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THE GAUGE PROPERTIES OF THE DUAL MODEL POMERON-REGGEONVERTEX: THEIR DERIVATION AND THEIR CONSEQUENCES

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A B S T R A C T

Study of the non-planar orientable single dual loop diagrams in 26 space-time dimensions has revealed an infinite positive definite spectrum of "Pomeron" intermediate states which couple to Reggeons via a bilinear Pomeron - Reggeon vertex operator. General algebraic techniques are developed to derive the behaviour of this vertex with respect to the Virasoro gauge operators. A reflection and transmission behaviour is found, reminiscent of the behaviour of a wave incident at the interface between two different media (in this case Reggeonic and Pomeronic). These gauge properties are such as to guarantee the desired "good properties", namely completeness of the transverse Reggeon states when coupled between physical Reggeon states on one side, and on the other side, either physical Pomeron states or else physical Reggeon state created via an intermediate Pomeron. This is yet another example of the amazing and gratifying self-consistency of the dual model with respect to duality, transversality and unitarity.

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INTRODUCTION

A useful tool for understanding the algebraic properties of dual models has been the construction of the projection operator ¹⁾ onto the space of "transverse" states, in the case where the dimension of space-time is the critical one (26 in the conventional model, 10 in the Neveu-Schwarz model) and the leading trajectory intercept is one. This projection operator enables one to prove directly the no-ghost theorem for tree graphs ^{2),3)}. When single loop graphs are computed via Feynman's tree theorem ⁴⁾ using the projection operator, one shows that the Lovelace ⁵⁾ conjecture is verified. In that case, namely, the Pomeron singularity becomes a series of factorizable poles ^{6),7)}. It was then proved that the spectrum of the Pomeron states is transverse and thus ghost free ⁸⁾, and that the same states also factorize the Shapiro-Virasoro ⁹⁾ model.

In the proof of the no-ghost theorem for the Pomeron sector one used only the knowledge of how gauges in the Pomeron sector converted into gauges of the Reggeon sector ⁶⁾. The converse problem, of working Reggeon gauges into the Pomeron sector has, however, not yet been solved. The understanding of this problem is necessary if one wants to check that no spurious Reggeon states couple to the loop through various dualizations. When a loop diagram is computed in one configuration (see Fig. 1a) it is explicitly ¹⁰⁾ free of spurious states. It can, however be dualized into a different configuration (see Fig. 1b) so that spurious Reggeon states could reappear in the tree on either side of the Pomeron singularity. Secondly, if one wants to compute two-loop diagrams in the configuration of Fig. 2 a projection operator must be inserted on the Reggeon line in the middle, and to work out its contribution this problem has to be solved. Thirdly, as we shall see this problem has some analogies with the gauge properties of the fermion-meson vertex ¹¹⁾. It can also be considered as part of the general problem of working out gauges through loop diagrams, a problem common to dual models and gauge field theories.

In this paper we derive the gauge identities obeyed by the Reggeon-Pomeron vertex. It is found that although the first gauge operator L_1 converts into Pomeron gauges, only a combination of L_2 and of an infinite number of L_{-n} converts into Pomeron gauges. From these two gauge conditions one can derive a more general one involving L_m ($m \geq 1$) and an infinite linear combination of L_{-n} but we have not found it convenient to work with this.

The situation is analogous to the phenomenon of reflection and transmission at the boundary of two different media, and indeed the Pomeron-Reggeon vertex describes the coupling between two different spaces of oscillators. It is conjectured that this could be true for the meson-fermion vertex also. One shows finally that in spite of their oddity, for $D=26$ these gauges decouple the spurious Reggeon states which could have been coupled so that duality in the non-planar loop is fully consistent with transversality.

In the first part, we rewrite the Pomeron-Reggeon vertex in a form close to the fermion-meson vertex introducing the Fubini-Veneziano¹²⁾ field $P(x)$. In the second part we prove the gauge identities satisfied by the vertex, using algebraic techniques based on certain general formulae on the commutation of bilinear operators which are shown in Appendix A. In the third part, we prove the absence of spurious states in the Reggeon sector. The Appendix B contains the proof of a property used in Section 3.

1. THE REGGEON-POMERON VERTEX

In Ref. 6) it was shown that the non-planar orientable one-loop graph can be refactorized in the following way

$$F_{r,s} = \langle \text{tree}(k_i) | \frac{1}{L_0 - 1} \sum_T (P) \frac{1}{L_0 - 1} | \text{tree}(P_j) \rangle \quad (1.1)$$

where

$$\sum_T (P) = \langle 0_{bc} | V(a^\dagger, b, c) D_P(p^2) P_0 V(a, b^\dagger, c^\dagger) | 0_{bc} \rangle \quad (1.2)$$

$$P_0 = \int_0^{2\pi} d\Theta e^{2i\Theta(L_0^b - L_0^c)} \quad (1.3)$$

$$D_P(p^2) = \int_0^1 \frac{dq}{q} \prod_{n=1}^{\infty} (1 - q^{2n})^2 q^{L_0^b + L_0^c - 2} \quad (1.4)$$

$V(a^+, b, c)$ is the transition vertex between the Reggeon and Pomeron sectors. In order to obtain a simplified expression for V including the zero modes we introduce the following redefinition of oscillators

$$\begin{aligned}\alpha_n^\mu &= -i\sqrt{n} a_n^\mu & \beta_n^\mu &= -i\sqrt{n} b_n^\mu & \bar{\beta}_n^\mu &= -i\sqrt{n} c_n^\mu \\ \alpha_{-n}^\mu &= i\sqrt{n} a_n^{\mu\dagger} & \beta_{-n}^\mu &= i\sqrt{n} b_n^{\mu\dagger} & \bar{\beta}_{-n}^\mu &= i\sqrt{n} c_n^{\mu\dagger} \\ \alpha_0 &= -i p_\mu = 2\beta_0 = 2\bar{\beta}_0\end{aligned}\quad (1.5)$$

and

$$P_\gamma^\mu(x) = \sum_{n=-\infty}^{\infty} \gamma_n^\mu x^{-n} \quad (1.6)$$

where

$$\gamma_n^\mu = \{ \alpha_n^\mu \text{ or } \beta_n^\mu \text{ or } \bar{\beta}_n^\mu \}$$

In the following we shall forget the Lorentz index and keep only the indices α , β , $\bar{\beta}$. The vertex then takes the following form

$$V_0 = \langle 0_{bc} | V(a^+, b, c) = g \left(\frac{\pi}{2} \right)^{p^2} \pi^{-\alpha(p^2)} t(\alpha) \langle 0_{\beta\bar{\beta}} | \mathcal{V} | 0_\alpha \rangle \quad (1.7)$$

where

$$t(\alpha) = \exp \tilde{\Delta} \quad (1.8)$$

$$\tilde{\Delta} = \frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{x y} : P_\alpha(x^{-1}) P_\alpha(y^{-1}) : g(x-y) \quad (1.9)$$

$$g(x) = \ln \frac{\sin \pi x}{\pi x} \quad (1.10)$$

$$\mathcal{V} = \exp \left[\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}} + \mathcal{V}_{\beta\bar{\beta}} \right] \quad (1.11)$$

$$\mathcal{V}_{\alpha\beta} = \frac{1}{(2i\pi)^2} \oint \frac{dx dz}{xz} P_{\alpha}(x^{-1}) P_{\beta}(z) K(x, z) \quad (1.12)$$

$$\mathcal{V}_{\alpha\bar{\beta}} = \frac{1}{(2i\pi)^2} \oint \frac{dx dz}{xz} P_{\alpha}(x^{-1}) P_{\bar{\beta}}(z) K(-x, z) \quad (1.13)$$

$$\mathcal{V}_{\beta\bar{\beta}} = \frac{1}{(2i\pi)^2} \oint \frac{dz dz'}{zz'} P_{\beta}(z) P_{\bar{\beta}}(z') I(z, z') \quad (1.14)$$

where

$$K(x, z) = -\ln(1 - ze^{2i\pi x}) \quad (1.16)$$

$$I(z, z') = \ln(1 - zz') \quad (1.17)$$

The contours in Eqs. (1.9), (1.12), (1.13) and (1.14) are chosen to include the origin and exclude the singularities of the functions $g(x-y)$, $K(x, z)$, $K(x, -z)$, $I(z, z')$. Thus these functions can be expanded in positive powers of the variables x, y, z, z' , and the integrals evaluated trivially.

