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TRANSFORMATION PROPERTIES OF PROPER VERTICES
IN GAUGE THEORIES

Benjamin W. Lee *)

CERN - Geneva
and

Institute for Theoretical Physics, State
University of New York, Stony Brook +)

A B S T R A C T

The Ward-Takahashi identities for (single-particle irreducible) proper vertices are derived for non-Abelian gauge theories ; the renormalization of such theories, previously discussed in terms of the Ward-Takahashi identities for Green functions, can now be considerably simplified.

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+) Permanent address.

In discussing the renormalizability and unitarity of the S matrix in (spontaneously broken) gauge theories, the Ward-Takahashi (WT) identities satisfied by Green functions play a central rôle. The renormalization procedure is usually stated in terms of proper (i.e., single-particle irreducible) vertices, so that the WT identities written in terms of proper vertices would enormously simplify the discussion of the renormalization procedure. It is the purpose of this note to provide a derivation of such identities. In previous discussions^{1),2)} of the renormalizability of these theories, one had to contend with^{*)} the WT identities for Green functions, and a considerable amount of work was necessary to extract information on proper vertices therefrom.

We shall adopt the convention previously used³⁾ that all fields [gauge fields $b_\mu^\alpha(x)$ and scalar fields $\psi_a(x)$] are represented by ϕ_i where i stands for all attributes of the fields. Thus for the gauge field $b_\mu^\alpha(x)$ i stands for the group index α , the Lorentz index μ and the space-time variable x , collectively. Summation and integration over repeated indices shall be understood. Thus, for example,

$$\phi_i^2 = \phi_i \phi_i = \int d^4x \left\{ -\sum_\alpha b_\mu^\alpha(x) b_\nu^\alpha(x) g^{\mu\nu} + \sum_a \psi_a(x) \psi_a(x) \right\}$$

The infinitesimal local gauge transformation of ϕ_i may be written as

$$\phi_i \rightarrow \phi_i' = \phi_i + (\Lambda_i^\alpha + t_{ij}^\alpha \phi_j) \omega_\alpha$$

where $\omega_\alpha = \omega_\alpha(x_\alpha)$ is the space-time dependent parameter of a compact Lie group G . Λ_i^α is of the form

$$\begin{aligned} \Lambda_i^\alpha &= \left[\frac{1}{g} \right]^{\alpha\beta} \partial_\mu \delta^4(x_\alpha - x), \quad \text{if } \phi_i = b_\mu^\beta(x), \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

*) This was especially true in the first-cited reference in 1).

where $g_{\alpha\beta} = g_\alpha \delta_{\alpha\beta}$ is the gauge coupling matrix ^{*}). We have

$$t_{ik}^\alpha (t_{kj}^\beta \phi_j + \Lambda_k^\beta) - t_{ik}^\beta (t_{kj}^\alpha \phi_j + \Lambda_k^\alpha) = f_{\alpha\beta\gamma} (t_{ik}^\gamma \phi_k + \Lambda_i^\gamma)$$

where $f_{\alpha\beta\gamma}$ is the structure constant of the local gauge group. The invariance of the Lagrangian under this gauge transformation may be formulated as

$$(\Lambda_i^\alpha + t_{ij}^\alpha \phi_j) \frac{\delta L[\phi]}{\delta \phi_i} = 0. \quad (1)$$

We shall always use a real basis for ϕ_i , so that the representation matrix of the Lie algebra $[t^\alpha]_{ij} = t_{ij}^\alpha$ is real antisymmetric.

In quantizing such a Lagrangian, we must choose a gauge. We consider a gauge condition linear in ϕ :

$$F_\alpha[\phi] = F_{\alpha i} \phi_i = 0$$

Usually, one chooses $F_{\alpha i}$ to be

$$F_{\alpha i} = \frac{1}{\sqrt{\alpha}} g^{\alpha\beta} \Lambda_i^\beta$$

if ϕ_i is a gauge field,

$$= \frac{\sqrt{\alpha}}{\xi} c_i^\alpha \quad (2)$$

if ϕ_i is a scalar field, where α, ξ are gauge fixing parameters, and c_i^α are some constants. In this gauge, the Feynman rules for constructing Green functions are deduced from the effective action S_{eff} ⁴⁾

$$S_{\text{eff}}[\phi] = L[\phi] - \frac{1}{2} \{F_\alpha[\phi]\}^2 + \bar{c}_\alpha M_{\alpha\beta}[\phi] c_\beta \quad (3)$$

