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A MODEL STUDY IN HADRON STATISTICAL BOOTSTRAPR. Hagedorn and I. Montvay^{*)}

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ABSTRACT

In the framework of the statistical bootstrap the decay of a fireball is considered as an exact inverse of its statistical composition. This assumption leads to a bootstrap formulated in terms of integral equations for all kinds of distributions of the fireball's decay products. We obtain solutions of the equations in terms of power series and of K-transforms and determine in the general case their asymptotic behaviour for large fireball mass. Relations to a thermodynamical description are established and illustrated by effective temperatures. The approach to the asymptotic limits is easy to investigate in a simplified linear bootstrap where the K-transforms can be more explicitly calculated.

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1. INTRODUCTION

Highly excited hadronic matter (we call it a fireball) has many properties which make a statistical-thermodynamical description desirable. The attempt to equip such a description with strong interactions in a self-consistent way has led one to consider that a fireball is composed of a variable number of hadrons of all types including resonances, and to extrapolate the notion of resonance beyond the well-established and tabulated ones into the region of large masses, where resonances and fireballs are identified [1]. If only we could keep a fireball together for a long time, then its composition would be ever changing (with a time scale of $t_0 \approx 1/m_\pi$) because inside of it all possible high-energy reactions with creation and absorption of any number of all kinds of hadrons would go on forever. The general philosophy of the model is, then, that this ensemble of all possible high-energy reactions is a (hopefully) complete global image of strong interactions. This has been emphasized and made plausible in Refs [2] and [3] on the basis of the classical work of Beth and Uhlenbeck [4] and of Belenkij [5].

In reality we cannot produce fireballs in the strict sense of equilibrium configurations, because highly excited hadronic matter occurs in the laboratory only in high-energy collisions, where the final state remembers the fast initial relative motion via angular momentum conservation and phase relations between partial waves. A pragmatic way of dealing with this difficulty is to introduce collective motions in the collision direction and to apply the fireball picture to the situation as seen by a co-moving observer in his local environment. This semi-empirical technique has been described in detail in [6] and in subsequent articles on inclusive particle production spectra. We shall not enter into these questions here; on the contrary, we will assume that ideal fireballs do exist.

Our present aim is to study the general structure of a statistical bootstrap model in the simplest non-trivial versions; our results are meant to exhibit the common typical behaviour of such models, not to be compared to experiments. We shall also point to a few side issues which suggest interesting relations to field theory, to thermodynamics, and to critical phenomena in many-body physics.

The statistical thermodynamical bootstrap was first formulated in the convenient language of the canonical (or grand canonical) ensemble [1] with the then shocking result that the hadronic mass spectrum had to grow exponentially and that therefore a universal limiting temperature T_0 exists.

The analysis was then first reformulated in the micro-canonical (phase space) language by Frautschi [7], and on the basis of his work Nahm [8], Frautschi and Hamer [9, 10, 11], and Yellin [12] advanced our understanding considerably.

We shall base our present calculations on the work of Yellin because of its great formal simplicity (and in spite of it not being realistic); we hope that it thus may serve as a transparent model, illustrating the beautiful and important, more realistic work of Frautschi and Hamer and of Nahm.

We thus postulate the following model:

Hadrons are composite states of hadrons in the sense of the statistical-thermodynamical bootstrap [1, 2, 7, 12] with the following simplifications:

- i) there is one single, uncharged, scalar input particle of mass m , called "pion";
- ii) we use Boltzmann statistics;
- iii) we use Lorentz-invariant momentum space.

With these simplifying assumptions Yellin derived the first explicit analytical solution of the bootstrap; its asymptotic behaviour agreed with that found in earlier work [1, 7, 8, 9] where these simplifying assumptions were not or not all made^{*)}. As the asymptotic form of the mass spectrum is the most relevant feature, we consider the above simplifications as non-essential.

To the postulates (i) to (iii) we now add another one:

- iv) the decay structure of a fireball (i.e. the ensemble of all processes leading from a fireball to its ultimate pion distribution) is the exact counterpart of its composition.

*) In [1] and [7] a class of asymptotic solutions was found, to which Yellin's result belongs.

This principle -- henceforward called "reciprocity" -- has been the tacitly assumed basis of all attempts to calculate particle production from the decay of a fireball, in particular in Refs. [6, 7, 9, 10]. The first model, in which it was exactly fulfilled, was recently described by one of us [13]. Even if fireballs of a bootstrap structure should exist in nature, the reciprocity principle might be broken by final-state interactions during the decay; but it cannot be too badly broken. Here we assume its exact validity.

Let $\tau(M^2)$ be the level density with respect to invariant momentum space^{*)} of a fireball with mass M; then Yellin's version of Frautschi's [7] version of the bootstrap equation can be written ($\hbar = c = 1$)

$$\tau(r^2) = \delta_0(r^2 - m^2) + A \sum_{\ell=2}^{\infty} \frac{B^\ell}{\ell!} \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} \tau(r_i^2) d^4 r_i \quad (1.1)$$

It is tacitly understood that $\tau(r^2)$ contains a factor $\theta(r^2)\theta(r_0 - m)$ and that $\delta_0(r^2 - m^2) = \theta(r_0)\delta(r^2 - m^2)$. With these restrictions, Eq. (1.1) is still invariant under proper orthochronous Lorentz transformations. The constants A and B have dimensions m^2 and m^{-2} , respectively; here B plays the same role as the interaction volume V_0 divided by $h^3 = (2\pi)^3$ does in the non-invariant formulation; therefore^{**) with $r_0 := m^{-1}$}

$$B \approx B_0 := \frac{\pi r_0^2}{h^2} = \frac{1}{4\pi m^2} \quad (1.2)$$

$$\alpha := AB \approx 1$$

Of course τ is a function $\tau(M^2; m, \alpha, B)$, but as we consider m, α, B as given constants of nature, we do not explicitly write them out.

*) If $\rho(M)$ is the level density with respect to non-invariant momentum space, then $\rho(M)$ and $\tau(M^2)$ have the same asymptotic M-dependence.

**) In equations defining a new quantity we use the symbol $:=$ or $=:$, the colon (:) standing next to the new quantity.

If we divide Eq. (1.1) by $\tau(r^2)$ we obtain $1 = \text{sum of positive terms}$ which, in the spirit of the model, are to be interpreted as probabilities:

$$P_\ell(r; r_1, \dots, r_\ell) d^4 r_1 \dots d^4 r_\ell :=$$

$$= \frac{\alpha}{\ell! B\tau(r^2)} \cdot \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} B\tau(r_i^2) d^4 r_i \quad (1.3)$$

The reciprocity postulate now claims that this is not only the probability to find a given fireball of mass $M = \sqrt{r^2}$ in a state where it consists of ℓ sub-fireballs with four momenta $\{r_1, d^4 r_1\} \dots \{r_\ell, d^4 r_\ell\}$, but also the probability that such a fireball will split up into exactly this configuration in its first step of decay. In fact we should rather say it the other way round, because the composition becomes observable only through decay.

We close the introduction with the remark that the recently observed excess of large transverse momenta at the CERN-ISR [14] does not disprove the statistical bootstrap approach, because for most of the particles the slope of the p distribution is still governed by T_0 . Only less than one per cent of the pions show too large transverse momenta; they were probably produced before local equilibrium was established ("superheating" [6, 15]), because at ISR energies the collision time is more than an order of magnitude shorter than the typical strong interaction time unit $t_0 = 1/m_\pi$.

The present work uses notation and results of two previous papers by one of us (I.M., Refs. [13, 16]).

2. INTEGRAL EQUATION FOR THE GENERATING FUNCTIONAL

2.1 Distributions, generating functionals

In order to characterize the pion distribution (within the fireball or in the final state after the decay) it is sufficient to give the probability of the different pion configurations^{*)}, that is

$$V_\ell(r; k_1, \dots, k_\ell) d^4 k_1 \dots d^4 k_\ell \quad (2.1)$$

*) The δ -functions restricting the pion four-momenta to the positive energy mass shell ($k_j^2 = m^2$; $k_{0j} > 0$) are contained in $V_\ell(r; k_1 \dots k_\ell)$.

where ℓ is the number of pions, $k_1, \dots, k_\ell$ are the pion four-momenta, and r is the four-momentum of the fireball. The pion mass will be denoted by m and the mass of the fireball by M . In a true composite state ℓ is always at least 2. For notational simplicity we include, however, also $\ell = 1$ when the fireball consists of a single pion; that is:

$$V_1(r; k_1) = \begin{cases} \delta^4(r - k_1) & \text{if } r^2 = m^2; r_0 > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

The normalization condition is

$$\sum_{\ell=1}^{\infty} V_\ell(r^2) := \sum_{\ell=1}^{\infty} \int d^4 k_1 \dots d^4 k_\ell V_\ell(r; k_1, \dots, k_\ell) = 1 \quad (2.3)$$

where $V_\ell(r^2)$ is the total probability for ℓ -pion states.