2. GAUGE CONDITIONS FOR THE POMERON-REGGEON VERTEX

a) Commuting gauges with $t(\alpha)$

As we shall see in the next section it is not necessary in order to prove the absence of spurious states in the Reggeon sector that the vertex V_0 converts each gauge L_n^{α} into a combination of Pomeron gauges L_m^{β} , $L_m^{\bar{\beta}}$. Rather it is sufficient that L_n^{α} converts into a linear combination of Pomeron gauges together with a linear combination of L_{-m}^{α} so that the incident L_n^{α} gauge generates respectively a transmission and reflection of gauges in the Pomeronic and Reggeonic media. As we shall see the most general such gauge condition can be derived by repeated application

of those involving $n=1$ and $n=2$ and so we shall concentrate on these latter cases and formulate the problem mathematically as follows by introducing a general gauge

$$L_{\phi}^{\alpha} = \frac{1}{2(2i\pi)} \oint \frac{dx}{x} : p^2(x^{-1}) : \phi(x) \quad (2.1)$$

so if

$$\phi(x) = x^{-n} \quad L_{\phi}^{\alpha} = L_n^{\alpha} \quad (2.2)$$

The use of $P^{\alpha}(x^{-1})$ instead of the more normal $P^{\alpha}(x)$ in the integrand is because the expressions $t(\alpha)$ (1.8) and ψ (1.11) are in terms of just $P^{\alpha}(x^{-1})$. The formulae in Appendix A are easily transformed to have $P(x^{-1})$ instead of $P(x)$ by a suitable change of variable. We shall try to prove that some L_{ϕ}^{α} converts into Pomeron gauges through V_0 . Such relations are already known in the case when ϕ is analytic near $x=0$ as shown in Ref. 6) where Pomeron gauges are shown to convert through V_0 into Reggeon gauges made of L_{-n} . So we are specifically interested in the cases where $\phi(x)_{x \rightarrow 0} \sim x^{-1}$ or x^{-2} so as to obtain the action of L_1, L_2 through V_0 .

The most difficult part is obviously to commute such L_{ϕ} through the operator $t(\alpha)$, and it is interesting to treat this problem separately. Suppose for instance that one considers the pole in the Pomeron ground state: $p^2 = -8$. Then

$$Frs \propto \frac{1}{\alpha(p^2)} \langle tree(k_i) | t(\alpha) | 0 \rangle \langle 0 | t^{\dagger}(\alpha) | tree(p_j) \rangle$$

Then, one has to check that $t(\alpha)$ verifies equations of the type

$$(c + L_{\phi}^{\alpha}) t(\alpha) | 0, \alpha \rangle = 0 \quad (2.3)$$

where c is a c number. We must allow for a c number in (2.3) if L_{ϕ}^{α} contains an L_0^{α} . Then the amount of L_0^{α} in L_{ϕ}^{α} determines the c number.

We shall consider first that $t(\alpha)$ is given by Eqs. (1.8), (1.9), but we shall not specify the form of the function $g(x)$ and only assume it is analytic at $x=0$ and even in x .

It is shown in the Appendix A that under these assumptions one has

$$[L_{\varphi}^{\alpha}, t(\alpha)] = t(\alpha) \left[-\frac{D}{2} g''(0) \oint \frac{dx}{2\pi i} \varphi(x) \cdot x + \frac{1}{2(2\ln)^2} \oint \frac{dx dy}{xy} : P^{\alpha}(x^{-1}) P^{\alpha}(y^{-1}) : H(x, y) \right] \quad (2.4)$$

where

$$H(x, y) = (x\varphi(x) - y\varphi(y)) g'(x-y) + \oint \frac{dz \cdot z}{2\ln} \varphi(z) g'(x-z) g'(y-z) \quad (2.5)$$

Let us first set $\varphi(x) = x^{-1}$, i.e., $L_{\varphi} = L_1$. Then we find obviously

$$[L_1^{\alpha}, t(\alpha)] = 0 \quad (2.6)$$

which reflects the invariance of $g(x-y)$ with respect to simultaneous translations of x and y .

So let us consider an L_{φ} such that $\varphi \underset{x \rightarrow 0}{\sim} (1/x^2) + \text{cte}$, i.e., L_{φ} is a combination of L_2 and of a sum of L_{-m} ($m = 0, 1, \dots$). Equation (2.4) then reads

$$[L_{\varphi}^{\alpha}, t(\alpha)] = t(\alpha) \left[-\frac{D}{2} g''(0) + \frac{1}{2(2\ln)^2} \oint \frac{dx dy}{xy} : P^{\alpha}(x^{-1}) P^{\alpha}(y^{-1}) : H(x, y) \right]$$

where

$$H(x, y) = (x\varphi(x) - y\varphi(y)) g'(x-y) + g'(x) g'(y) \quad (2.7)$$

Let us now set

$$g(x) = \ln \frac{f(x)}{x} \quad (2.8)$$

as in Eq. (1.10), but without specifying f . f has to behave like x near $x=0$ and be odd in x . Then

$$H = - \frac{x\varphi(x) - y\varphi(y)}{x-y} + (x\varphi(x) - y\varphi(y)) \frac{f'(x-y)}{f(x-y)} + \frac{f'(x)f'(y)}{f(x)f(y)} - \frac{1}{x} \frac{f'(y)}{f(y)} - \frac{1}{y} \frac{f'(x)}{f(x)} + \frac{1}{xy} \quad (2.9)$$

Then

$$L\varphi t(\alpha) = t(\alpha) \left[-\frac{D}{2} g''(0) + \frac{1}{2(2\pi)^2} \int_{|x| < |y|} \frac{dx dy}{xy} : P^\alpha(x^{-1}) P^\alpha(y^{-1}) : \tilde{H}(x, y) \right] \quad (2.10)$$

$$\tilde{H}(x, y) = \frac{2x\varphi(x)}{y-x} - \frac{1}{x} \frac{f'(y)}{f(y)} - \frac{1}{y} \frac{f'(x)}{f(x)} + \frac{1}{xy} + \sigma(x, y) \quad (2.11)$$

$$\sigma(x, y) = (x\varphi(x) - y\varphi(y)) \frac{f'(x-y)}{f(x-y)} + \frac{f'(x)}{f(x)} \frac{f'(y)}{f(y)} \quad (2.12)$$

one sees easily that since $|x| < |y|$ the Laurent expansion of the first term in H has only negative powers in y , hence the integration in y will yield destruction operators and this term will give zero on the vacuum because of the normal ordering. Similarly, the three next terms in Eq. (2.11) annihilate the vacuum.

Since $\varphi(x) \underset{x \rightarrow 0}{\sim} x^{-2}$, $f(x) \underset{x \rightarrow 0}{\sim} x$ and f is odd one sees that σ is analytic in x, y , near $x=y=0$ owing to a cancellation between the two terms. Thus its integration yields bilinears in creation operations, which will never annihilate the vacuum, or give a constant unless we can choose $\varphi(x)$ such that:

$$\sigma(x, y) = \lambda \quad \lambda \text{ is a constant}$$

Hence if we can set

$$x\varphi(x) - y\varphi(y) = \left[\lambda - \frac{f'(x)}{f(x)} \frac{f'(y)}{f(y)} \right] \frac{f(x-y)}{f'(x-y)} \quad (2.13)$$

then we have shown that Eq. (2.3) holds. This is not possible in general for a given f since the equation relates a function of one variable to a function of two variables. Thus f must satisfy the equation:

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \left\{ \left[\lambda - \frac{f'(x)}{f(x)} \frac{f'(y)}{f(y)} \right] \frac{f(x-y)}{f'(x-y)} \right\} = 0 \quad (2.14)$$

This shows that the operator $t(\alpha)$ is almost determined by the requirement that Eq. (2.3) holds. Although we have not found the general solution of Eq. (2.14), we have shown that it depends on at most two additional parameters, beyond λ .

In particular if we set $f(x) = (1/\pi) \sin \pi x$ as in Eq. (1.10), we see that the functional equation has a solution if $\lambda = -\pi^2$:

$$\begin{aligned} x\varphi(x) - y\varphi(y) &= (\lambda - \pi^2 \cot \pi x \cot \pi y) \frac{1}{\pi \cot \pi(x-y)} \\ &= \frac{1}{\pi} (\lambda - \pi^2 \cot \pi x \cot \pi y) \frac{\cot \pi y - \cot \pi x}{1 + \cot \pi x \cot \pi y} = \pi \cot \pi x - \pi \cot \pi y \end{aligned} \quad (2.15)$$

we choose the solution:

$$\varphi(x) = \frac{\pi}{x} \cot \pi x \quad (2.16)$$

Hence

$$L_\varphi^\alpha = L_2^\alpha + \sum_{n=0}^{\infty} c_n L_{-n}^\alpha \quad \text{where} \quad \sum_{n=0}^{\infty} c_n x^n = \frac{\pi}{x} \cot \pi x - \frac{1}{x^2} \quad (2.17)$$

It is remarkable and gratifying to the calculations that starting with a general L_φ and a general function f one gets a functional equation for f which is indeed satisfied in our case. In this case where:

$$g(x) = \ln \frac{\sin \pi x}{\pi x} \underset{x \rightarrow 0}{\sim} - \frac{\pi^2 x^2}{6}$$

$$g''(0) = - \frac{\pi^2}{3}$$

we have shown that $t(\alpha)$ obeys the following equation

$$\left(L_\varphi^\alpha - \frac{D\pi^2}{6}\right) t(\alpha) = t(\alpha) \frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : P^\alpha(x^{-1}) P^\alpha(y^{-1}) : \tilde{H}(x, y) \quad (2.18)$$

$$|x| < |y|$$

where

$$\tilde{H}(x, y) = \frac{2x\varphi(x)}{y-x} - \frac{1}{x} \frac{f'(y)}{f(y)} - \frac{1}{y} \frac{f'(x)}{f(x)} + \frac{1}{xy} - \pi^2 \quad (2.19)$$

