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POSSIBLE BORN AMPLITUDE FOR PIONS

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A B S T R A C T

A new multipion amplitude is suggested. For four pions it coincides with an earlier proposal; the six pion amplitude bootstraps at the internal pion pole; a symmetry of the amplitude is associated with decoupling of the rho-trajectory tachyon and of an ancestor above the pion trajectory.

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During the last five years, considerable progress has been made towards a possible theory of strong interactions based on real crossing-invariant Born amplitudes for linear Regge trajectories. One outstanding problem is to write, even for external spin-zero pions, a consistent Born amplitude with physically-acceptable intercepts and mass spectrum.

Recently a method of classifying Born amplitudes ¹⁾⁻⁶⁾ was developed ⁷⁾, making use of the symmetric group. At the same time it was pointed out how the symmetric group could be used to construct new amplitudes. This approach led to general four-meson amplitudes of the form

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{\gamma}{2}} \phi_4 \left(\begin{matrix} \alpha_s \\ \beta_s(x) \end{matrix} ; \begin{matrix} \alpha_t \\ \beta_t(x) \end{matrix} ; \begin{matrix} \alpha_u \\ \beta_u(x) \end{matrix} \right) \quad (1)$$

where

$$\gamma = \alpha_s + \alpha_t + \alpha_u + 1 \quad (2)$$

and where ϕ_4 satisfies the two requirements ^{*)} that :

(i) it is invariant under arbitrary permutations of the three argument pairs $\{\alpha_i, \beta_i\}$ with $i=s, t, u$ and where

$$\beta_s(x) = x(1-x+x^2)^{-1} \quad (3)$$

$$\beta_t(x) = \beta_s(1-x) \quad (4)$$

$$\beta_u(x) = \beta_s\left(-\frac{x}{1-x}\right) \quad (5)$$

(ii) it is an analytic function both in α_i and in β_i .

A particularly attractive and simple form for ϕ_4 , that was selected in Ref. 7), is

$$\phi_4 = \frac{1}{3} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (6)$$

^{*)} A general analysis of ϕ_4 , based on these two requirements, has been made in Ref. 8).

Let us take $\alpha_p(0) - \alpha_\pi(0) = \frac{1}{2}$ and define

$$\alpha = \gamma - 2 = 1 - \alpha_p(0) \quad (7)$$

then the corresponding A_4 becomes explicitly

$$A_4 = \int_0^1 dx \, x^{-\alpha-1} (1-x)^{-\alpha-1} (1-x+x^2)^{\alpha/2} \cdot \frac{1}{3} [\alpha_s x + \alpha_t (1-x) - \alpha_u x(1-x)] \quad (8)$$

We now propose a possible generalization of the A_4 of Eq. (8) to an A_N for general N (= even integer ≥ 4). This new multiplication amplitude will be presented in two equivalent ways.

We first write, using partial differential operators, and treating z differences as independent ^{*})

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \sum_{\{q_1, q_2, \dots, q_N\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} \dots \frac{\partial}{\partial(z_{q_{N-1}} - z_{q_N})} \cdot \left[\prod_{i \neq j} (z_i - z_j)^{-p_i p_j} (S_N^{(1)})^{\alpha/2} \right] \quad (9)$$

A second alternative is obtained by introducing ^{**}) N complex variables ϕ_i satisfying the Grassmann algebra

$$\{\phi_i, \phi_j\}_+ = 0 \quad (10)$$

whereupon the amplitude is obtained by starting from the integral ⁶⁾

$$\int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i p_j} (S_N^{(1)})^{\alpha/2} \quad (11)$$

^{*}) In Eq. (9), $dV_3 = dz_a dz_b dz_c (z_a - z_b)^{-1} (z_b - z_c)^{-1} (z_c - z_a)^{-1}$ is the Haar measure; the sum is over all permutations of $\{q_1 q_2 \dots q_N\} = 1, 2, \dots, N$;

$$S_N^{(1)} = \sum_{\{q_1, q_2, \dots, q_N\}} \prod_{k=1}^N (z_{q_k} - z_{q_{k+1}})^{-2};$$

and the integration domain keeps the z_i in cyclic order on a circle with z_a, z_b, z_c (chosen arbitrarily) held fixed.

^{**}) Such complex variables have been used in Refs. 9), 10) to establish a connection between the amplitudes of Refs. 2) and 4).

and then extending it, using the Grassmann algebra, to the new amplitude

$$A_N = \int \prod_{i=1}^N \frac{d\phi_i}{\phi_i^2} \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j - \phi_i \phi_j)^{-p_i \cdot p_j} \left[\sum_{\{q_1, q_2, \dots, q_N\}} \prod_{k=1}^N (z_{q_k} - z_{q_{k+1}} - \phi_{q_k} \phi_{q_{k+1}})^{-2} \right]^{a/2} \quad (12)$$

where the $\oint_i$ integrals are understood to be Cauchy integrals encircling the origin.

The equivalence of Eqs. (9) and (12) is evident once one realizes that the $\oint_i$ integrals amount to taking the first term of a Taylor expansion with respect to the z differences. Note that if we extrapolate the present amplitude to the unphysical value $a = 0$, we obtain the Neveu-Schwarz dual pion model ⁴⁾.