Instead of using the individual functions $V_\ell(r; k_1, \dots, k_\ell)$ it is more convenient to introduce the generating functional [17] (we shall call it the "pion functional"):

$$V[r; \phi(k)] := \sum_{\ell=1}^{\infty} \int d^4 k_1 \dots d^4 k_\ell V_\ell(r; k_1, \dots, k_\ell) \phi(k_1) \dots \phi(k_\ell) \quad (2.4)$$

The normalization condition Eq. (2.3) is then $V(r; \phi \equiv 1) = 1$. The "exclusive" probabilities $V_\ell(r; k_1, \dots, k_\ell)$ and "inclusive" distributions $D_n(r; k_1, \dots, k_n)$ are given by

$$V_\ell(r; k_1, \dots, k_\ell) = \frac{1}{\ell!} \frac{\delta^\ell}{\delta \phi(k_1) \dots \delta \phi(k_\ell)} V[r; \phi(k)]_{\phi=0} \quad (2.5)$$

$$D_n(r; k_1, \dots, k_n) := \frac{\delta^n}{\delta \phi(k_1) \dots \delta \phi(k_n)} V[r; \phi(k)]_{\phi=1}$$

In particular, the single-particle distribution is^{*)}

$$\mathcal{D}_1(r; k) d^4 k = : \mathcal{D}(r^2, r \cdot k) \delta_0(k^2 - m^2) d^4 k = \mathcal{D}(r^2, r \cdot k) \frac{d^3 k}{2k_0} \quad (2.6)$$

In the special case when the function ϕ is put equal to a constant h , the generating functional goes over into the generating function of the multiplicity distribution

$$\Psi(r^2, h) := V[r; \phi \equiv h] \quad (2.7)$$

and therefore

$$V_\ell(r^2) := \frac{1}{\ell!} \frac{\partial^\ell}{\partial h^\ell} \Psi(r^2, h)_{h=0} \quad (2.8)$$

$$D_m(r^2) := \frac{\partial^m}{\partial h^m} \Psi(r^2, h)_{h=1} = \sum_{\ell=1}^{\infty} \ell(\ell-1)\dots(\ell-m+1) V_\ell(r^2)$$

2.2 The generalized bootstrap equation

In the introduction we defined the probability that a fireball of four-momentum r is composed of (or will decay into) ℓ sub-fireballs with four-momenta $\{r_1, d^4 r_1\} \dots \{r_\ell, d^4 r_\ell\}$. Including for convenience the case $\ell = 1$, we have

$$P_\ell(r, r_1, \dots, r_\ell) d^4 r_1 \dots d^4 r_\ell = \frac{\alpha}{\ell! B\tau(r^2)} \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} B\tau(r_i^2) d^4 r_i \quad (2.9)$$

$$P_1(r, r_1) d^4 r_1 := \begin{cases} \delta^4(r - r_1) d^4 r_1 & \text{if } r^2 = m^2; r_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

*) Note the difference of a factor of 2 in the normalization of $D(r^2, r \cdot k)$ with respect to Ref. [13, 16].

From the fundamental assumption that all the strong interaction is hidden in the mass spectrum $\tau(r^2)$ via a highly non-linear bootstrap equation of type (1.1), it follows that at each instant the ℓ constituents of a fireball are formally free particles, having independently of each other the same internal (and decay) structure as their parent fireball. Therefore the total pion functional is, at fixed composition, the product of the functionals of the constituents; more generally a weighted sum of such products:

$$V[r; \phi(k)] = \sum_{\ell=1}^{\infty} \int P_{\ell}(r; r_1, \dots, r_{\ell}) \prod_{i=1}^{\ell} V[r_i; \phi(k)] d^4 r_i \quad (2.10)$$

so that, with the bootstrap equation (1.1), we find the "generalized bootstrap equation"

$$\begin{aligned} B\tau(r^2) V[r; \phi(k)] &= B\delta_0(r^2 - m^2) \phi(r) + \\ &+ \alpha \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} B\tau(r_i^2) V[r_i; \phi(k)] d^4 r_i \end{aligned} \quad (2.11)$$

From the normalization condition $V[r, \phi(k) \equiv 1] = 1$ it follows that if in Eq. (2.11) we put $\phi \equiv 1$, then it reduces to the bootstrap equation (1.1) for the level density $\tau(M^2)$; with $\phi(k) \equiv h$ we obtain from Eq. (2.7) the bootstrap equation for the multiplicity distribution [note that we assume the pion to be non-degenerate, thus the parameter D of Eq. (21) of Ref. [13] is here put equal to one]. The first functional derivative at $\phi = 1$ gives the bootstrap equation for the single-particle distribution, etc.

Hence Eq. (2.11) is the most transparent expression of our reciprocity postulate: this single equation describes the composition of a fireball together with all its decay properties. Therefore it should be obvious that here the final pion distribution is not the result of a particular decay chain assumed *ad hoc* (for instance a two-body decay chain [9, 10, 11, 16]) but of a superposition of all possible decay sequences.

2.3 Solution of the generalized bootstrap equation

In order to stress the relation to statistical thermodynamics, where the partition function is the Laplace transform of the level density, we use here a covariant Laplace transformation to solve the generalized bootstrap equation (2.11). To this end we define a time-like four-vector b , a temperature T , and its inverse β by

$$\begin{aligned} b &= (b_0, \vec{b}) \\ b^2 &> 0 \quad ; \quad b_0 > 0 \\ \beta &:= \frac{1}{T} := \sqrt{b^2} \end{aligned} \tag{2.12}$$

and introduce the "partition functionals" of the fireball and the one-pion states, respectively [remember Eq. (2.2)]:

$$\begin{aligned} Z[b; \phi(k)] &:= \int d^4k e^{-b \cdot k} B \tau(k^2) V[k; \phi(k)] \\ z[b; \phi(k)] &:= \int d^4k e^{-b \cdot k} B \delta_0(k^2 - m^2) \phi(k) \end{aligned} \tag{2.13}$$

By the same covariant Laplace transformation Eq. (2.11) goes over into

$$Z[b; \phi] = z[b; \phi] + \alpha \left\{ e^{Z[b; \phi]} - Z[b; \phi] - 1 \right\} \tag{2.14}$$

Interpreting this equation as a relation between Z and z , where $z[b, \phi(k)]$ is a known functional, we write

$$\begin{aligned} G(z[b; \phi(k)]) &:= Z[b; \phi(k)] \\ z &= (1 + \alpha) G(z) - \alpha e^{G(z)} + \alpha \end{aligned} \tag{2.15}$$

The last equation is plotted in Fig. 1.

Equations of this type have been used first by Nahm [8] and recently by Yellin [12]. The solution of the bootstrap equation is equivalent to solving Eq. (2.15) for $G(z)$ -- which is simple enough graphically but not so simple analytically. Assume $G(z)$ to be expressed as a power series (explicit expressions for its coefficients g_ℓ are given in Appendix C):

$$G(z) = \sum_{\ell=1}^{\infty} g_\ell z^\ell \quad (2.16)$$

then we have

$$\begin{aligned} Z[b; \phi(k)] &= G(z[b; \phi(k)]) = \\ &= \sum_{\ell=1}^{\infty} g_\ell \int \prod_{i=1}^{\ell} \{ d^4 r_i e^{-b r_i} B \delta_0(k_i^2 - m^2) \phi(k_i) \} = \end{aligned} \quad (2.17)$$

$$= \int d^4 r e^{-b r} \left\{ \sum_{\ell=1}^{\infty} g_\ell \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} d^4 r_i B \delta_0(k_i^2 - m^2) \phi(k_i) \right\}$$

Comparing with (2.13) yields the power series solution of the generalized bootstrap equation (2.11):

$$\begin{aligned} B\tau(k^2) V[r; \phi(k)] &= \\ &= \sum_{\ell=1}^{\infty} g_\ell \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \left\{ \prod_{i=1}^{\ell} d^4 r_i B \delta_0(k_i^2 - m^2) \phi(k_i) \right\} \end{aligned} \quad (2.18)$$

For $\phi \equiv 1$ this reduces to Yellin's [12] explicit solution for $\tau(r^2)$; this being thus known, we can divide Eq. (2.18) by $\tau(r^2)$ and obtain $V[r; \phi(k)]$.

From this expression it is easy to obtain all kinds of pion distributions. For instance, the "exclusive" probabilities defined in Eq. (2.5) are

$$V_\ell(r; k_1, \dots, k_\ell) = \frac{g_\ell}{B\tau(r^2)} \delta^4(r - \sum_{i=1}^{\ell} k_i) \prod_{i=1}^{\ell} B \delta_0(k_i^2 - m^2) \quad (2.19)$$

This is the same as Eq. (24) in Ref. [13] (given there without proof). It should be pointed out that the form of Eq. (2.19) is very natural in any kind of model assuming statistical decay of some excited objects (see, for instance, [18]). It states that the pion distribution is proportional to phase space, the coefficients being given by g_ℓ . In other words, it says that the averaged squared transition matrix element between a fireball and an ℓ -pion state is

$$|\langle k_1, \dots, k_\ell | T | r \rangle|^2_{av} = \frac{g_\ell B^\ell}{B\tau(r^2)} \quad (2.20)$$

In the statistical bootstrap model the coefficients g_ℓ are determined by Eq. (2.15), but in more general kinds of models g_ℓ can be something else (perhaps depending also on r^2). In that case $\tau(r^2)$ no longer has the meaning of a density of states but it is simply a normalization factor of the probability distribution.