We shall now look how the right-hand side of Eq. (2.18) commutes through

$$\langle 0_{\beta\bar{\beta}} | \mathcal{V} | 0_\alpha \rangle$$

b) Commuting gauges through $\mathcal{V}$

We rederive here firstly for the sake of completeness the commutation of L_1^α , through $\mathcal{V}$, although this relation was already derived in Ref. 6)

$$[L_1^\alpha, \mathcal{V}_{\alpha\beta}] = \frac{1}{(2i\pi)^2} \oint \frac{dx dz}{xz} P^\alpha(x^{-1}) P^\beta(z) \frac{\partial}{\partial x} K(x, z) \quad (2.20)$$

$$= \frac{2i\pi}{(2i\pi)^2} \oint \frac{dx dz}{xz} P^\alpha(x^{-1}) P^\beta(z) \frac{z e^{2i\pi x}}{1 - z e^{2i\pi x}} \quad (2.21)$$

$$= \frac{2i\pi}{(2i\pi)^2} \oint \frac{dx dz}{x z} p^\alpha(x^{-1}) p^\beta(z) z \frac{\partial}{\partial z} K(x, z) \quad (2.22)$$

$$= 2i\pi [\mathcal{V}_{\alpha\beta}, L_0^\beta] = 2i\pi [\mathcal{V}_{\alpha\beta}, L_0^\beta - \bar{L}_0^\beta] \quad (2.23)$$

Here we have used another functional property of K , i.e.,

$$\frac{\partial}{\partial x} K(x, z) = 2i\pi z \frac{\partial}{\partial z} K(x, z)$$

One can check easily that

$$[L_0^\beta - \bar{L}_0^\beta, \mathcal{V}_{\beta\bar{\beta}}] = 0 \quad (2.24)$$

Hence the gauge condition

$$L_1^\alpha V_0 = 2i\pi V_0 (L_0^\beta - \bar{L}_0^\beta) \quad (2.25)$$

We now want to show that similarly [see Eq. (2.10)]

$$\frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : p^\alpha(x^{-1}) p^\alpha(y^{-1}) : \tilde{H}(x, y) \langle 0_{\beta\bar{\beta}} | \mathcal{V}_1 | 0_\alpha \rangle = \langle 0_{\beta\bar{\beta}} | \mathcal{V}_1 | 0_\alpha \rangle [L_\psi^\beta + \bar{L}_{\bar{\psi}}^\beta] \quad (2.26)$$

One must first commute the bilinear in P^α through $\mathcal{V}_1$ until it reaches $|0_\alpha\rangle$. Then we shall use the fact that:

$$\frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : p^\alpha(x^{-1}) p^\alpha(y^{-1}) : \tilde{H}(x, y) |0_\alpha\rangle = -\frac{\pi^2}{2} \alpha_0^2 |0_\alpha\rangle = -2\pi^2 \beta_0^2 |0_\alpha\rangle \quad (2.27)$$

From the formulae in Appendix A we can show that the first commutator yields:

$$I_1 = \left[\frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : p^\alpha(x^{-1}) p^\alpha(y^{-1}) : \tilde{H}(x, y), \mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}} \right] = \quad (2.28)$$

$$= \frac{1}{2(2i\pi)^2} \oint \frac{dx dz}{x z} \left[\rho^\alpha(x^{-1}) \cdot \rho^\beta(z) \rho(x, z) + \rho^\alpha(x^{-1}) \rho^{\bar{\beta}}(z) \rho(-x, z) \right] \quad (2.29)$$

where

$$\rho(x, z) = \frac{1}{2i\pi} \oint_{|x| < |y|} dy \tilde{H}(x, y) \frac{\partial}{\partial y} K(y, z) + \frac{1}{2i\pi} \oint_{|y| < |x|} dy \frac{\partial}{\partial y} K(y, z) \tilde{H}(y, x) \quad (2.30)$$

Similarly, the second commutator with $\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}$ yields

$$\begin{aligned} I_2 = \frac{1}{2} [I_1, \mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}] &= \frac{1}{4(2i\pi)^3} \oint \frac{dz_1}{z_1} \frac{dz_2}{z_2} dx \left[P_\beta(z_1) \rho(x, z_1) \right. \\ &\quad \left. + P_{\bar{\beta}}(z_1) \rho(-x, z_1) \right] \frac{\partial}{\partial x} \left[P_\beta(z_2) K(x, z_2) + P_{\bar{\beta}}(z_2) K(-x, z_2) \right] \end{aligned} \quad (2.31)$$

So we get:

$$\begin{aligned} &\frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : \rho^\alpha(x^{-1}) \cdot \rho^\alpha(y^{-1}) : \tilde{H}(x, y) \langle 0_{\beta\bar{\beta}} | \mathcal{V} | 0_\alpha \rangle \\ &= \langle 0_{\beta\bar{\beta}} | \mathcal{V} \left[-2\pi^2 \beta_\alpha^2 + I_1 + I_2 \right] | 0_\alpha \rangle \end{aligned} \quad (2.32)$$

We compute now the function $\rho(x, z)$ with the actual values of the function K and $\tilde{H}$

$$\tilde{H}(x, y) = \frac{2x\varphi(x)}{y-x} - \frac{y}{x} \varphi(y) - \frac{x}{y} \varphi(x) + \frac{1}{xy} - \pi^2 \quad (2.33)$$

Distorting in the integration contour in Eq. (2.30) we get

$$\begin{aligned}
 P(x,z) &= \frac{\partial}{\partial x} K(x,z) 2x \varphi(x) + \frac{1}{2i\pi} \oint_{|y| < |x|} dy \frac{\partial}{\partial y} K(y,z) [\tilde{H}(x,y) + \tilde{H}(y,x)] \\
 &= 2x \varphi(x) \frac{\partial}{\partial x} K(x,z) + \frac{1}{2i\pi} \oint_{|y| < |x|} dy \frac{\partial}{\partial y} K(y,z) \left[\frac{2x^2 \varphi(x)}{y(y-x)} + \frac{2y^2 \varphi(y)}{x(x-y)} + \frac{2}{xy} - 2\pi^2 \right] \\
 &= 2x \varphi(x) \left[\frac{\partial}{\partial x} K(x,z) - \frac{\partial}{\partial y} K(y,z) \right]_{y=0} + \frac{2}{x} \frac{\partial}{\partial y} K(y,z) \Big|_{y=0} \quad (2.34) \\
 &= 4i\pi^2 z \operatorname{ctg} \pi x \left[\frac{e^{2i\pi x} - 1}{(1-z)(1-ze^{2i\pi x})} \right] + \frac{2}{x} \frac{2i\pi z}{1-z}
 \end{aligned}$$

Using

$$\operatorname{ctg} \pi x = i \frac{e^{2i\pi x} + 1}{e^{2i\pi x} - 1}$$

we obtain

$$P(x,z) = -4\pi^2 \frac{z(z+1)}{1-z} \frac{e^{2i\pi x}}{1-ze^{2i\pi x}} - 4\pi^2 \frac{z}{1-z} + \frac{2}{x} \frac{2i\pi z}{1-z} \quad (2.35)$$

$$P(x,z) = -4\pi^2 \frac{z(z+1)}{1-z} \frac{\partial}{\partial z} K(x,z) - 4\pi^2 \frac{z}{1-z} + \frac{2}{x} \frac{2i\pi z}{1-z} \quad (2.36)$$

Apart from the last two terms we have succeeded in writing $P(x,z)$ as a function of z multiplied by $(\partial/\partial z)K(x,z)$ which is what we want in order to recognize the expression as the vertex multiplied by a gauge operator in the β space. Integrating the last two terms with respect to x we get:

$$\begin{aligned}
 I, |0_\alpha\rangle &= -2\pi^2 \left[\frac{1}{(2i\pi)^2} \oint \frac{dx dz}{xz} P^\alpha(x^{-1}) \frac{z(z+1)}{1-z} \left[P_\beta(z) \frac{\partial}{\partial z} K(x,z) \right. \right. \\
 &\quad \left. \left. + P_\beta(z) \frac{\partial}{\partial z} K(-x,z) \right] + \frac{2\beta_0}{2i\pi} \oint \frac{dz}{1-z} (P_\beta(z) + P_{\bar{\beta}}(z)) \right] |0_\alpha\rangle \quad (2.37)
 \end{aligned}$$

where we have used $\alpha_1 |0_\alpha\rangle = 0$ and $\alpha_0 = 2\beta_0$. Let us now introduce:

$$L_\psi^\beta = \frac{1}{2(2i\pi)^2} \oint \frac{dz_1 dz_2}{z_1 z_2} \frac{z_1 \psi(z_1) + z_2 \psi(z_2)}{z_2 - z_1} \cdot P^\beta(z_1) \cdot P^\beta(z_2); \quad (2.38)$$

$|z_1| < |z_2|$

Since

$$\left[\frac{1}{2i\pi} \oint \frac{dz}{z} p^\beta(z) k(z), L_\psi^\beta \right] = \frac{1}{2i\pi} \oint \frac{dz}{z} p^\beta(z) z \psi(z) \frac{\partial}{\partial z} k(z) \quad (2.39)$$

we see that choosing

$$\psi(z) = \frac{z+1}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n \quad (2.40)$$