^{*}) If the group G is a direct product of n simple groups, $G_1 \otimes G_2 \otimes \dots \otimes G_n$, there are in general n gauge couplings $g_1, g_2, \dots, g_n$. Within the same factor group G_i , of course $g_\alpha = g_\beta$.

where $\bar{c}$ and c are fictitious anticommuting fields which generate the so-called de Witt-Faddeev-Popov ghost loops ^{5),6)} and $M_{\alpha\beta}$ is given by

$$M_{\alpha\beta}[\phi] = F_{\alpha i}(\Lambda_i^\beta + t_{ij}^\beta \phi_j). \quad (3')$$

The operator M is in general not Hermitian so that the ghost line is orientable.

In the following we shall be dealing with unrenormalized fields and coupling constants, and dimensionally regularized ⁷⁾ quantities. As has been shown elsewhere ³⁾, the generating functional $W[J]$ of Green functions satisfies the WT identity ^{*)}

$$\left\{ -F_{\alpha} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] + J_i \left(\Lambda_i^\beta + t_{ij}^\beta \frac{1}{i} \frac{\delta}{\delta J_j} \right) M^{-1}_{\beta\alpha} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} \times W[J] = 0. \quad (4)$$

The quantity $M^{-1} \left[\frac{\delta}{i} \frac{\delta J}{\delta J} \right] W[J]$ is the Green function for the fictitious field c in the presence of the external source J .

The functional $Z[J]$ defined by

$$W[J] = e^{iZ[J]}$$

is the generating functional of the connected Green functions. The Legendre transform $\Gamma[A]$ of $Z[J]$:

$$\begin{aligned} \Gamma[A] &= Z[J] - J_i A_i, \\ A_i &= \delta Z[J] / \delta J_i, \\ -J_i &= \delta \Gamma[A] / \delta A_i, \end{aligned} \quad (5)$$

is the generating functional of proper vertices ⁸⁾. For later use we define $X_{ij}[A]$ by

*) An equivalent diagrammatic derivation of Eq. (4) is given in Ref. 2).

$$\begin{aligned} X_{ij}[A] &= -\delta A_i / \delta J_j \\ X_{ik}[A] \delta^2 \Gamma[A] / \delta A_k \delta A_j &= \delta_{ij} \end{aligned} \quad (6)$$

i.e., $X_{ij}[A]$ are the propagators when the fields ϕ are constrained to have the vacuum expectation values A .

Now we are in a position to write down the WT identity for $\Gamma[A]$. First observe that

$$M_{\beta\alpha}^{-1} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] e^{iZ[J]} = e^{iZ[J]} M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right] \quad (7)$$

where

$$A_i + \frac{1}{i} \frac{\delta}{\delta J_i} = A_i + i X_{ij}[A] \frac{\delta}{\delta A_j} = e^{-iZ[J]} \frac{1}{i} \frac{\delta}{\delta J_i} e^{iZ[J]}.$$

Recalling that F_α is linear in ϕ , we write Eq. (4) as

$$\begin{aligned} -F_\alpha[A] - \frac{\delta \Gamma[A]}{\delta A_i} \left[\Lambda_i^\beta + t_{ij}^\beta \left(A_j + i X_{jk}[A] \frac{\delta}{\delta A_k} \right) \right] \\ \times M_{\beta\alpha}^{-1} \left[A + i X \frac{\delta}{\delta A} \right] = 0. \end{aligned} \quad (8)$$

Now define $\Gamma_0[A]$ by

$$\Gamma[A] = \Gamma_0[A] - \frac{1}{2} \{F_\alpha[A]\}^2 \quad (9)$$

Then from Eq. (8) follows

$$0 = \frac{\delta \Gamma_0[A]}{\delta A_i} \left[\Lambda_i^\beta + t_{ij}^\beta \left(A_j + \frac{1}{i} \frac{\delta}{\delta J_j} \right) \right] M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right] \quad (10)$$

We must now examine

$$\begin{aligned}
 & t_{ij}^{\beta} \frac{1}{i} \frac{\delta}{\delta J_j} M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right] \\
 &= i t_{ij}^{\beta} X_{jk} [A] \frac{\delta}{\delta A_k} M_{\beta\alpha}^{-1} \left[A + i X \frac{\delta}{\delta A} \right] \\
 &\equiv \gamma_i^{\beta} [A] M_{\beta\alpha}^{-1} \left[A + i X \frac{\delta}{\delta A} \right]. \tag{11}
 \end{aligned}$$