Once the pion distributions $V_\ell(r; k_1, \dots, k_\ell)$ or the pion functional V are known, it is also easy to calculate any kind of distributions of inclusive type. From Eq. (2.18) one obtains an infinite series representation for these quantities.

2.4 Qualitative discussion of the bootstrap equation

If in Eqs. (2.13) we put $\phi \equiv 1$ the functionals become invariant functions of b^2 , that is, of T . We can then choose a frame of reference where $b = (1/T, 0, 0, 0)$. Furthermore we introduce a factor $1 = \int \delta_0(M^2 - r^2) dM^2$ into the first of equations (2.13) and obtain with $r = (\sqrt{\vec{p}^2 + M^2}, \vec{p})$:

$$Z(T, B) = \int_m^\infty B \tau(M^2) dM^2 \int_0^\infty \exp \left[- \frac{\sqrt{\vec{p}^2 M^2}}{T} \right] \frac{2\pi \vec{p}^2 dp}{\sqrt{\vec{p}^2 + M^2}} \quad (2.21)$$

$$z(T, m) = \int_0^\infty B \exp \left[- \frac{\sqrt{\vec{p}^2 + m^2}}{T} \right] \frac{2\pi \vec{p}^2 dp}{\sqrt{\vec{p}^2 + m^2}} = 2\pi m B T K_1 \left(\frac{m}{T} \right)$$

where K_1 is the first modified Hankel function. Since the momentum integral is the same in both cases, we have

$$Z(T, B) = \int_m^\infty B \tau(M^2) 2\pi M T K_1 \left(\frac{M}{T} \right) dM^2 \equiv \int_m^\infty \tau(M^2) z(T, M) dM^2 \quad (2.22)$$

We call z and Z "invariant partition functions" of the single pion and of the fireball, respectively.

We now rewrite Eq. (2.15) in the variables $Z(T) \equiv G(z)$ and $z(T, m) \equiv z$

$$z = (1 + \alpha) Z - \alpha e^Z + \alpha \quad (2.23)$$

Starting from zero we let Z grow to large positive values; z first also grows, reaches a maximum z_0 , and then shoots down exponentially to $-\infty$ [Fig. 1]. At the maximum the second derivative is non-zero, hence Z has a square root singularity^{*)} at z_0 :

$$Z(T, B) = G(z) \approx Z_0 \pm \text{const} \sqrt{z_0 - z} + \dots$$

$$Z_0 =: G_0 = G(z_0) = \ln \left(1 + \frac{1}{\alpha} \right) \quad (2.24)$$

$$z_0 = (1 + \alpha) \ln \left(1 + \frac{1}{\alpha} \right) - 1$$

*) The importance of this fact was first remarked by Nahm [8].

As $z(T)$ grows monotonously with T , there exists a maximum value $T_0(\alpha, B, m)$ such that

$$z_0 = z(T_0, m) = 2\pi m B T_0 K_1\left(\frac{m}{T_0}\right) \quad (2.25)$$

The physical branch of $Z(T)$ then behaves near T_0 like

$$Z(T \lesssim T_0, B) = Z_0 - \text{const} \sqrt{T_0 - T} + \dots \quad (2.26)$$

Equation (2.22) can yield a singularity at T_0 only if $\tau(M^2)$ grows asymptotically like $M^{-a} \exp(M/T_0)$. Using the expansion of K_1 for large argument we obtain from Eq. (2.22)

$$Z(T \rightarrow T_0, B) = F(T_0) + \text{const} \int_m^\infty e^{-M \frac{T_0 - T}{T T_0}} \cdot M^{\frac{3}{2} - a} dM \quad (2.27)$$

A square root singularity requires $a = 3$; we thus re-obtain the results of Nahm [8] and of Yellin [12]:

$$\tau(M^2) \xrightarrow{\text{large } M} e^{\frac{M}{T_0}} \cdot \left(\frac{C_3 m}{M^3} + \dots \right) \quad (2.28)$$

The constant C_3 and the next term in the expansion in inverse powers of M are calculated in Appendix A, Eq. (A.22). We furthermore obtain a simple expression relating T_0 to the free parameters α, B, m [Eqs. (2.24, 25)]:

$$2\pi m B T_0 K_1\left(\frac{m}{T_0}\right) = (1 + \alpha) \ln\left(1 + \frac{1}{\alpha}\right) - 1 \quad (2.29)$$

The relation between m/T_0 and B is shown in Fig. 2 where we assume $\alpha = 1$, while B varies in the neighbourhood of $B_0 = \pi r_0^2 / h^2 = (4\pi m^2)^{-1}$. We do not think that B and m can be independent of each other; m as the lowest mass in the spectrum determines the range of interaction and thus B ; therefore B should not greatly differ from B_0 and consequently T_0 must be approximately equal to m . The value of T_0 belonging to $B = B_0$ is called the "natural limiting temperature"; it is used in all figures.

We close this section with the remark that the covariant Laplace transform can be inverted in much the same way as the ordinary one; in fact Eqs. (2.21, 22) show that it is essentially a K transform. Its well-known inversion allows us to supplement the power series expansion (2.18) of the pion functional by an integral representation. Such representations are derived in Appendix A for $\tau(M^2)$. The K-transform method applies in the same way for the n-particle inclusive distributions or for the generating function of the multiplicity distribution (see at the end of Appendix A). The integral representations obtained are particularly useful for the determination of the asymptotic behaviour. This is shown in detail in the example of $\tau(M^2)$ in Appendix A.

2.5 Relation to standard statistical thermodynamics

Our partition functions Z and z are K transforms of densities of states with fixed total momentum zero, while those of standard statistical thermodynamics (SST) are Laplace transforms of densities of states with non-fixed total momentum which vanishes only on the average. For systems with large mass-energy $M \gg T$ the differences become irrelevant.

It is easy to show that an SST system of total energy M (including rest mass) has a total momentum $\vec{P}$ fluctuating around zero with $\Delta P^2/(2M) \approx (3/2)T$, a very plausible result [see, for example, Ref. [3], Section 3.5].

Instead of the usual level density $\sigma(M,V)$ with non-fixed total momentum, we introduce the level density at zero momentum $\sigma_0(M,V)$, and compensate this by allowing the system of mass M to move freely around in the volume V , i.e. we replace the SST partition function (subscript s for SST)

$$Z_s(T,V) = \int_0^\infty \sigma(M,V) e^{-\frac{M}{T}} dM \quad (2.30)$$

by

$$\begin{aligned}
 Z_s(T, V) &= \frac{V}{(2\pi)^3} \int_0^\infty \sigma_o(M, V) e^{-\frac{\sqrt{\vec{p}^2 + M^2}}{T}} d^3P dM \\
 &= \frac{V}{(2\pi)^3} \int_0^\infty \sigma_o(M, V) 4\pi M^2 T K_2\left(\frac{M}{T}\right) dM
 \end{aligned} \tag{2.31}$$

Usually $M \gg T$; the asymptotic expansion of K_2 yields

$$Z_s(T, V) = \int_0^\infty \sigma_o(M, V) \left[\frac{V}{(2\pi)^3} (2\pi M T)^{3/2} \right] e^{-\frac{M}{T}} dM \tag{2.32}$$

so that in view of Eq. (2.30)

$$\sigma(M, V) \approx \sigma_o(M, V) \left\{ \frac{V}{(2\pi)^3} (2\pi M T)^{3/2} \right\} \tag{2.33}$$

The factor in curly brackets counts the extra levels due to the free movement of M in V ; in fact this factor is very nearly equal to $[V/(2\pi)^3] \cdot \Delta P^3 \cdot 4\pi/3$ with the above $\Delta P \approx \sqrt{3MT}$.