Equation (2.37) can be written as:

$$I_1 |0_\alpha\rangle = -2\pi^2 \left\{ [\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}, L_\psi^\beta + L_{\bar{\psi}}^{\bar{\beta}}] + \frac{2\beta_0}{2i\pi} \oint \frac{dz}{1-z} (p_\beta(z) + p_{\bar{\beta}}(z)) \right\} |0_\alpha\rangle \quad (2.41)$$

We now come back to the problem of evaluating the double commutator I_2 . We integrate with respect to x in Eq. (2.31). Only the term in x^{-1} contributes in $\mathcal{P}(x, z)$:

$$\mathcal{P}(x, z) \underset{x \rightarrow 0}{\sim} \frac{2}{x} \frac{2i\pi z}{1-z} \quad (2.42)$$

and

$$\begin{aligned} I_2 &= \frac{(4i\pi)(2i\pi)}{4(2i\pi)^2} \oint \frac{dz_1 dz_2}{z_1 z_2} (p_\beta(z_1) - p_{\bar{\beta}}(z_1)) \frac{z_1}{1-z_1} (p_\beta(z_2) - p_{\bar{\beta}}(z_2)) \frac{z_2}{1-z_2} \\ &= -2\pi^2 \left[\oint \frac{dz}{2i\pi(1-z)} (p_\beta(z) - p_{\bar{\beta}}(z)) \right]^2 \end{aligned} \quad (2.43)$$

So we get finally the result:

$$\begin{aligned} &\frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : p^\alpha(x^{-1}) p^\alpha(y^{-1}) : \bar{H}(x, y) \langle 0_{\beta\bar{\beta}} | \mathcal{V} | 0_\alpha \rangle \\ &= -2\pi^2 \langle 0_{\beta\bar{\beta}} | \mathcal{V} \left[\beta_0^2 + \left[\frac{1}{2i\pi} \oint \frac{dz}{1-z} (p_\beta(z) - p_{\bar{\beta}}(z)) \right]^2 + \frac{2\beta_0}{2i\pi} \oint \frac{dz}{1-z} (p_\beta(z) + p_{\bar{\beta}}(z)) \right. \right. \\ &\quad \left. \left. + [(\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}), L_\psi^\beta + L_{\bar{\psi}}^{\bar{\beta}}] \right] | 0_\alpha \rangle \end{aligned} \quad (2.44)$$

We now want to show that this is nothing else than

$$J = -2\pi^2 \langle 0_{\beta\bar{\beta}} | \mathcal{V} | 0_{\alpha} \rangle (L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}) \quad (2.45)$$

We note that since ψ is analytic in z at zero we need only to commute once $L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}$ with $\mathcal{V}_{\alpha\beta}$, $\mathcal{V}_{\alpha\bar{\beta}}$, $\mathcal{V}_{\beta\bar{\beta}}$. Thus

$$J = -2\pi^2 \langle 0_{\beta\bar{\beta}} | [L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}} + [(\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}), L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}] + [\mathcal{V}_{\beta\bar{\beta}}, L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}]] \mathcal{V} | 0_{\alpha} \rangle \quad (2.46)$$

Using the general formula

$$[\mathcal{V}_{\beta\bar{\beta}}, L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}] = \frac{1}{(2i\pi)^2} \oint \frac{dz dz'}{z z'} P_{\beta}(z) \cdot P_{\bar{\beta}}(z') \mu(z, z') \quad (2.47)$$

$$\mu(z, z') = z \psi(z) \frac{\partial}{\partial z} I(z, z') + z' \psi(z') \frac{\partial}{\partial z'} I(z, z') \quad (2.48)$$

we obtain by explicit computation

$$\mu(z, z') = - \frac{2zz'}{(1-z)(1-z')} \quad (2.49)$$

and get

$$J = -2\pi^2 \langle 0_{\beta\bar{\beta}} | [L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}} + [(\mathcal{V}_{\alpha\beta} + \mathcal{V}_{\alpha\bar{\beta}}), L_{\psi}^{\beta} + \bar{L}_{\psi}^{\bar{\beta}}] - 2 \oint \frac{dz}{2i\pi} \frac{P^{\beta}(z)}{1-z} \oint \frac{dz'}{2i\pi} \frac{P^{\bar{\beta}}(z')}{1-z'}] \mathcal{V} | 0_{\alpha} \rangle \quad (2.50)$$

Let us now compute finally $\langle 0_{\beta\bar{\beta}} | L_{\psi}^{\beta}$

$$\langle 0_{\beta\bar{\beta}} | L_{\psi}^{\beta} = \frac{1}{2(2i\pi)^2} \langle 0_{\beta\bar{\beta}} | \oint \frac{dx dy}{xy} \frac{x\psi(x) + y\psi(y)}{y-x} : p^{\beta}(x) \cdot p^{\beta}(y) : \quad (2.51)$$

$|x| < |y|$

$$= \frac{1}{2(2i\pi)^2} \langle 0_{\beta\bar{\beta}} | \oint \frac{dx dy}{xy} \frac{y\psi(y) - x\psi(x)}{y-x} : p^{\beta}(x) \cdot p^{\beta}(y) : \quad (2.52)$$

$$= \frac{1}{2(2i\pi)^2} \langle 0_{\beta\bar{\beta}} | \oint \frac{dx dy}{xy} \frac{1+x+y-xy}{(1-y)(1-x)} : p^\beta(x) \cdot p^\beta(y) : \quad (2.53)$$

$$= \langle 0_{\beta\bar{\beta}} | \left[\frac{1}{2} \beta_0^2 + \frac{1}{(2i\pi)^2} \oint \frac{dx dy}{xy} \frac{y+x-xy}{(1-x)(1-y)} : p^\beta(x) \cdot p^\beta(y) : \right. \quad (2.54)$$

$$\left. + \left(\frac{1}{2i\pi} \oint \frac{dx}{1-x} p^\beta(x) \right)^2 \right] \quad (2.55)$$

Adding the contribution of $L_{\psi}^{\bar{\beta}}$ we see that Eq. (2.50) is now identical to Eq. (2.44). We have now proved that

$$\begin{aligned} & \frac{1}{2(2i\pi)^2} \oint \frac{dx dy}{xy} : p^\alpha(x) \cdot p^\alpha(y) : \bar{H}(x,y) \langle 0_{\beta\bar{\beta}} | \psi | 0_\alpha \rangle \\ & = -2\pi^2 \langle 0_{\beta\bar{\beta}} | \psi | 0_\alpha \rangle (L_\psi^\beta + L_\psi^{\bar{\beta}}) \end{aligned} \quad (2.56)$$

The second gauge condition satisfied by V_0 therefore reads as follows:

$$\left(L_\psi^\alpha - \frac{D-26}{6} \pi^2 \right) V_0 = -2\pi^2 V_0 (L_\psi^\beta + L_\psi^{\bar{\beta}}) \quad (2.57)$$

where

$$\psi(x) = \frac{\pi}{x} \cotg \pi x = \sum_{n=0}^{\infty} c_n x^n + \frac{1}{x^2} \quad (2.58)$$

$$\psi(x) = \frac{x+1}{1-x} \quad (2.59)$$

Using $c_0 = -(\pi^2/3)$ we put the gauge condition in its final form

$$\begin{aligned} & \left[L_2^\alpha - \frac{\pi^2}{6} (D-26) + \sum_{n=0}^{\infty} c_n (L_{-n}^\alpha + \delta_{n0}) \right] V_0 \\ & = -2\pi^2 V_0 \left[L_0^\beta + L_0^{\bar{\beta}} - 2 + 2 \sum_{n=1}^{\infty} (L_n^\beta + L_n^{\bar{\beta}}) \right] \end{aligned} \quad (2.60)$$

Once relations (2.60) and (2.25) are found we obtain an infinite number of gauge relations involving higher L_n by computing the multiple commutator

$$[L_1^\alpha, [L_1^\alpha, \dots [L_1^\alpha, (L_2^\alpha - \frac{\pi^2}{6}(D-26) + \sum_{n=0}^{\infty} c_n(L_{-n}^\alpha + \delta_{n0}))] \dots] V_0$$

using the Virasoro algebra and relations (2.60), (2.25). One gets:

$$\left(p! L_{p+2}^\alpha + \sum_{m=-1}^{\infty} c_{m+p} \frac{\Gamma(m+p+2)}{\Gamma(2+m)} L_{-m}^\alpha \right) V_0 = -4\pi^2 V_0 \sum_{n=1}^{\infty} (2\ln n)^p [(1)^p L_n^\beta + L_n^{\bar{\beta}}] \quad (2.61)$$

$p \geq 1$

3. DECOUPLING OF SPURIOUS STATES IN THE REGGEON SECTOR

We now want to check the completeness of the transverse Reggeon states when coupled between physical Reggeons on one side and on the other side, either a physical Pomeron state or else physical Reggeon states created via an intermediate Pomeron. The appropriate residues to consider are:

$$F_r = \langle \text{tree}(k_j) | m_N V_0 | \text{Pomeron physical state} \rangle \quad (3.1)$$

$$F_{rs} = \langle \text{tree}(k_j) | m_N V_0 D_p(p^2) | \Psi \rangle \quad (3.2)$$

where m_N is the projection operator on the mass shell $p^2 = 2(N-1)$ of the Reggeon considered adjacent to the Pomeron Reggeon vertex, and

$$|\Psi\rangle = P_0 V_0^T \frac{1}{L_0 - 1} | \text{tree}(p_i) \rangle \quad (3.3)$$