Let us examine the case $N=4$. We may rewrite

$$A_4 = \int \prod_{i=1}^4 dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} (S_4^{(1)})^{a/2} \cdot \sum_{\{q_1, q_2, q_3, q_4\}} (-1)^p \frac{\theta_{q_1, q_2, q_3, q_4}}{(z_{q_1} - z_{q_2})(z_{q_3} - z_{q_4})} \quad (13)$$

where $\theta_{q_1, q_2, q_3, q_4}$ is given by

$$\begin{aligned} \theta_{q_1, q_2, q_3, q_4} = & (-2p_{q_1} \cdot p_{q_2})(-2p_{q_3} \cdot p_{q_4}) \\ & + (-2p_{q_1} \cdot p_{q_2}) (S_4^{(1)})^{-a/2} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} (S_4^{(1)})^{a/2} \\ & + (-2p_{q_3} \cdot p_{q_4}) (S_4^{(1)})^{-a/2} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} (S_4^{(1)})^{a/2} \\ & + (S_4^{(1)})^{-a/2} \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} (S_4^{(1)})^{a/2} \end{aligned} \quad (14)$$

After some algebra we find that

$$A_4 = \int_0^1 dx \ x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{\alpha}{2}+1} \phi_4 \quad (15)$$

where

$$\phi_4 = \hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u - \frac{3a}{2} \beta_s \beta_t \beta_u \quad (16)$$

in which we have defined the quantities

$$\hat{\alpha}_s = 2p_1 \cdot p_2 + a \left(1 - \frac{(1-x)^2}{2(1-x+x^2)} \right) = \alpha_s + \frac{a \beta_t \beta_u}{2 \beta_s} \quad (17)$$

$$\hat{\alpha}_t = 2p_2 \cdot p_3 + a \left(1 - \frac{x^2}{2(1-x+x^2)} \right) = \alpha_t + \frac{a \beta_u \beta_s}{2 \beta_t} \quad (18)$$

$$\hat{\alpha}_u = 2p_1 \cdot p_3 + a \left(1 - \frac{1}{2(1-x+x^2)} \right) = \alpha_u + \frac{a \beta_s \beta_t}{2 \beta_u} \quad (19)$$

and have introduced the fully S_3 symmetric variable

$$\beta_s \beta_t \beta_u = - \frac{x^2 (1-x)^2}{(1-x+x^2)^3} \quad (20)$$

In deriving Eqs. (15), (16) it is necessary to use the identities

$$\beta_s \beta_t + \beta_t \beta_u + \beta_u \beta_s = 0 \quad (21)$$

$$\frac{\beta_s^2 \beta_t^2}{\beta_u} + \frac{\beta_t^2 \beta_u^2}{\beta_s} + \frac{\beta_u^2 \beta_s^2}{\beta_t} = 3 \beta_s \beta_t \beta_u \quad (22)$$

Let us agree to use the symbol $\equiv$ to mean equality of two forms of ϕ_4 in Eq. (15) after integration. Then the following equalities hold

$$\hat{\alpha}_s \beta_s \equiv \hat{\alpha}_t \beta_t \equiv \hat{\alpha}_u \beta_u \quad (23)$$

$$\equiv \frac{1}{3} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (24)$$

$$\equiv \hat{\alpha}_s^2 \beta_s + \hat{\alpha}_t^2 \beta_t + \hat{\alpha}_u^2 \beta_u - \frac{3a}{2} \beta_s \beta_t \beta_u \quad (25)$$

Once these results have been achieved [note that Eqs. (23)-(25) are valid for any intercept $\alpha_p(0)$] one deduces that Eqs. (15) and (16) lead to

$$A_4 = \int_0^1 dx \, x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{a}{2}+1} \frac{1}{3} (\alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u) \quad (26)$$

as required.

To study bootstrap consistency, let us consider the limit

$$\lim_{\alpha_{123} \rightarrow 0} (\alpha_{123} A_6) \quad (27)$$

of the amplitude A_6 which can be written in the form

$$\begin{aligned} A_6 = & \int \left[\prod_{i=1}^6 dz_i (dV_3)^{-1} \prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} \prod_{i=1}^6 (z_i - z_{i+1})^{\frac{a}{2}} \prod_{i < j} (z_i - z_j)^{(-1)^{i \cdot j}} \right] \\ & \cdot \left[\prod_{i=1}^6 (z_i - z_{i+1})^2 S_6^{(1)} \right]^{\frac{a}{2}} \\ & \cdot \left[\prod_{i < j} (z_i - z_j)^{-(-1)^{i \cdot j}} \sum_{\{q_1, q_2, q_3, q_4, q_5, q_6\}} (-1)^P \frac{\theta_{q_1, q_2, q_3, q_4, q_5, q_6}}{(z_{q_1} - z_{q_2})(z_{q_3} - z_{q_4})(z_{q_5} - z_{q_6})} \right] \end{aligned} \quad (28)$$

The first of the three square brackets in Eq. (28) has well-known factorization properties since it is simply the generalized Euler B function integrand.