We now compare our invariant Z with Z_s ; of course we have to choose equal volumes, hence

$$V = V_o = \frac{4\pi}{3} (4\pi B_o)^{3/2} \tag{2.34}$$

Then, from Eqs. (2.22, 31)

$$\begin{aligned}
 Z(T, B_o) &= \int_0^\infty B_o \mathcal{I}(M^2) 4\pi M^2 T K_1\left(\frac{M}{T}\right) dM \\
 Z_s(T, V_o) &= \int_0^\infty \left[\frac{4\pi}{3} \left(\frac{B_o}{\pi}\right)^{3/2} \right] \sigma_o(M, V) 4\pi M^2 T K_2\left(\frac{M}{T}\right) dM
 \end{aligned} \tag{2.35}$$

For $M \gg T$ the K-functions have the same asymptotics, hence we find

$$\begin{aligned}
 Z(T, B_0) &\longleftrightarrow Z_s(T, V_0) \\
 \frac{4\pi}{3} (4\pi B_0)^{3/2} &\longleftrightarrow V_0 \\
 \frac{\tau(M^2)}{\sqrt{B_0}} &\longleftrightarrow \frac{4}{3\sqrt{\pi}} \sigma_0(M, V_0)
 \end{aligned} \tag{2.36}$$

From this it follows that and how the usual thermodynamic quantities can be obtained from $Z(T, B)$, where then B may assume any macroscopic value [B_0 only serves to define the bootstrap of $\tau(M^2)$]. In particular the average fireball mass and its relative fluctuations for $T \rightarrow T_0$ are given by [see Eq. (2.26)]

$$\begin{aligned}
 \langle M \rangle_T &:= T^2 \frac{\partial}{\partial T} \ln Z(T, B_0) \approx \text{const} \cdot T_0 \cdot \sqrt{\frac{T_0}{T_0 - T}} \\
 \frac{\Delta M}{M} &:= \frac{1}{M} \sqrt{\langle \Delta M^2 \rangle} \approx \frac{1}{\text{const}} \sqrt{\frac{\langle M \rangle_T}{2T_0}}
 \end{aligned} \tag{2.37}$$

Carlitz [19] was the first to remark that these diverging fluctuations make the canonical description inadequate for many questions, e.g. for exact momentum spectra.

We note in passing that in our approach $T = 1/\sqrt{b^2}$ as well as the partition functions are Lorentz invariants, while the partition functionals, from which particle momentum distributions can be calculated, are covariant objects, depending on the four-vector b . This four-vector reduces, in the rest frame of the fireball, to $b_r = (1/T, 0, 0, 0)$; for a fireball moving with four-momentum r , four-velocity v^*),

$$k = M \cdot (\gamma, \vec{\beta} \gamma) = M \cdot v \tag{2.38}$$

it becomes

$$b = \frac{k}{MT} = \left(\frac{\gamma}{T}, \frac{\vec{\beta} \gamma}{T} \right) = \frac{v}{T} \tag{2.39}$$

*) The $\vec{\beta}$ in $v \equiv (\gamma, \vec{\beta} \gamma)$ has nothing to do with $\beta = 1/T$ [Eq. (2.12)].

With respect to the old controversy about the "right" transformation properties of thermodynamic quantities [20]; we thus adopt here the most natural viewpoint of Touscheck [21].

It is worth while to remark that our invariant single-pion partition function is related to the Feynman propagator $\Delta_F(x^2, m^2)$ for space-like $x^2 = -b^2$. From Eq. (2.21) we obtain for $x^2 < 0$:

$$Z\left(T = \frac{1}{\sqrt{-x^2}}, m\right) = \frac{2\pi m B}{\sqrt{-x^2}} K_1(m\sqrt{-x^2}) = (2\pi)^3 B \Delta_F(x^2, m^2) \quad (2.40)$$

Furthermore with Eq. (2.22) and the notation of Ref. [13]

$$Z\left(T = \frac{1}{\sqrt{-x^2}}\right) = (2\pi)^3 \int_m^\infty B\tau(M^2) \Delta_F(x^2, M^2) dM^2 =: B \Delta_H(x^2) \quad (2.41)$$

The "superpropagator" $\Delta_H(x^2)$ then becomes singular at a space-like distance $1/T_0$ from the light-cone. With these remarks we have related our present method to the ones of Refs. [12, 13]. The use of covariant Laplace transforms avoids dealing with products of singular functions (distributions [12]) or formal manipulations with non-existing Fourier transforms. For further speculations about relations between hadron statistical bootstrap and field theory, see Ref. [12].

Finally we mention the close analogy of the singularity of $Z(T)$ at T_0 with phase transitions in SST. The important difference is that in SST the number of particles is constant and their nature fixed (fixed pressure, no chemical reactions); therefore, if for instance the boiling point is reached, the temperature will stay there for some time while more energy is fed in, but eventually all liquid will be evaporated and the temperature must increase again. Thus in SST the partition function Z_s would follow curves like $\{a_{1c}\}$ or $\{a_{1d}\}$ in Fig. 1, while in our case the curve bends to the left or has a pole [see Section 3] so that $T > T_0$ is definitely excluded. The reason is, of course, that neither the number nor the mass of particles created at the "hadronic boiling point T_0 " are limited. As V.F. Weisskopf once put it: "We are boiling off the vacuum, its vapour consisting of stable hadrons".

Nevertheless there should be close similarities to the theory of phase transitions.

2.6 Composition of a fireball, or its first decay step

The probability distribution in Eq. (2.9) determines the distribution of the constituent fireballs inside the fireball. The integral of $P_\ell(r; r_1, \dots, r_\ell)$ over $r_1, \dots, r_\ell$ gives the total probability for ℓ -fireball states; therefore the generating function $\phi(r^2, h)$ of the fireball multiplicity distribution (not of the final pion distribution) is determined by

$$\begin{aligned} B\tau(r^2)\phi(r^2, h) &= h B\delta_0(r^2 - m^2) + \\ &+ \alpha \sum_{\ell=2}^{\infty} \frac{h^\ell}{\ell!} \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} B\tau(r_i^2) d^4 r_i \end{aligned} \quad (2.42)$$

The mean multiplicity $\ell(r^2)$, for example, is the derivative of $\phi(r^2, h)$ at $h = 1$, therefore

$$\begin{aligned} B\tau(r^2)\ell(r^2) &= B\delta_0(r^2 - m^2) + \\ &+ \alpha \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} \int \delta^4(r - \sum_{i=1}^{\ell} r_i) \prod_{i=1}^{\ell} B\tau(r_i^2) d^4 r_i \end{aligned} \quad (2.43)$$

In order to determine the asymptotic behaviour of $\ell(M^2)$ for large M let us calculate the Laplace transform of $\tau(r^2)\ell(r^2)$:

$$\begin{aligned} \lambda(T) &:= \int d^4 r e^{-Tr} B\tau(r^2)\ell(r^2) = \\ &= z(T, m) + \alpha Z(T, B) \cdot (e^{Z(T, B)} - 1) \end{aligned} \quad (2.44)$$

Here $z(T, m)$ and $Z(T)$ are defined in Eq. (2.21). Using Eq. (2.23) we can eliminate αe^Z and find

$$\lambda(T) = z(T, m)(1 - Z(T, B)) + (1 + \alpha) Z^2(T, B) \quad (2.45)$$

Now the asymptotic behaviour of $\tau(r^2)\ell(r^2)$ can be obtained from the singularity structure of $\lambda(T)$ near T_0 . It is obvious from Eq. (2.45) that $\lambda(T)$ has the same square root branch point at T_0 as $Z(T)$, the difference being only in the numerical factor in front of the square

root. This means that asymptotically $\tau(r^2)\ell(r^2)$ is proportional to $\tau(r^2)$. In other words, $\ell(r^2)$ tends to a finite asymptotic limit: ℓ_∞ . There is no difficulty in working out the value of ℓ_∞ from Eqs. (A.4), (2.44) and (2.45). It turns out to be

$$\ell_\infty = 1 + (1 + \alpha) \ln\left(1 + \frac{1}{\alpha}\right) = 2.0 + 2 \quad (2.46)$$

The numerical value of ℓ_∞ at $\alpha = 1$ is ~ 2.4 . If α tends to infinity then ℓ_∞ tends to 2; that is, in this limit the fireball consists of two fireballs and the "truncation" of the bootstrap equation as proposed by Chiu [22] becomes exact. In any case, for not too small values of α , ℓ_∞ is not very large. This is in accordance with the numerical results of Ref. [7]*), where $\ell_\infty \approx 2.4$ was found. So far the constant α was an additional free parameter; Eq. (2.46) attaches a physical interpretation to it and suggests, in comparison with Ref. [7] and because of experimental support for dominance of two-body decay, that one should choose $\alpha \approx 1$.