The mathematical result to be proved is that the above amplitudes (3.1), (3.2) vanish when m_N is replaced by $m_N(\tau - 1)$ and Ψ is the physical state projection operator. To facilitate these proofs we shall use the result proved in Appendix B, that

$$\langle \text{tree}(k_j) | m_N(\tau-1) = \langle \phi_1 | L_1^\alpha + \langle \phi_2 | (L_2^\alpha + \frac{3}{2} L_1^{\alpha^2}) \quad (3.4)$$

when $|\phi_1\rangle$ and $|\phi_2\rangle$ satisfy the equations

$$L_n^\alpha |\phi_1\rangle = 0 \quad n \geq 0 \quad (3.5)$$

$$(L_n^\alpha + \delta_{n0}) |\phi_2\rangle = 0 \quad n \geq 0 \quad (3.6)$$

Then it is clear that we shall just need to use the following gauge identities:

$$L_1^\alpha V_0 = 2i\pi V_0 (L_0^\beta - L_0^{\bar{\beta}}) \quad (3.7)$$

$$[L_2^\alpha + \sum_{n=0}^{\infty} c_n (L_{-n}^\alpha + \delta_{n0})] V_0 = -2\pi^2 V_0 [L_0^\beta + L_0^{\bar{\beta}} - 2 + 2 \sum_{n=1}^{\infty} (L_n^\beta + L_n^{\bar{\beta}})] \quad (3.8)$$

$$(L_n^\beta - L_{-n}^{\bar{\beta}}) |\psi\rangle = 0 \quad (3.9)$$

$$-\frac{1}{2i\pi} \sum_{m=1}^{\infty} \frac{(2i\pi n)^{m+1}}{(m+1)!} W_{-m}^\alpha V_0 = V_0 [L_n^\beta - L_{-n}^{\bar{\beta}} - L_0^\beta + L_0^{\bar{\beta}} + n] \quad (3.10)$$

Equations (3.9) and (3.10) were proved in Ref. 6).

As a consequence all that remains to be shown are the results

$$\langle \phi_2 | L_2^\alpha V_0 | \text{Pomeron physical state} \rangle = 0 \quad (3.11)$$

$$\langle \phi_2 | L_2^\alpha V_0 D_p(p^2) |\psi\rangle = 0 \quad (3.12)$$

since the L_1^α terms are dealt with trivially. Equation (3.11) is a trivial consequence of the Ward identity (3.8), Eq. (3.6) and the conditions satisfied by the Pomeron physical states as shown by the work of ⁸⁾.

$$(L_n^\beta - \delta_{n0}) | \text{Pomeron physical state} \rangle = 0 \quad n \geq 0$$

$$(L_n^{\bar{\beta}} - \delta_{n0}) | \text{Pomeron physical state} \rangle = 0 \quad n \geq 0$$

The proof of (3.12) is more complicated: taking matrix elements of Eq. (3.8) between $\langle \phi_2 |$ and $V_0 D_p(p^2) | \Psi \rangle$ and using (3.6) yields:

$$\begin{aligned} \langle \phi_2 | L_2^\alpha V_0 D_p(p^2) | \Psi \rangle = & -2\pi^2 \langle \phi_2 | V_0 [L_0^\beta + L_0^{\bar{\beta}} - 2 \\ & + 2 \sum_{n=1}^{\infty} (L_n^\beta + L_n^{\bar{\beta}})] D_p(p^2) | \Psi \rangle \end{aligned} \quad (3.13)$$

Let us compute

$$\begin{aligned} \langle \phi_2 | V_0 (L_n^\beta + L_n^{\bar{\beta}}) q^{L_0^\beta + L_0^{\bar{\beta}}} | \Psi \rangle &= q^n \langle \phi_2 | V_0 q^{L_0^\beta + L_0^{\bar{\beta}}} (L_n^\beta + L_n^{\bar{\beta}}) | \Psi \rangle \\ &= q^n \langle \phi_2 | V_0 q^{L_0^\beta + L_0^{\bar{\beta}}} (L_{-n}^\beta + L_{-n}^{\bar{\beta}}) | \Psi \rangle = q^{2n} \langle \phi_2 | V_0 (L_{-n}^\beta + L_{-n}^{\bar{\beta}}) q^{L_0^\beta + L_0^{\bar{\beta}}} | \Psi \rangle \end{aligned} \quad (3.14)$$

It is a consequence of Eq. (3.6) that

$$\left(-\frac{1}{2i\pi} \sum_{m=1}^{\infty} \frac{(2i\pi n)^{m+1} - (-2i\pi n)^{m+1}}{(m+1)!} W_{-m}^\alpha \right) V_0 = V_0 (L_n^\beta + L_n^{\bar{\beta}} - L_{-n}^\beta - L_{-n}^{\bar{\beta}} + 2n) \quad (3.15)$$

Since

$$\langle \phi_2 | W_{-m}^\alpha = \langle \phi_2 | (-L_0 + 1) = 2 \langle \phi_2 | \quad (3.16)$$

one gets

$$\langle \phi_2 | V_0 (L_n^\beta + L_n^{\bar{\beta}} - L_{-n}^\beta - L_{-n}^{\bar{\beta}} - 2n) = 0 \quad (3.17)$$

Hence

$$\begin{aligned} \langle \phi_2 | V_0 (L_n^\beta + \bar{L}_n^{\bar{\beta}}) q^{L_0^\beta + L_0^{\bar{\beta}}} | \psi \rangle &= q^{2n} \langle \phi_2 | V_0 (L_n^\beta + \bar{L}_n^{\bar{\beta}}) q^{L_0^\beta + L_0^{\bar{\beta}}} | \psi \rangle \\ - 2n q^{2n} \langle \phi_2 | V_0 q^{L_0^\beta + L_0^{\bar{\beta}}} | \psi \rangle &= - \frac{2n q^{2n}}{1 - q^{2n}} \langle \phi_2 | V_0 q^{L_0^\beta + L_0^{\bar{\beta}}} | \psi \rangle \end{aligned} \quad (3.18)$$

And

$$\begin{aligned} \langle \phi_2 | L_2^\alpha V_0 D_p(p^2) | \psi \rangle &= -2\pi^2 \langle \phi_2 | V_0 \int_0^1 \frac{dq}{q} q^{L_0^\beta + L_0^{\bar{\beta}} - 2} \prod_{n=1}^{\infty} (1 - q^{2n})^2 [L_0^\beta + L_0^{\bar{\beta}} - 2 - 4 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1 - q^{2n}}] | \psi \rangle \\ &= -2\pi^2 \langle \phi_2 | V_0 \int_0^1 dq \frac{\partial}{\partial q} \left[q^{L_0^\beta + L_0^{\bar{\beta}} - 2} \prod_{n=1}^{\infty} (1 - q^{2n})^2 \right] | \psi \rangle = 0 \end{aligned} \quad (3.19)$$

There remains now to be shown that this result holds true also when applied on a pole non-adjacent to the Pomeron propagator. The proof follows exactly the same lines and one needs only to show the additional result that:

$$\langle 0 | V(k_2) \dots V(k_r) (L_0^\alpha - 1) \frac{1}{L_0^\alpha - 1} V_0 D_p(p^2) | \psi \rangle = 0 \quad (3.20)$$

This is indeed true because when computing:

$$\langle 0, k_1 | V(k_2) \dots V(k_r) x^{L_0} V_0 D_p(p^2) | \psi \rangle$$

one finds a factor $(\sin \pi x)^{-k_1 \cdot k_r}$ in the integrand which vanishes at $x=1$ in the appropriate region for $k_1 \cdot k_r$.

Doubts can be raised as to the validity of the proof of (3.12) in the case where a Reggeon mass shell coincides with a Pomeron mass-shell. In that case the last step in Eq. (3.19) is dubious since L_0^β and $L_0^{\bar{\beta}}$ can take the value 1. Similarly, when deriving the completeness of transverse states on a pole non-adjacent to the Pomeron propagator, Eq. (3.20) has to be used an infinite number of times. Although we have not been able to clear up these points we feel optimistic that it will prove possible.

APPENDIX A

EVALUATION OF COMMUTATORS

As we have seen in this paper most of the fundamental quantities in dual theory, e.g., the Virasoro generators and the exponents of vertex operators are either linear or bilinear in annihilation operators. Further they can be expressed in terms of these operators via the Fubini-Veneziano fields $P^\mu(x)$ which possess the elegant commutation relations most conveniently expressed in the form

$$P_\mu(x) P_\nu(y) = : P_\mu(x) P_\nu(y) : + \frac{g_{\mu\nu} xy}{(x-y)^2} \quad |x| > |y| \quad (A.1)$$

where the normal ordering refers to all modes but the zero one. Since the normal ordered product is well defined for all x, y ($\neq 0$) this equation implies that the ordinary product is well defined only in the region indicated, and is in fact singular at $x=y$.