Let us define integration variables $0 \leq x, w, y \leq 1$ corresponding to the cross-ratios

$$x = (z_1, z_6; z_2, z_3) \quad (29)$$

$$w = (z_1, z_6; z_3, z_4) \quad (30)$$

$$y = (z_1, z_6; z_4, z_5) \quad (31)$$

where $(a, b; c, d) = (a-c)(b-d)(a-d)^{-1}(b-c)^{-1}$.

Then concerning the second bracket in Eq. (28) we find

$$\lim_{w \rightarrow 0} \left[\prod_{i=1}^6 (z_i - z_{i+1})^2 S_6^{(1)}(z) \right] = 96 (1-x+x^2) (1-y+y^2) \quad (32)$$

consistent with factorization.

The factorization of the third and final bracket of Eq. (28) is more subtle. We find, within an over-all numerical factor

$$\lim_{\alpha_{123} \rightarrow 0} (\alpha_{123} A_6) = \int_0^1 dx x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} (1-x+x^2)^{\frac{\gamma}{2}} \int_0^1 dy y^{-\alpha_{56}-1} (1-y)^{-\alpha_{45}-1} (1-y+y^2)^{\frac{\gamma}{2}} \mathcal{F}(x,y) \quad (33)$$

with, after some algebra

$$\mathcal{F}(x,y) = \frac{1}{3} (\alpha_{12} \beta_{12} + \alpha_{23} \beta_{23} + \alpha_{13} \beta_{13}) \cdot \frac{1}{3} (\alpha_{56} \beta_{56} + \alpha_{45} \beta_{45} + \alpha_{46} \beta_{46}) \quad (34)$$

in which we defined

$$\beta_{12}(x) = (1-x)(1-x+x^2)^{-1} ; \quad \beta_{23} = \beta_{12}(1-x) ; \quad \beta_{13} = \beta_{12}(1-1/x) \quad (35)$$

$$\beta_{56}(y) = (1-y)(1-y+y^2)^{-1} ; \quad \beta_{45} = \beta_{56}(1-y) ; \quad \beta_{46} = \beta_{56}(1-1/y) \quad (36)$$

This exhibits the bootstrap consistency, as required.

Concerning the amplitude A_N we would like to mention a symmetry which is higher than the symmetric group invariance used in its construction. It seems possible to rewrite the amplitude in the form

$$A_N = \int \prod_{i=1}^N dz_i (dV_3)^{-1} \frac{1}{(z_a - z_b)} \cdot \sum_{\{q_1, q_2, \dots, q_{N-2}\}} (-1)^P \frac{\partial}{\partial(z_{q_1} - z_{q_2})} \frac{\partial}{\partial(z_{q_3} - z_{q_4})} \dots \frac{\partial}{\partial(z_{q_{N-3}} - z_{q_{N-2}})} \cdot \left[\prod_{i \neq j} (z_i - z_j)^{-p_i \cdot p_j} (S_N^{(1)})^{a/2} \right] \quad (37)$$

where z_a, z_b are chosen arbitrarily from the z_i and the sum is over all permutations $q_1, q_2, q_3, \dots, q_{N-2} = 1, 2, 3, \dots, N$ (excluding a, b). The sum in Eq. (37) contains only $(N-3)!! = (N-3) \cdot (N-5) \cdot (N-7) \dots 3 \cdot 1$ different terms compared to the $(N-1)!!$ different terms in Eq. (9). We have checked at $N=4$ that this gives rise to the equivalent forms

$$\phi_4 = \hat{\alpha}_s \beta_s \equiv \hat{\alpha}_t \beta_t \equiv \hat{\alpha}_u \beta_u \quad (38)$$

dependent on the choice of z_a, z_b . Further, we have shown that A_5 defined by Eq. (37) bootstraps correctly for all choices of a, b . This higher symmetry is associated with the tachyon-decoupling factors in the integrand and can be shown to be the underlying mechanism [in the form of the equivalences in Eq. (38)] for removal of an ancestor spin-one state at $\alpha_{123} = 0$ in A_6 ; it seems appropriate, therefore, to refer to this special symmetry as spin-lowering symmetry.

In concluding, we should remark that our four-pion amplitude A_4 is satisfactory phenomenologically even at the daughter level. At the ρ mass (760 MeV) there is no ϵ resonance, where experimentally there is certainly no narrow resonance ¹¹⁾. At the f mass (1300 MeV) there is no p wave resonance (ρ') agreeing with its experimental absence ; but there is an s wave (ϵ') resonance of elastic width ~ 260 MeV in nice agreement with the recent pion phase shift analysis ^{12),*)}.

Concerning our multipion amplitude, one would like further consistency checks, in particular concerning the multiparticle factorization properties. This ambitious project will lead to further evidence concerning whether our proposal is the right model to use for describing physical strangeness-zero mesons.

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*) By contrast, the competing model of Ref. 3) has a broad ϵ resonance, a strongly coupled ρ' and a slightly ghostlike ϵ' . Note that on the parent trajectory the predictions of the present A_4 and those of the Lovelace-Shapiro model coincide.

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