2.7 Pion spectrum first generation

We obtain the invariant single-pion momentum spectrum in the first generation by allowing in the probability distribution $P_\ell(r; r_1, \dots, r_\ell)$ [Eq. (2.9)] any one of the $r_1, \dots, r_\ell$ to refer to a pion with momentum k , i.e. by replacing $\tau(r_i^2)$ by $\delta_0(k^2 - m^2)$ one after the other for $i = 1, \dots, \ell$, integrating over the remaining r_i , finally summing over all $\ell = 2, \dots, \infty$. Thus

$$\begin{aligned} B\tau(r^2)W(k)\delta_0(k^2 - m^2)d^4k : &= \\ &= B\delta_0(k^2 - m^2)d^4k \left[\alpha \sum_{\ell=2}^{\infty} \frac{\ell}{\ell!} \int \delta^4(r - k - r_2 - \dots - r_\ell) \prod_{i=2}^{\ell} B\tau(r_i^2) d^4r_i \right] \end{aligned} \quad (2.47)$$

*) It can be shown that the result in Eq. (2.46) is valid also in the linear bootstrap of Section 3. As a consequence, the chain-decay approximation of Ref. [16] becomes exact at α tending to infinity and is a good approximation at $\alpha \gtrsim 1$.

where $W(k) \equiv \sqrt{k^2 + m^2} (d^3N/dk^3)$ is the invariant distribution in some normalization. Introducing $r' \equiv r - k$, relabelling the terms in the sum, separating out the first term and applying the bootstrap equation (2.11) with $\phi \equiv 1$ to eliminate the sum, we obtain

$$W(k) = \frac{B [(\alpha+1) \tau(r'^2) - \delta_0(r'^2 - m^2)]}{\tau(k^2)} \quad (2.48)$$

This equation shows that $W(k)$ is an invariant function of $r^2 = M^2$ and $r'^2 = M'^2$, where M' is the missing mass. We write $\tau(r'^2) = \int \delta^4(r - k - p) \tau(p^2) d^4p$ and correspondingly for $\delta_0(r'^2 - m^2)$; then we Laplace transform:

$$\begin{aligned} W(k) \int e^{-kr} B \tau(r^2) d^4r &= \\ &= W(k) Z(T) = B e^{-\frac{\sqrt{k^2 + m^2}}{T}} [(\alpha+1) Z(T) - z(T, m)] \end{aligned} \quad (2.49)$$

hence

$$W(k) = e^{-\frac{\sqrt{k^2 + m^2}}{T}} f(T) \quad (2.50)$$

Thus the invariant single-pion spectrum in the first decay generation appears to be a pure Boltzmann spectrum with $T \leq T_0$. This is, however, the "canonical" situation, where not the mass M of the decaying fireball but its temperature T is given, i.e. (2.50) is the pion decay spectrum (1st generation) from a canonical ensemble of fireballs (we have integrated over d^4r -- neither the mass nor the momentum is fixed any more). The "microcanonical" situation as given by Eq. (2.48) looks quite different. For the sake of simplicity we drop the $\delta_0(r'^2 - m^2)$, i.e. we disregard the endpoint of the pion spectrum, where $M \rightarrow 2$ pions. Normalization does not interest us either, hence

$$W(k) = \text{const} \frac{\tau(M'^2)}{\tau(M^2)} \quad (2.51)$$

In the rest frame of the decaying fireball we have $M' = (M^2 + m^2 - 2ME)^{1/2}$ and then $W(k)$ becomes a function of E only. We then define

$$d^{(1)}(M, E) := W(k) \text{ [in rest frame of fireball]}. \quad (2.52)$$

The superscript (1) stands for "1st generation". With the asymptotic mass spectrum Eq. (A.22) we find

$$d^{(1)}(M, E) = \text{const.} \cdot \frac{M^3 \left(1 - \frac{C_4 T_0}{M'}\right)}{M'^3 \left(1 - \frac{C_4 T_0}{M}\right)} \exp\left(-\frac{M - M'}{T_0}\right) \quad (2.53)$$

$$M' := \sqrt{M^2 + m^2 - 2ME}$$

If one would expand the square root and neglect the E -dependence of the terms in front of the exponential, one would obtain a pure Boltzmann function with temperature T_0 . In fact this is a poor approximation; $d^{(1)}(M, E)$ deviates appreciably from a simple Boltzmann distribution. This is best discussed with the help of effective temperatures.

2.8 Effective temperatures; first generation

There are many ways of introducing effective temperatures. The most convenient is the following one: for a pure Boltzmann function we have the identity

$$T \equiv - \left[\frac{d}{dE} \ln \left(e^{-\frac{E}{T}} \right) \right]^{-1} \quad (2.54)$$

Accordingly we define

$$T_{\text{eff}}^{(1)}(M, E) := - \left[\frac{d}{dE} \ln \left(d^{(1)}(M, E) \right) \right]^{-1} \quad (2.55)$$

Furthermore one can ask for the average value of T_{eff} between $E' = m$ and $E' = E$; this average should be taken with a weight function proportional to the number of particles emitted with energy E' ; hence we define

$$T_{\text{av}}^{(1)}(M, E) := \frac{\int_m^E T_{\text{eff}}^{(1)}(M, E') d^{(1)}(M, E') \sqrt{E'^2 - m^2} dE'}{\int_m^E d^{(1)}(M, E') \sqrt{E'^2 - m^2} dE'} \quad (2.56)$$

The factor $\sqrt{E'^2 - m^2}$ is due to integration over $d^3k/2k_0$ [see Eq. (2.6)].

The result of a numerical evaluation of $T_{\text{eff}}^{(1)}$ and $T_{\text{av}}^{(1)}$ is shown in Figs. 3 and 4. It is striking to see that $T_{\text{eff}}^{(1)}$ for small energies lies considerably above T_0 and drops below T_0 only at about $E \approx 3T_0$. This behaviour was first noticed by Carlitz [19] and then studied by Frautschi and Hamer [10]. It is easy to understand why all $T_{\text{eff}}^{(1)}$ curves cross T_0 at about $E \approx 3T_0$. Expanding Eq. (2.55) up to terms of order T_0/M and E/M (both considered to be of same order) one finds

$$T_{\text{eff}}^{(1)}(M, E) = T_0 \left(1 - \frac{E - 3T_0}{M} - \dots \right) \quad (2.57)$$

so that $T_{\text{eff}}^{(1)}$ decreases approximately linearly with slope T_0/M and crosses T_0 at $E \approx 3T_0$. At that place the maximum of the distribution $d^{(1)}(M, E)$ has already been passed and therefore most first generation pions are emitted at $T_{\text{eff}}^{(1)} > T_0$ so that even $T_{\text{av}}^{(1)}(M, E)$ stays above T_0 for all E up to the kinematic limit.

It is interesting to compare this with the corresponding situation in the linear bootstrap (Figs. 12, 13).

2.9 Asymptotic distributions

The asymptotic limit of the pion distributions is investigated in a rather general class of statistical bootstrap models in Appendix B (in the simplest cases the results coincide with those obtained in Refs. [13, 16] within specific models). Here we mention only a characteristic feature of the results in our model defined by Eq. (2.11). The multiplicity distribution obtained from Eq. (2.19) by integrating over $d^4k_1, \dots, d^4k_\ell$ is

$$V_\ell(r^2) = \frac{g_\ell B^\ell}{B\tau(r^2)} \rho^{(\ell)}(r^2, m^2, \dots, m^2) \quad (2.58)$$

where $\rho^{(\ell)}$ is the ℓ -pion invariant phase space. It is shown in Appendix C that the coefficients g_ℓ behave for large ℓ as

$$g_\ell \sim z_0^{-\ell} \ell^{-3/2} \sqrt{\frac{z_0}{2\pi(1+\alpha)}} \quad (2.59)$$

with z_0 given by Eq. (2.24). If as a first approximation the pion mass is neglected, then the phase space $\rho^{(\ell)}$ becomes

$$\rho^{(\ell)}(M^2, 0, \dots, 0) = \frac{\pi}{2\Gamma(\ell)\Gamma(\ell-1)} \left(\frac{\pi M^2}{2} \right)^{\ell-2} \quad (2.60)$$

and the ℓ -distribution in Eq. (2.58) peaks at

$$\ell_0 = M \sqrt{\frac{\pi B}{2z_0}} \approx \frac{M}{2T_0} \quad (2.61)$$

Here the value of T_0 is taken from Eq. (2.29) at zero pion mass, to be consistent in the approximation. The shape of the distribution around ℓ_0 is approximated by

$$V_\ell(M^2) \approx \text{const} e^{-\frac{(\ell-\ell_0)^2}{\ell_0}} \quad (\ell \text{ near } \ell_0) \quad (2.62)$$

This means that as M grows the pion multiplicity distribution centred about ℓ_0 gets broader proportionally to $\sqrt{M}$, while ℓ_0 shifts proportionally to M . Furthermore, it follows that for any fixed $\ell \ll \ell_0$ the probability to observe exactly ℓ pions in the final state vanishes like

$$V_\ell(M^2) \approx \text{const} \cdot M^{2\ell-1} e^{-\frac{M}{T_0}} \quad (M \gg 2\ell T_0) \quad (2.63)$$

3. LINEAR BOOTSTRAP MODEL

In Ref. [13] it came as a real surprise that the asymptotic limits of the complete statistical bootstrap happened to coincide with those of Ref. [16] for the simple approximate chain decay, where the fireball emits in each decay step one single pion. This situation is clarified by the results of Appendix B showing that the asymptotic forms are the same in a rather general class of models. In the framework of our assumption that composition and decay of fireball are the inverse of each other, the chain decay corresponds to a situation where the fireball is a bound state of another fireball and a single pion. In a more general situation the fireball would be composed of another fireball and any number of pions. This model (which we shall consider in the present section) is probably only a crude approximation to reality, but it is still worth considering because of its mathematical simplicity. It is a special case of the class of "truncated statistical bootstrap models" of which another special case has recently been considered by Chiu [22].