The fundamental operation with the above mentioned linear or bilinear quantities is commutation and this Appendix deals a systematic treatment of this based on Wick's theorem with the above equation providing the basic contraction.

Commutation of quantities linear in $P^\mu(x)$

First consider the commutator $[A_1, A_2]$ where

$$A_i = \oint \frac{dx}{2\pi i x} P^\mu(x) G_\mu^i(x) \quad i = 1, 2 \quad (A.2)$$

and the $G_\mu^i(x)$ commute with $P^\mu(x)$ as well as mutually, and are analytic in a neighbourhood of $x=0$ except for a possible isolated singularity there. The contour of integration encircles the origin and can otherwise be freely distorted within the region of analyticity. In order to use (A.1) we must take the integration after the operator product of fields. In order that the latter is well defined we must treat the two terms of the commutator separately with the contour associated with the left-hand factor always enclosing the contour associated with the right-hand factor. So, using (A.1)

and the symmetry property of the normal ordered product

$$[A_1, A_2] = \oint_{|x| > |y|} \frac{dx}{2\pi i} \oint_{|y| > |x|} \frac{dy}{2\pi i} G_1^\mu(x) G_2^\nu(y) \left[:P_\mu(x) P_\nu(y): + \frac{g_{\mu\nu} x \cdot y}{(x-y)^2} \right]$$

Now we can try to distort the contour $|x| > |y|$, $|y| > |x|$ into each other and are impeded only by the $x=y$ singularity in the contraction whose effect can be evaluated by the residue theorem to give

$$\begin{aligned} [A_1, A_2] &= \oint_0 \frac{dy}{2\pi i} \oint_y \frac{dx}{2\pi i} \frac{G_{1\mu}(x) G_{2\mu}(y)}{(x-y)^2} \\ &= - \oint \frac{dy}{2\pi i} G_{1\mu}(y) \frac{d}{dy} G_{2\mu}(y) \end{aligned} \quad (A.3)$$

This result was used to derive Eq. (2.20). The subsequent equations of Sections 2, 3, can be derived similarly or with the further techniques about to be described.

Classification of normal ordered quantities bilinear in $P^\mu(x)$

Now define

$$B_g = \oint_{|x| > |y| > 0} \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} g(x, y) : P^\mu(x) P_\mu(y) : \quad (A.4)$$

within a neighbourhood of the origin the function $g(x, y)$ will be analytic except for possible isolated singularities at $x=0$, $y=0$ and $x=y$. In fact all cases of interest will be covered by the following two possibilities for the behaviour at $x=y$

- i) $g(x, y)$ is analytic at $x=y$;
- ii) $g(x, y)$ has a simple pole at $x=y$. (This will be the Virasoro case.)

In case i) the x and y contours can be interchanged by distortion, and so we find that only the part of $g(x, y)$ symmetric under x, y interchange contributes. Notice how the annihilation creation operator content of B_g depends on the analyticity properties of $g(x, y)$ at 0 and ∞ in x and y . Thus if $g(x, y)$ is analytic at 0, B_g contains no creation operators whereas if it is analytic at ∞ it contains no destruction operators. This latter is the case usually occurring in the

text and there the change of variables $x \rightarrow 1/x$, $y \rightarrow 1/y$ has been made in order to focus attention back on the origin.

As we shall now see the Virasoro generators constitute the most general possibility occurring in case ii). Splitting the numerator of $(x-y)^{-1}$ into parts symmetric and antisymmetric under x, y interchange we have

$$g(x,y) = \frac{S(x,y)}{x-y} + \frac{A(x,y)}{x-y}$$

The second term will be discarded since it is analytic and symmetric and so belongs to case i). Because of the singularity at $x=y$ it is now significant whether the y contour lies inside the x contour or vice versa. Hence, we define

$$B_{<} = \oint \frac{dx}{2\pi i x} \oint \frac{dy}{2\pi i y} \frac{S(x,y)}{x-y} : P_{\mu}(x) P^{\mu}(y) : \\ |x| > |y| > 0 \\ |y| > |x| > 0$$

Interchange of the integration variables x and y shows us that

$$B_{>} = -B_{<}$$

on the other hand, by pulling the x contour across the pole at $x=y$ we see

$$B_{>} = B_{<} + \oint \frac{dy}{2\pi i y} \tilde{g}(y) : P(y)^2 :$$

where

$$y \tilde{g}(y) = S(y,y)$$

is the residue of the pole in $g(x,y)$ at $x=y$. Thus

$$B_g = B_{>} = \frac{1}{2} \oint \frac{dy}{2\pi i y} \tilde{g}(y) : P(y)^2 : \quad (A.5)$$

and this is the familiar form of the Virasoro generator L_n if $\tilde{g}(y) = +y^n$.

It is not strictly necessary to use the form (A.4) with the arguments x and y separated out but it looks more mathematically rigorous and will provide a more unified treatment.

Evaluation of commutators of quantities bilinear in $P^\mu(x)$

We now wish to evaluate commutators $[B_f, B_g]$. There are three possibilities according to our classification; i)-i), i)-ii) and ii)-ii) commutators. The general idea follows the treatment of the linear case. We see that we must now express the product of four P^μ fields in terms of normal ordered products and contractions and this is precisely what Wick's theorem does:

$$\begin{aligned}
 P_{\mu_1}(x_1) P_{\mu_2}(x_2) P_{\mu_3}(x_3) P_{\mu_4}(x_4) &= : P_{\mu_1}(x_1) P_{\mu_2}(x_2) P_{\mu_3}(x_3) P_{\mu_4}(x_4) : \\
 &+ \frac{g_{\mu_1\mu_2} x_1 x_2}{(x_1 - x_2)^2} : P_{\mu_3}(x_3) P_{\mu_4}(x_4) : + \frac{g_{\mu_1\mu_3} x_1 x_3}{(x_1 - x_3)^2} : P_{\mu_2}(x_2) P_{\mu_4}(x_4) : \\
 &+ \frac{g_{\mu_1\mu_4} x_1 x_4}{(x_1 - x_4)^2} : P_{\mu_2}(x_2) P_{\mu_3}(x_3) : + \frac{g_{\mu_2\mu_3} x_2 x_3}{(x_2 - x_3)^2} : P_{\mu_1}(x_1) P_{\mu_4}(x_4) : \\
 &+ \frac{g_{\mu_2\mu_4} x_2 x_4}{(x_2 - x_4)^2} : P_{\mu_1}(x_1) P_{\mu_3}(x_3) : + \frac{g_{\mu_3\mu_4} x_3 x_4}{(x_3 - x_4)^2} : P_{\mu_1}(x_1) P_{\mu_2}(x_2) : \\
 &+ \frac{g_{\mu_1\mu_2} g_{\mu_3\mu_4} x_1 x_2 x_3 x_4}{(x_1 - x_2)^2 (x_3 - x_4)^2} + \frac{g_{\mu_1\mu_3} g_{\mu_2\mu_4} x_1 x_2 x_3 x_4}{(x_1 - x_3)^2 (x_2 - x_4)^2} \\
 &+ \frac{g_{\mu_1\mu_4} g_{\mu_2\mu_3} x_1 x_2 x_3 x_4}{(x_1 - x_4)^2 (x_2 - x_3)^2}
 \end{aligned} \tag{A.6}$$

Applying this to the two terms in the commutator we have:

$$\begin{aligned}
 [B_f, B_g] &= \oint_{\Gamma} \frac{dx_1}{2\pi i x_1} \frac{dx_2}{2\pi i x_2} \frac{dy_1}{2\pi i y_1} \frac{dy_2}{2\pi i y_2} \{ (x_1, x_2) g(y_1, y_2) \left[\frac{x_1 y_1}{(x_1 - y_1)^2} : P(x_2) \cdot P(y_2) : \right. \right. \\
 &+ \frac{x_1 y_2}{(x_1 - y_2)^2} : P(x_2) \cdot P(y_1) : + \frac{x_2 y_1}{(x_2 - y_1)^2} : P(x_1) \cdot P(y_2) : + \frac{x_2 y_2}{(x_2 - y_2)^2} : P(x_1) \cdot P(y_1) : \\
 &\left. \left. + D x_1 x_2 y_1 y_2 \left(\frac{1}{(x_1 - y_1)^2 (x_2 - y_2)^2} + \frac{1}{(x_1 - y_2)^2 (x_2 - y_1)^2} \right) \right] \right\}
 \end{aligned} \tag{A.7}$$

and

$$\Gamma = \Gamma_{>} - \Gamma_{<} = |x_1| > |x_2| > |y_1| > |y_2| - |y_1| > |y_2| > |x_1| > |x_2|$$

The final step is to distort contour $\Gamma_{>}$ into $\Gamma_{<}$. This is impeded by the singularities of the contractions and also by the possible singularities of f and g at $x_1=x_2$ and $y_1=y_2$ if they are case ii). Notice that some terms in (A.6) (terms 1, 2, 7 and 8) have not contributed since the distortion of $\Gamma_{>}$ into $\Gamma_{<}$ was trivial since there were no intervening singularities. Notice also that the last two terms in (A.7) are c number contributions proportional to the space time dimension D .

