1 - \sqrt{\frac{y-1}{x-1}}\right)$$

we obtain three corresponding terms in an obvious way. We shall choose the z contour which has to be in $|z| < 1$ to consistently enclose the points x and y which are possible singularities of the integrands of the individual terms but not of their sum. We see immediately that the third term integrates to zero since the contour can be distorted to infinity. The first integral is

$$\sqrt{xy} \oint_{|z|>|x|} \frac{dz}{2\pi i z^N} \frac{1}{z-x} \left[\frac{1 - \sqrt{\frac{y-1}{z-1}}}{y-z} \right] \quad (\text{A.7})$$

The square bracket being analytic in z on and inside the contour can be expanded as a power series in z

$$\sum_{n=0}^{\infty} z^n f_n(y) = \frac{1}{\sqrt{zy}} \alpha(z, y)$$

But

$$\oint_{|z|>|x|} \frac{dz}{2\pi i z^N} \frac{1}{z-x} z^n = \begin{cases} 0 & \text{if } n < N \\ x^{n-N} & \text{if } n \geq N \end{cases}$$

This is proved by distorting the contour to ∞ if $n < N$ and to zero if $n \geq N$, picking up the residue at x in the later case. So the integral (A.6) is

$$\sqrt{xy} \sum_{n=N}^{\infty} y^{n-N} f_n(y) = \sqrt{xy} \sum_{n=0}^{\infty} y^n f_{n+N}(y)$$

This yields the first term of the right-hand side of (A.5). The second term works out similarly. Equation (4.16) is derived similarly by considering, instead of (A.6), the following

$$\oint \frac{dz}{2\pi i z^{N+1}} \left[\left(\frac{N}{2} + z \frac{\partial}{\partial z} \right) \alpha(x, z) \right] \propto (z, y)$$

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