The key relation for the probability distribution of the fireball constituents is now, instead of Eq. (2.9),

$$\begin{aligned} P_\ell(r; r', k_2, \dots, k_\ell) &= \\ &= \frac{\alpha \ell}{\ell! B\tau(r^2)} \cdot B\tau(r'^2) \delta^4(r - r' - \sum_{i=2}^{\ell} k_i) \prod_{i=2}^{\ell} B \delta_0(k_i^2 - m^2) \end{aligned} \quad (3.1)$$

The corresponding equation for the pion functional is instead of Eq. (2.11)

$$B\tau(r^2)V[r;\phi(k)] = B\delta_0(r^2-m^2)\phi(r) + \alpha \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} \times$$

$$\times \int B\tau(r'^2)V[r';\phi(k)] d^4r' \delta^4(r-r'-\sum_{i=2}^{\ell} k_i) \prod_{i=2}^{\ell} d^4k_i B\delta_0(k_i^2-m^2)\phi(k_i) \quad (3.2)$$

This equation has the great advantage of being linear in the unknown functional. Following the procedure of Section 2.3, we now obtain

$$G(z) = z + \alpha G(z) (e^z - 1) \quad (3.3)$$

and therefore

$$G(z) = \frac{z}{1+\alpha - \alpha e^z} \quad (3.4)$$

This function is drawn in Fig. 1. It has a simple pole at

$$z_0 = \ln\left(1 + \frac{1}{\alpha}\right) \quad (3.5)$$

Comparing with Eq. (2.24) corresponding to the complete bootstrap, the position of the singularity differs only in the dependence on the parameter α . The expression (2.29) for the limiting temperature T_0 now becomes [see Fig. 2]

$$z_0 = \ln\left(1 + \frac{1}{\alpha}\right) = 2\pi m T_0 B K_1\left(\frac{m}{T_0}\right) \quad (3.6)$$

3.1 Explicit solutions (leading pole approximation)

The asymptotic forms of the solutions can now be expressed by inverting the K-transforms, which in this case is rather easy owing to the pole of $Z(T) \equiv G[Z(T, m)]$ at $\beta_0 = 1/T_0$. The pole contribution according to Eq. (A.9) is exponentially greater than the neglected contributions; that is, if $F(M^2)$ is a function obtained by exactly inverting the K-transform and $F_p(M^2)$ is the pole contribution, then

$$F(M^2) = F_p(M^2) \cdot [1 + O(e^{-M(\beta_0 - \beta_1)})] \quad (3.7)$$

where β_1 is the real part of the next singularity left to β_0 . Therefore in the following formulae it is sensible to keep the Bessel functions in their full form instead of using their asymptotic expansions (whose correction terms vanish not exponentially but only as inverse powers of M). In this way we obtain asymptotic formulae which are good approximations even at non-asymptotic M values. We also give the expressions obtained by using only the leading term in the asymptotic expansions of the Bessel functions; these expressions will be marked by a superscript (∞) , they are truly asymptotic.

We note for further use

$$\begin{aligned} z_0 &= z(\beta_0, m) = \ln(1 + \frac{1}{\alpha}) = 2\pi m \beta_0^{-1} B K_1(m\beta_0) \\ z'_0 &= \frac{d}{d\beta} z(\beta, m) \Big|_{\beta_0} = -2\pi m^2 \beta_0^{-1} B K_2(m\beta_0) \\ \frac{z_0}{z'_0} &= - \frac{K_1(m\beta_0)}{m K_2(m\beta_0)} \end{aligned} \quad (3.8)$$

With the help of the pole formula (A.9) we now obtain

$$\begin{aligned} \tau_p(M^2) &= - \frac{\beta_0^2 z_0 I_1(M\beta_0)}{2\pi M B (1+\alpha) z'_0} \\ \tau_p^{(\infty)}(M^2) &= -m \frac{z_0}{z'_0} \cdot \frac{B_0}{B(1+\alpha)} \cdot (m\beta_0)^{3/2} \cdot \frac{2}{\sqrt{2\pi}} \cdot \frac{e^{M\beta_0}}{\sqrt{m M^3}} \\ B_0 &= (4\pi m^2)^{-1} \end{aligned} \quad (3.9)$$

Further expansion of the Bessel function $I_1(M\beta_0)$ results in multiplying $\tau_p^{(\infty)}(M)$ by $[1 - 3/(8M\beta_0)]$. Figure 5 shows all three approximations for $B = B_0$ and $\alpha = 1$.

For the mean multiplicity $n(M^2)$ and the invariant single-pion distribution $D(M^2, M'^2) \equiv D[r^2, (r - k)^2]$ we follow the same procedure and obtain expressions $[\tau(M^2)n(M^2)]_{\text{pole}}$ and $[\tau(M^2)D(M^2, M'^2)]_{\text{pole}}$. We then define $n_p(M^2)$ and $D_p(M^2, M'^2)$ by dividing the $[\dots]_{\text{pole}}$ expressions by $\tau_p(M^2)$. The result is

$$n_p(M^2) = - \left[1 + \frac{z_0}{\beta_0 z'_0} + z_0 + M \cdot \frac{z_0}{z'_0} \cdot \frac{I_0(M\beta_0)}{I_1(M\beta_0)} \right] \quad (3.10)$$

$$n_p^{(\infty)}(M^2) = - M \cdot \frac{z_0}{z'_0}$$

Figure 6 shows $n_p(M^2)/M$ and $n_p^{(\infty)}(M^2)/M$ together with the value $[m + (3/2)T_0]^{-1}$ which is the asymptotic limit in the crudest thermodynamic guess.

Finally, we have

$$D_p(M^2, M'^2) = - \frac{BM}{M'z'_0 \beta_0 I_1(M\beta_0)} \times \\ \times \left[(1 + \beta_0 z'_0) \cdot I_1(M'\beta_0) + M'\beta_0 I_0(M'\beta_0) \right] \quad (3.11)$$

$$D_p^{(\infty)}(M^2, M'^2) = - \frac{BM}{z'_0} e^{-E\beta_0}$$

Here E is the pion energy in the rest frame of the decaying fireball; thus

$$M' = \sqrt{M^2 + m^2 - 2ME} \quad (3.12)$$

If we define the functions with argument E corresponding to $D_p(M^2, M'^2)$ by

$$d_p(M, E) := \mathcal{D}_p(M^2, M'^2) \quad (3.13)$$

then Figs. 7 and 8 show the behaviour of $d_p(M, E)/d_p^{(\infty)}(M, E)$ and of $\ln [d_p(M, E)/d_p(M, m)]$ for several fireball masses M . These figures show that the full decay process with all recoils exactly taken into account leads to a distribution which decreases faster than a Boltzmann spectrum and therefore cannot explain the deviation recently measured at the CERN ISR [14], which goes in the opposite direction.

3.2 Effective temperatures

We use the same definition of effective and average temperatures as we did in Section 2.8. Hence, identifying the distribution $d(M, E)$ with a Boltzmann distribution, the latter must have an effective temperature

$$T_{\text{eff}}(M, E) := \left[-\frac{d}{dE} \ln d(M, E) \right]^{-1} \quad (3.14)$$

Accordingly, the average temperature becomes

$$T_{\text{av}}(M, E) := \frac{\int_m^E T_{\text{eff}}(M, E') d(M, E') \sqrt{E'^2 - m^2} dE'}{\int_m^E d(M, E') \sqrt{E'^2 - m^2} dE'} \quad (3.15)$$

Figures 9 and 10 show the result of a numerical evaluation of $T_{\text{eff}}(M, E)$ and $T_{\text{av}}(M, E)$ for several fireball masses M . Note the difference between these temperatures (valid for the final distributions) and those calculated for the pion distribution in the first generation (Figs. 12 and 13).

For large E the average temperature reaches a limit, because the weight $d(M, E)$ decreases exponentially. We define

$$T_{av}(M) := \lim_{E \rightarrow \frac{M}{2}} T_{av}(M, E) \quad (3.16)$$

and compare it to the canonical temperature T which would yield M as the value of the mean energy if the system is put in a heat bath of that temperature.

We have for the mean energy

$$\langle M \rangle_T = -\frac{\partial}{\partial \beta} \ln Z(\beta, B) \quad (3.17)$$

Now with $Z(\beta, B) = G[z(m, \beta)]$ and Eqs. (3.4), (3.5) and (3.8) we obtain

$$\langle M \rangle_T = m \frac{K_2\left(\frac{m}{T}\right)}{K_1\left(\frac{m}{T}\right)} \cdot \left[1 + \frac{ze^z}{e^{z_0} - e^z} \right]$$

$$z(T, m) = 2\pi m T B K_1\left(\frac{m}{T}\right) \quad (3.18)$$

$$z_0 = z(T_0, m) = \ln\left(1 + \frac{1}{\alpha}\right)$$

The two temperatures cannot yet be compared, because in the microcanonical situation the lowest mass which can decay is $M = 2m$; for this value the two decay particles do not carry any kinetic energy, hence remain at rest; this corresponds to zero temperature. We therefore must renormalize the canonical temperature T such that for $\langle M \rangle_T = 2m$ it becomes zero; in other words: one has to consider a situation in which two particles are brought in before one starts heating. The recipe for doing this renormalization has been given in Ref. [6], Appendix A. The result is, approximately,

$$M_{T, \text{renorm}} = \langle M \rangle_T + \frac{3}{2} T + 2m \quad (3.19)$$

We then consider M and $M_{T, \text{renorm}}$ as equivalent.