The treatment of the distortion of $\Gamma_{>}$ into $\Gamma_{<}$ will depend upon which of the three types of commutator we consider. For definiteness we shall first take B_f to be a type ii) bilinear [so $f(x_1, x_2)$ has a pole at $x_1=x_2$] and write it as $B_{f(ii)}$ to emphasize the fact, and make no special assumption about B_g .

First notice that the terms 1 and 3, 2 and 4, 5 and 6 in (A.7) form natural pairs in that the integrand of each pair is antisymmetric under interchange of x_1 and x_2 . In fact we can apply the same argument that leads from (A.4) to (A.5) to rewrite (A.7) as

$$[B_{f(ii)}, B_g] = \oint_{\Gamma'} \frac{dx}{2\pi i x} \frac{dy_1}{2\pi i y_1} \frac{dy_2}{2\pi i y_2} \tilde{f}(x) g(y_1, y_2) \quad (A.8)$$

$$\left[\frac{xy_1}{(x-y_1)^2} : P(x) \cdot P(y_2) : + \frac{xy_2}{(x-y_2)^2} : P(x) \cdot P(y_1) : + \frac{Dx^2 y_1 y_2}{(x-y_1)^2 (x-y_2)^2} \right]$$

where $f(x_1, x_2) \sim x_1 \tilde{f}(x_1)/(x_1 - x_2)$ near $x_1 = x_2$ and

$$\Gamma' = \Gamma'_{>} - \Gamma'_{<} = |x| > |y_1| > |y_2| - |y_1| > |y_2| > |x|$$

If now we assume that g is also class ii) we can repeat the above argument in the variables y_1 and y_2 and obtain

$$[B_{f(ii)}, B_{g(ii)}] = \oint \frac{dx}{2\pi i x} \oint \frac{dy}{2\pi i y} \tilde{f}(x) \tilde{g}(y) \left[- \frac{xy}{(x-y)^2} : P(x) \cdot P(y) : + \frac{Dx^2 y^2}{2(x-y)^4} \right]$$

$$|x| > |y| - |y| > |x|$$

. The x integration here is evaluated simply by taking the residue at $x=y$

$$\begin{aligned} [B_{f(ii)}, B_{g(ii)}] &= \frac{1}{2} \oint \frac{dy}{2\pi i} [\tilde{f}'(y) \tilde{g}(y) - \tilde{f}(y) \tilde{g}'(y)] : P(y)^2 : \\ &- \frac{D}{24} \oint \frac{dy}{2\pi i} [(\gamma \tilde{f}(y))'' (\gamma \tilde{g}(y))' - (\gamma \tilde{f}(y))' (\gamma \tilde{g}(y))''] \end{aligned} \quad (A.9)$$

where the dash denotes differentiation. If $\tilde{f}(y) = +y^m$, $\tilde{g}(y) = +y^m$ this gives the Virasoro algebra together with the correct c number terms.

If, instead of assuming that g is class ii) we assume that it is class i) we see that in order to evaluate (A.8) we must distort $\Gamma'_>$ into $\Gamma'_<$ by moving the x contour. The only obstacles are the double poles at y_1 and y_2 whose residues yield the result

$$\begin{aligned} [B_{f(iii)}, B_{g(i)}] &= - \oint \frac{dy_1}{2\pi i} \frac{dy_2}{2\pi i} : P(y_1) \cdot P(y_2) : \left[y_1 \tilde{f}(y_1) \frac{\partial}{\partial y_1} + y_2 \tilde{f}(y_2) \frac{\partial}{\partial y_2} \right] g(y_1, y_2) \\ &+ D \oint \frac{dy_1}{2\pi i} \frac{dy_2}{2\pi i} \left[\frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g(y_1, y_2) \right] \left[\frac{y_1 \tilde{f}(y_1) - y_2 \tilde{f}(y_2)}{y_1 - y_2} \right] \end{aligned} \quad (A.10)$$

Since the integrands are analytic at $y_1 = y_2$ we need no longer specify whether the y_1 contour lies inside the y_2 or not.

The only case remaining to be treated is that in which f and g are both class i) and to do this we must go back to Eq. (A.7). Now the only singularities of the integrands impeding the distortion of $\Gamma'_>$ into $\Gamma'_<$ are the singularities of the contractions. Taking account of these by the residue theorem we find that

$$[B_{f(i)}, B_{g(i)}] = B_{h(i)} - D \oint \frac{dx}{2\pi i} \frac{dy}{2\pi i} \frac{h(x, y)}{(x-y)^2} \quad (A.11)$$

where

$$h(x, y) = \oint \frac{dz}{2\pi i} \left[f(x, z) \overset{\longleftrightarrow}{\frac{\partial}{\partial z}} g(z, y) + f(y, z) \overset{\longleftrightarrow}{\frac{\partial}{\partial z}} g(z, x) \right] \quad (A.12)$$

and is explicitly symmetric under x, y interchange and antisymmetric under f, g interchange. Notice that the result is a class 1) bilinear as indicated in the equation. So also was the operator part of the right-hand side of (A.10). Notice that in both equations (A.10) and (A.11) [but not (A.9)] the c number term can be totally absorbed into the operator part simply by removing the normal ordering. Also note that if f and g are both analytic at the origin h vanishes since the z integration can be contracted to zero. This simply reflects the fact that functions of destruction operators commute if no creation operators are present.

Derivation of Eq. (2.5)

Finally we wish to illustrate how results (A.10)-(A.12) can be used to derive Eq. (2.5) in the text. First, by the Baker-Hausdorff theorem:

$$[L_\phi, e^{\tilde{\Delta}}] = e^{\tilde{\Delta}} \left\{ [L_\phi, \tilde{\Delta}] + \frac{1}{2} [[L_\phi, \tilde{\Delta}], \tilde{\Delta}] + \dots \right\} \quad (\text{A.13})$$

In order to agree with the notation of the Appendix we must change variables from x to $1/x$, etc., as compared with the text

$$L_\phi = \oint \frac{dx}{2\pi i x} : P(x)^2 : \frac{1}{2} \phi\left(\frac{1}{x}\right)$$

$$\tilde{\Delta} = \oint \frac{dx}{2\pi i x} \frac{dy}{2\pi i y} : P(x) P(y) : \frac{1}{2} g\left(\frac{1}{x} - \frac{1}{y}\right)$$

According to (A.10) the first commutator in (A.13) is given by

$$\begin{aligned} & -\frac{1}{2} \oint \frac{dy_1}{2\pi i y_1} \frac{dy_2}{2\pi i y_2} : P(y_1) \cdot P(y_2) : \left[\phi\left(\frac{1}{y_1}\right) y_1 \frac{\partial}{\partial y_1} + \phi\left(\frac{1}{y_2}\right) y_2 \frac{\partial}{\partial y_2} \right] g\left(\frac{1}{y_1} - \frac{1}{y_2}\right) \\ & + \frac{D}{2} \oint \frac{dy_1}{2\pi i} \frac{dy_2}{2\pi i} \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g\left(\frac{1}{y_1} - \frac{1}{y_2}\right) \frac{y_1 \phi(1/y_1) - y_2 \phi(1/y_2)}{y_2 - y_1} \end{aligned}$$

Now changing back variables $y_i \rightarrow 1/y_i$ this becomes:

$$\begin{aligned} & \frac{1}{2} \oint \frac{dy_1}{2\pi i y_1} \frac{dy_2}{2\pi i y_2} : P(1/y_1) \cdot P(1/y_2) : \left[\phi(y_1) y_1 \frac{\partial}{\partial y_1} + \phi(y_2) y_2 \frac{\partial}{\partial y_2} \right] g(y_1 - y_2) \\ & + \frac{D}{2} \oint \frac{dy_1}{2\pi i} \frac{dy_2}{2\pi i} \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} g(y_1 - y_2) \frac{y_2 \phi(y_1) - y_1 \phi(y_2)}{y_2 - y_1} \end{aligned}$$

The c number term is evaluated by assuming $|y_1| > |y_2|$ and splitting it into two parts one with $\phi(y_1)$, the other with $\phi(y_2)$. These two parts are singular at $y_1 = y_2$. The first part vanishes since the integrand inside the y_2 contour is analytic. The second part is evaluated by the residue theorem since the only singularity inside the y_1 contour is the pole at y_2 : (we are assuming g is analytic and that ϕ may be singular at 0)

$$- \frac{D}{2} g''(0) \oint \frac{dy_1}{2\pi i} y_1 \phi(y_1)$$

Using again the analyticity of g and the results (A.11) and (A.12) we soon see that the second commutator in (A.13) gives the remaining term in (A.5) and that all subsequent commutators vanish.