Upon defining ^{*)}

$$G_{\beta\alpha}^{-1} [A] = M_{\beta\alpha}^{-1} \left[A + i X \frac{\delta}{\delta A} \right] 1$$

so that $G_{\beta\alpha}^{-1} [A]$ is the ghost propagator when the fields ϕ are constrained to have the vacuum expectation values A , we can solve Eq. (11) for $\gamma_i^{\alpha} [A]$:

$$\gamma_i^{\alpha} [A] = -i t_{ij}^{\beta} X_{jk} [A] G_{\beta\gamma} [A] \frac{\delta}{\delta A_k} G_{\gamma\alpha}^{-1} [A] \tag{12}$$

Now Eq. (10) may be written as

$$L_{\alpha i} [A] \frac{\delta}{\delta A_i} \Gamma_0 [A] = 0 \tag{13}$$

where

$$L_{\alpha i} [A] = \Lambda_i^{\alpha} + t_{ij}^{\alpha} A_j + \gamma_i^{\alpha} [A] \tag{14}$$

and

$$G_{\beta\alpha}^{-1} [A] = F_{\beta i} L_{\alpha i} [A] \tag{15}$$

^{*)} Here one must distinguish between an operator equation such as $(1+x\partial)^2 = 1+x\partial+x^2\partial^2$ and the equation resulting from its action on a certain function of x , such as $(1+x\partial)^2 1 = 1$.

Note that $G^{-1} [A]$ is the generating functional of proper vertices with two ghost lines, so that

$$G^{-1}_{\beta\alpha} \equiv G^{-1}_{\beta\alpha} [0]$$

is the inverse ghost propagator :

$$\gamma_{\beta\alpha,i} \equiv \frac{\delta^2 G^{-1}_{\beta\alpha} [0]}{\delta A_i}$$

is the proper vertex of two ghosts at α and β and the field at i , etc.

The system, Eqs. (12)-(15), is our main result. Equation (13) defines a non-linear gauge transformation which leaves $\Gamma_0[A]$ invariant. It is a generalization of Eq. (1) and reduces to it in the tree approximation.

On the basis of Eqs. (12)-(15), the considerations of Ref. 1) can be generalized readily to any compact Lie groups and arbitrary representations of scalar fields ^{*}). Furthermore it can be shown that the divergent parts of the renormalization constants are insensitive to the parameter ξ in Eq. (2) and therefore the renormalized S matrix is independent of the gauges ^{**),***)}. The details have been worked out and will be reported elsewhere.

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^{*}) The discussions in Refs. 1) were tailored to the group $SU(2)$ and other completely reducible compact Lie groups (i.e., groups in which the product of two reducible representations R and R' contains a third R'' at most once).

^{**}) The renormalizability in the most general case, and the gauge independence of the renormalized S matrix have also been worked out, prior to me, by G. 't Hooft and M. Veltman by means of their diagrammatic analysis (private communication) and by J. Zinn-Justin (unpublished manuscript) on the basis of Eq. (4).

^{***}) I have not fully investigated the case in which F_α is of dimension two and not linear in fields.

REFERENCES

- 1) B.W. Lee and J. Zinn-Justin - Phys.Rev. D5, 3121, 3137, 3155 (1972).
- 2) G. 't Hooft and M. Veltman - Nuclear Phys. B50, 318 (1972).
- 3) B.W. Lee and J. Zinn-Justin - Phys.Rev. D7, 1049 (1973).
- 4) G. 't Hooft - Nuclear Phys. B33, 173 (1971).
- 5) B.S. de Witt - Phys.Rev. 162, 1195, 1239 (1967).
- 6) L.D. Faddeev and V. Popov - Phys.Letters 25B, 29 (1967).
- 7) G. 't Hooft and M. Veltman - Nuclear Phys. B44, 189 (1972).
- 8) G. Jona-Lasinio - Nuovo Cimento 34, 1790 (1964).

E R R A T U M

On page 5, lines 5 and 6, G^{-1} should have been G .

Thus, lines 5 and 6 should read :

" $G_{\beta\alpha} [A] = \dots$

so that $G_{\beta\alpha} [A]$ is..."