Figure 11 shows $T_{\text{av}}(M)$ according to Eq. (3.16) (full curve) and T as function of $M = M_{T, \text{renorm}}$ according to Eq. (3.19). The near coincidence of $T_{\text{av}}(M)$ with the thermodynamical temperature T corresponding to the same mass is striking in view of the small average number of particles (at $M = 8$ GeV, Fig. [6] tells us that there are only about 15 particles). It is only for large particle numbers that energy-momentum conservation becomes good in the canonical situation, and only then that the two temperatures should become asymptotically equal. That they never are very different may be an explanation for the success of the thermodynamical model in fitting momentum distributions [6,23].

3.3 The first generation distribution

Here again we can consider the pion spectrum in the first generation. The method is the same as in Sections 2.6-2.8 and it leads to the same distribution function as in Eq. (2.48) with the only difference that now we use the pole approximation for $\tau(r^2)$. In Eq. (2.53) we would have to replace M^3 and M'^3 by $M^{3/2}$ and $M'^{3/2}$; similarly, in Eq. (2.57) we will have $E - \frac{3}{2}T_0$ instead of $E - 3T_0$. Figures 12 and 13 show what $T_{\text{eff}}^{(1)}(M, E)$ and $T_{\text{av}}^{(1)}(M, E)$ now become. The crossing of the $T_{\text{eff}}^{(1)}$ curves is now at $E \approx \frac{3}{2}T_0$. The great difference compared to Figs. 3 and 4 is, however, that now $T_{\text{av}}^{(1)}$ can drop below T_0 if the energy is large enough, while that was not so in the full bootstrap case. This leads one to suspect that in the full bootstrap case $T_{\text{av}}(M, E)$ for the final distribution might stay above T_0 . We have not attempted the rather lengthy calculation necessary to decide that. In any case we see that while the general results of the full and the linear bootstrap are similar, significant differences exist in numerical detail.

4. CONCLUSIONS

a) Our results are in qualitative agreement with those of Frautschi and Hamer [10,11] though not in all details. We are working in invariant momentum space; we do not introduce explicit resonances such as ρ , ω , etc.; furthermore, in each single decay step we allow all multiplicities and leave it to the bootstrap to decide their distribution -- all this is different from their approach and probably more than enough to account for quantitative differences between their and our figures.

b) Our calculations suggest (and in accordance with Refs. 6, 10 and 11 that the anomalously large transverse momenta measured at the CERN ISR cannot be understood as kinematical distortion of statistical decays: the exact final pion distribution $d(M,E)$ which contains the full decay tree with all kinematical constraints (recoils) lies below a Boltzmann distribution; and this the more the higher the pion energy -- exactly opposite to what has been measured [14]. This anomaly therefore indicates the existence of some non-statistical mechanism, which may govern particle production before some sort of equilibrium has been reached, i.e. for times shorter than 10^{-24} sec (superheating [6,15]).

c) The use of purely thermodynamic formulae referring to the canonical situation is not such a bad approximation as one might think; at least in the linear bootstrap model the average temperature nearly coincides with the canonical one; this may explain the success of the thermodynamical model [6,23].

d) The reciprocity-postulate is very powerful; it allows one to coin the integral equations for all distribution functions and the bootstrap equation into one single equation.

e) Our covariant formulation leads automatically to the definition of an "inverse temperature four vector" and from there straight to covariant thermodynamics.

f) It seems quite clear that the numerical changes brought about by using a completely covariant formulation are minimal but the gain in formal simplicity is enormous. Even though we do not claim that the one is right and the other wrong, we would advocate abandoning the non-covariant description.

g) The use of covariant Laplace transforms and their inversion via inverse K-transforms leads to explicit asymptotic expansions of the solutions to the bootstrap equations; all constants appearing in these solutions are calculable in terms of the one fundamental constant m , the lowest mass (pion mass) of the spectrum. Even the "free parameters" $\alpha = AB$ and B cannot deviate much from their natural values $\alpha = 1$ and $B_0 = (4\pi m^2)^{-1}$, because m also sets the scale for the range of the strong interaction. In the last instance then, m -- the lowest hadron mass -- is the only physical constant in this model. In a more realistic version, there must be three such masses (mesons, baryons, and strange mesons) but there will still be one limiting temperature T_0 .

h) It is shown that quite general models can be set up, which all have a very similar behaviour if one single condition is fulfilled; namely that a maximum temperature T_0 exists [see Appendix B].

At the time of finishing this paper we became acquainted with the work of Ilgenfritz and Kripfganz [24] who independently noticed the importance of the reciprocity postulate and of the covariant Laplace transform; using these methods, they derived and solved the bootstrap equations for the generating functionals in the same way as we do here. While our papers touch at this point, we feel there is on the whole sufficiently little overlap to justify publishing our work.

Acknowledgements

All figures (except Fig. 1) are hand-drawn copies (original size) of graphic output of SIGMA (System for Interactive Graphical Mathematical Applications; Ref. 30).

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ASYMPTOTIC SOLUTIONS FROM K TRANSFORMS

In order to obtain the solution of integral equations such as Eq. (2.11) in the form of an integral representation [which is in general better suited for numerical studies than the infinite series representation in Eq. (2.18)], it is useful to study in more detail the inversion of the K transformation. Let us consider for simplicity the problem for the state density $\tau(M^2)$ corresponding to $\phi(k) = 1$.

In this case we have to deal with Eqs. (2.21, 22, 23). From Eq. (2.22), slightly rewritten, we see that $Z(T)$ is related to $\tau(M^2)$ by a K transformation [25], namely*)

$$Z(\beta) = 4\pi \int_m^\infty dM B \tau(M^2) \left(\frac{M}{\beta}\right)^{3/2} K_1(M\beta) \sqrt{M\beta} \quad (A.1)$$

with the inverse

$$B \tau(M^2) = \frac{1}{4\pi^2 i} \int_{c-i\infty}^{c+i\infty} d\beta Z(\beta) \left(\frac{\beta}{M}\right)^{3/2} I_1(M\beta) \sqrt{M\beta} \quad (A.2)$$

The integration path goes right to the singularities β_s of $Z(\beta)$, i.e. $c > \text{Re } \beta_s$ for all s . If the singularities are known, then c may be shifted to the left to some c' in order to pick up the leading contributions from the singularities, taking advantage of the exponential behaviour of $I_1(M\beta)$:

$$I_1(M\beta) \approx \frac{e^{M\beta}}{\sqrt{2\pi M\beta}} \left(1 - \frac{3}{8M\beta} + \dots\right) \quad (A.3)$$

valid for $\text{Re } (M\beta)$ large and positive.

*) We shall use the same letters Z, z for the partition functions irrespective of whether we consider them as functions of T or of $\beta = 1/T$.

Thus if the singularities β_s are branch points or poles, we have the following situation:

The contributions of a simple pole at some β_s and of a cut starting from some β_s are

$$B\tau_{pole}(M^2) = \frac{\gamma_s}{2\pi} \beta_s^2 M^{-1} I_1(M\beta_s) \quad (A.4)$$

$$B\tau_{cut}(M^2) = -\frac{1}{4\pi^2 i} \int_{c'}^{\beta_s} d\beta \text{Disc}(Z(\beta)) \beta^2 M^{-1} I_1(M\beta)$$

where γ_s is the residue of the pole. If we can show that there is one rightmost singularity, then in the limit $M \rightarrow \infty$ the contributions of the others are exponentially smaller and can be neglected.

The singularities of $Z(\beta) \equiv G(z)$ are those of $G(z)$ because the relation between z and β does not contain singularities for $\beta > 0$. In our model $G(z)$ has one real singularity (a pole or a square root branch point) $z_0 > 0$; from Eq. (2.21), namely

$$\frac{z(\beta, m)}{2\pi m B} = \beta^{-1} K_1(m\beta) \quad (A.5)$$

it follows that the singularities in the β plane are the roots β_s of the equation

$$\frac{z_0}{2\pi m B} = \beta_s^{-1} K_1(m\beta_s) \quad (A.6)$$

and there are infinitely many of them. To show that they have the required property, we use the integral representation of $K_1(m\beta)$, valid for $\text{Re } \beta > 0$,

$$\beta^{-1} K_1(m\beta) = \int_1^{\infty} dx \sqrt{x^2 - 1} e^{-mx\beta} \quad (\text{A.7})$$

The β_s are then the roots of the two equations following from separation of real and imaginary parts of Eqs. (A.6,7):

$$\begin{aligned} \frac{z_0}{2\pi m B} &= \int_1^{\infty} dx \sqrt{x^2 - 1} e^{-mx \text{Re} \beta_s} \cos(mx \cdot \text{Im} \beta_s) \\ 0 &= \int_1^{\infty} dx \sqrt{x^2 - 1} e^{-mx \text{Re} \beta_s} \sin(mx \cdot \text{Im} \beta_s) \end{aligned} \quad (\text{A.8})$$

The exact positions of the roots β_s can be found numerically; we are only interested in their relative position. From Eqs. (A.8) it follows that

- i) if β_s is a solution, then its complex conjugate β_s^* is also a solution;
- ii) there is one and only one solution on the positive real axis -- we call this solution β_0 ;
- iii) for $\text{Im } \beta > 0$ the oscillations of the cosine in the first integral tend to make the integrand smaller, in order to maintain the value of the integral, $\text{Re } \beta$ must decrease; hence all other solutions have a smaller real part as β_0 .