APPENDIX B

Here we shall give an explicit proof of Eq. (3.4) which we rewrite in the form

$$|x\rangle = (\tau-1) m_N |tree\rangle = L_{-1} |\phi_1\rangle + (L_{-2} + \frac{3}{2} L_{-1}^2) |\phi_2\rangle \quad (B.1)$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ satisfy Eqs. (3.5) and (3.6). Goddard and Thorn³⁾ showed that the right-hand side of the above equation is both spurious and physical (and has hence zero norm). It is not difficult to see that the left-hand side is also both spurious and physical. [Spurious because of the structure of τ^{-1} explained below, physical because

$$L_n \tau = 0 \quad L_n m_N |tree\rangle = 0 \quad n \geq 1 \quad (B.2)$$

Lacking a proof that all spurious physical states can be written in the form of the right-hand side of the above equation we shall prove it explicitly for the above state $|x\rangle$ by examining the detailed form of τ

$$\tau = \oint \frac{dy}{2\pi i y} y^E \quad (B.3)$$

where the integration encircles the origin and

$$E = (L_0 - 1)(D_0 - 1) + \sum_{n=1}^{\infty} (L_{-n} D_n + D_{-n} L_n) \quad (B.4)$$

L_n is the usual Virasoro operator and

$$D_n = \oint \frac{dx}{2\pi i x} x^n \frac{1}{k \cdot P(x)} \quad (B.5)$$

where k is a certain lightlike vector. Following the method of Ref. 1) we see that

$$|x\rangle = \sum_{n=1}^{\infty} L_{-n} D_n \oint \frac{dy}{2\pi i y} \int_1^y dx x^{E-1} m_N |tree\rangle \quad (B.6)$$

and this demonstrates the spurious nature of $|\chi\rangle$. Now according to the Virasoro algebra each L_{-n} $n \geq 2$ can be expressed as a multiple $(n-2^{\text{th}})$ commutator of L_{-1} with L_{-2} :

$$L_{-n} = \frac{1}{(n-2)!} [L_{-1}, [L_{-1}, \dots [L_{-1}, L_{-2}] \dots]] = \frac{1}{(n-2)!} [L_{-1}, L_{-2}]_{n-2} \quad (\text{B.7})$$

But by the Baker-Hausdorff theorem

$$e^{xA} B e^{-xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} [A, B]_n \quad (\text{B.8})$$

we see that

$$\begin{aligned} \sum_{n=1}^{\infty} x^n L_{-n} &= x L_{-1} + x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} [L_{-1}, L_{-2}]_{n-2} \\ &= x L_{-1} + x^2 e^{xL_{-1}} L_{-2} e^{-xL_{-1}} \end{aligned} \quad (\text{B.9})$$

Hence we can rewrite $|\chi\rangle$ as

$$|\chi\rangle = [L_{-1} D_1 + \oint \frac{dx}{2\pi i x} e^{xL_{-1}} L_{-2} e^{-xL_{-1}} \frac{x^2}{k \cdot P(x)}] |\eta\rangle \quad (\text{B.10})$$

where

$$|\eta\rangle = \oint \frac{dy}{2\pi i y} \int_1^y dx x^{E-1} m_N |tree\rangle \quad (\text{B.11})$$

and this shows how $|\chi\rangle$ is decomposed into two parts, one beginning with L_{-1} and one with L_{-2} . Comparing with Eq. (B.1) we have explicitly that

$$|\phi_2\rangle = \oint \frac{dx}{2\pi i x} e^{-xL_{-1}} \frac{x^2}{k \cdot P(x)} |\eta\rangle \quad (\text{B.12})$$

and we now want to show that this indeed satisfies Eq. (3.6). After that we shall show indirectly that $|\phi_1\rangle$ will satisfy Eq. (3.5). For the proof of Eq. (3.6) we need the following formulae

$$\begin{aligned}
 L_n |\eta\rangle &= L_n \oint \frac{dy}{2\pi i y} \int_1^y dx x^{E-1} m_N |tree\rangle \\
 &= \oint \frac{dy}{2\pi i y} \int_1^y dx x^{E-n-1} L_n m_N |tree\rangle \\
 &= \oint \frac{dy}{2\pi i y} (y^{E-n} - 1) \frac{1}{E-n} L_n m_N |tree\rangle = 0
 \end{aligned} \tag{B.13}$$

where we used that $L_n E = (E-n)L_n$ and the fact that the eigenvalue of $E \leq 0$ together with (B.2). Analogously we can prove that

$$(L_0 - 1) |\eta\rangle = 0 \tag{B.14}$$

With the help of Eq. (B.8) one also finds

$$L_n e^{-xL-1} = e^{-xL-1} \sum_{k=0}^{n+1} \binom{n+1}{k} (-x)^k L_{n-k} \quad n \geq -1 \tag{B.15}$$

Finally, we need

$$\begin{aligned}
 [L_n, \frac{1}{k \cdot P(x)}] &= \frac{1}{k \cdot P(x)} [k \cdot P(x), L_n] \frac{1}{k \cdot P(x)} \\
 &= -\frac{n x^n}{k \cdot P(x)} + x^{n+1} \frac{d}{dx} \frac{1}{k \cdot P(x)}
 \end{aligned} \tag{B.16}$$

since

$$[L_n, p^\mu(x)] = x \frac{d}{dx} [x^n p^\mu(x)] \tag{B.17}$$

Consider so for $n > 0$, using (B.13), (B.14), (B.15) and (B.16)

$$\begin{aligned}
 L_n |\phi_2\rangle &= \oint \frac{dx}{2\pi i x} x^2 L_n e^{-xL-1} \frac{1}{k \cdot P(x)} |\eta\rangle \\
 &= \sum_{k=0}^{n+1} \binom{n+1}{k} \oint \frac{dx}{2\pi i x} x^2 e^{-xL-1} (-x)^k L_{n-k} \frac{1}{k \cdot P(x)} |\eta\rangle
 \end{aligned}$$

$$\begin{aligned}
&= \oint \frac{dx}{2\pi i} x^2 e^{-xL-1} \left[(-x)^{n+1} L_{-1} \frac{1}{k \cdot P(x)} + (n+1) (-x)^n \frac{1}{k \cdot P(x)} \right. \\
&\quad \left. + \sum_{k=0}^n \binom{n+1}{k} (-x)^k \left[- \frac{(n-k) x^{n-k}}{k \cdot P(x)} + x^{n-k+1} \frac{d}{dx} \frac{1}{k \cdot P(x)} \right] \right] | \eta \rangle \\
&= \oint \frac{dx}{2\pi i} x^2 e^{-xL-1} \left[(-x)^{n+1} L_{-1} \frac{1}{k \cdot P(x)} + (n+1) (-x)^n \frac{1}{k \cdot P(x)} \right. \\
&\quad \left. + (-x)^n \frac{1}{k \cdot P(x)} + (-1)^n x^{n+1} \frac{d}{dx} \frac{1}{k \cdot P(x)} \right] | \eta \rangle \\
&= (-1)^n \oint \frac{dx}{2\pi i} \frac{d}{dx} \left[x^{n+2} e^{-xL-1} \frac{1}{k \cdot P(x)} \right] = 0
\end{aligned} \tag{B.18}$$

Similarly, one finds

$$(L_0 + 1) | \phi_2 \rangle = 0 \tag{B.19}$$

To prove that

$$L_n | \phi_1 \rangle = 0 \quad n \geq 0 \tag{B.20}$$

one can proceed as above by extracting $| \phi_1 \rangle$ out of (B.10) and then use the technique above. This is, however, very complicated and we shall proceed in a somewhat more indirect way. We have verified that the state $| \phi_2 \rangle$ defined by (B.12) satisfies

$$(L_n + \delta_{n0}) | \phi_2 \rangle = 0 \quad n \geq 0 \tag{B.21}$$

Goddard and Thorn ³⁾ showed that this implies that

$$(L_n - \delta_{n0}) (L_{-2} + \frac{3}{2} L_{-1}^2) | \phi_2 \rangle = 0 \tag{B.22}$$

Since $(L_n - \delta_{n0}) | x \rangle = 0$ it follows that

$$(L_n - \delta_{n0}) L_{-1} |\phi_i\rangle = 0 \quad n \geq 0 \quad (\text{B.23})$$

Using the Virasoro algebra one deduces from this that

$$L_{-1} L_0 |\phi_i\rangle = 0 \quad (\text{B.24})$$

$$L_{-1} L_1 |\phi_i\rangle = -2L_0 |\phi_i\rangle \quad (\text{B.25})$$

$$L_{-1} L_2 |\phi_i\rangle = -L_1 |\phi_i\rangle \quad (\text{B.26})$$

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Our final result (B.20) will follow from these equations via the lemma

$$L_{-1} |\psi\rangle = 0 \quad \Rightarrow \quad |\psi\rangle = 0 \quad (\text{B.27})$$

which we now prove.

Goddard and Thorn ³⁾ showed that any state, such as $|\psi\rangle$ can be expressed as a linear combination of the linearly independent set

$$L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \dots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \dots K_{-m}^{\mu_m} A_{-1}^{\nu_1} \dots A_{-l}^{\nu_l} |0\rangle \quad (\text{B.28})$$

where $K_n = k \cdot \alpha_n$ and A_{-n}^i are the Del Giudice-Di Vecchia-Fubini operators ¹³⁾. But $L_{-1} |\psi\rangle$ is a similar sum of linearly independent states with the same numerical coefficients entering. It is zero only if their numerical coefficients are all zero and this in turn implies that $|\psi\rangle$ is zero.

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fig. 1a

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fig. 1b

fig. 2