As the integral (A.7) is analytic at $\beta = \beta_0$, the next root β_s lies at finite distance from β_0 . Thus for $M \rightarrow \infty$ we have to care only for the contribution of the singularity at β_0 . Hence, the pole case yields

$$B\tau_p(M^2) \approx \frac{\delta_0}{2\pi} \beta_0^2 M^{-1} I_1(M\beta_0) \quad (\text{A.9})$$

where γ_0 is the residue of the pole and β_0 is given by Eq. (3.6):

$$\ln\left(1 + \frac{1}{z}\right) = 2\pi m B \beta_0^{-1} K_1(m\beta_0) \quad (\text{A.10})$$

The result is reported in Eq. (3.9). If one works out the asymptotic expansion using Eqs. (3.8) and (A.3) one finds

$$\begin{aligned} \tau_p(M^2) &= C_p \cdot \frac{e^{M/T_0}}{\sqrt{m M^3}} \left(1 - \frac{3T_0}{8M} + \dots\right) \\ C_p &= \frac{B_0}{B(1+\alpha)} \cdot \frac{2}{\sqrt{2\pi}} \left(\frac{m}{T_0}\right)^{3/2} \cdot \frac{K_1(m/T_0)}{K_2(m/T_0)} \end{aligned} \quad (\text{A.11})$$

With $B = B_0 = (4\pi m^2)^{-1}$ and $\alpha = 1$ we find $C_p (\alpha = 1, B = B_0) = 0.07212$.

The cut case is somewhat more complicated. First we note that in the integral (A.4) we can, with exponentially negligible error, replace c' by zero:

$$B \tau_{\text{cut}}(M^2) \approx \frac{1}{4\pi^2 i} \int_0^{\beta_0} d\beta \text{Disc}(Z(\beta)) \beta^2 M^{-1} I_1(M\beta) \quad (\text{A.12})$$

If we wish to push the asymptotic expansion to the second term in $1/M$, then we must

- i) use the asymptotic formula (A.3) for $I_1(M\beta)$;
- ii) expand $Z(\beta)$ up to terms of order $(\beta - \beta_0)^{3/2}$, thus obtaining two cut contributions due to $(\beta - \beta_0)^{1/2}$ and $(\beta - \beta_0)^{3/2}$;
- iii) express the resulting integrals by the confluent hypergeometric function and use its asymptotic expansion up to the second term.

We only sketch the rather long calculation. From Section 2.4 we know that $Z(\beta) \equiv G(z)$ is the solution of

$$z = (1+\alpha) G(z) - \alpha (e^{G(z)} - 1) \quad (\text{A.13})$$

The function $G(z)$ has its real singularity where $dz/dG = 0$; there

$$\begin{aligned} z_0 &= (1+\alpha) \ln(1+\frac{1}{\alpha}) - 1 \\ G_0 &= G(z_0) = \ln(1+\frac{1}{\alpha}) \end{aligned} \quad (\text{A.14})$$

We expand $G(z)$ around z_0 :

$$G(z) = G_0 - \sqrt{2} \sqrt{\frac{z_0 - z}{1+\alpha}} - \frac{1}{3} \frac{z_0 - z}{1+\alpha} - \frac{1}{36} \left(2 \frac{z_0 - z}{1+\alpha} \right)^{3/2} - \dots \quad (\text{A.15})$$

Only the terms $(z_0 - z)^{1/2}$ and $(z_0 - z)^{3/2}$ will give contributions to the integral over the discontinuity. We have with $\Delta := \beta - \beta_0$

$$\begin{aligned} z_0 - z &= -z'_0 \Delta - \frac{1}{2} z''_0 \Delta^2 - \dots \\ z'_0 &= \left. \frac{dz}{d\beta} \right|_{\beta_0} = - \frac{2\pi m^2 B}{\beta_0} K_2(m\beta_0) \\ z''_0 &= \left. \frac{d^2 z}{d\beta^2} \right|_{\beta_0} = \frac{2\pi m^4 B}{(m\beta_0)^2} \cdot [3K_2(m\beta_0) + m\beta_0 K_1(m\beta_0)] \end{aligned} \quad (\text{A.16})$$

Expanding $\sqrt{z_0 - z}$ to second order and $(z_0 - z)^{3/2}$ to first, we find the part of $Z(\beta)$ which contributes to the cut near β_0 :

$$Z_{\text{cut}}(\beta) = \gamma_{1/2} (\beta - \beta_0)^{1/2} + \gamma_{3/2} (\beta - \beta_0)^{3/2} + \dots$$

$$\gamma_{1/2} := - \sqrt{\frac{2}{1+\alpha}} \cdot \sqrt{-z'_0} \quad (\text{A.17})$$

$$\gamma_{3/2} := \frac{1}{4} \sqrt{\frac{2}{1+\alpha}} \left(\frac{z''_0}{\sqrt{-z'_0}} - \frac{2}{9} \frac{1}{1+\alpha} (-z'_0)^{3/2} \right)$$

The discontinuity due to a power cut is given by

$$\text{Disc} [\gamma_a (\beta - \beta_0)^a] = 2i \gamma_a (\beta_0 - \beta)^a \cdot \sin \pi a \quad (\text{A.18})$$

Thus with Eq. (A.12) and using Eq. (A.3)

$$\begin{aligned} B\tau_c(M^2) &\approx \\ &\approx -\frac{1}{2\pi^2 M} \sum_{a=\frac{1}{2}, \frac{3}{2}} \gamma_a \sin \pi a \int_0^{\beta_0} d\beta (\beta_0 - \beta)^a \beta^2 \frac{e^{\frac{M\beta}{\sqrt{2\pi M\beta}}}}{\sqrt{2\pi M\beta}} \left(1 - \frac{3}{8M\beta}\right) \end{aligned} \quad (\text{A.19})$$

The four resulting integrals can all be expressed by the confluent hypergeometric function [27]

$${}_1F_1(\alpha, \gamma, x) = \frac{x^{1-\gamma}}{B(\alpha, \gamma-\alpha)} \int_0^x e^{-t} t^{\alpha-1} (x-t)^{\gamma-\alpha-1} dt \quad (\text{A.20})$$

with the asymptotic expansion [28]

$${}_1F_1(\alpha, \gamma, x) \approx \frac{(\gamma-1)!}{(\alpha-1)!} e^x x^{\alpha-\gamma} \left[1 + \frac{(1-\alpha)(\gamma-\alpha)}{x} + \dots \right] \quad (\text{A.21})$$

The rest is straightforward.

Collecting all contributing terms up to $1/M^4$ we finally obtain the asymptotic expression for the full bootstrap spectrum

$$\begin{aligned} \tau(M^2) &\approx \frac{C_3^m}{M^3} e^{\frac{M}{T_0}} \left(1 - C_4 \frac{T_0}{M} + \dots \right) \\ C_3 &= \sqrt{\frac{B_0}{B(1+\alpha)}} \cdot \frac{1}{\sqrt{2\pi^2}} \cdot \left(\frac{m}{T_0} \right) \sqrt{K_2(m/T_0)} \\ C_4 &= \left[\frac{3}{2} - \frac{3}{8} \left(\frac{m}{T_0} \right) \frac{K_1(m/T_0)}{K_2(m/T_0)} + \frac{B}{B_0(1+\alpha)} \cdot \frac{1}{24} \cdot K_2\left(\frac{m}{T_0}\right) \right] \end{aligned} \quad (\text{A.22})$$

For $\alpha = 1$, $B = B_0$ we obtain numerically

$$\begin{aligned} C_3(\alpha=1; B=B_0) &= 0.2062\dots \\ C_4(\alpha=1; B=B_0) &= 1.423\dots \end{aligned} \quad (\text{A.23})$$

The constant C_3 can be compared to the constant found by Frautschi and Hamer [9] in their corresponding iterative solution^{*)}. They calculate, however, $\rho(M)$ and not $\tau(M^2)$, these two differing by dimension. The correspondence is obviously

$$\begin{aligned} B_0 \tau(M^2) &\iff \frac{V_0}{(2\pi)^3} \rho(M) \\ \tau(M^2) &\iff \frac{2}{3\pi m} \rho(M) \end{aligned} \quad (\text{A.24})$$

Writing $\rho(M) = C_{FH} e^{M/T_0}/M^3$ and comparing with Eq. (A.22) we find

$$C_{FH} \iff \frac{3\pi m^2}{2} C_3 = 4.7 C_3 m^2 = 0.97 m^2$$

while Frautschi and Hamer give the value

$$C_{FH} = (1.1 \pm 0.3) m^2$$

The agreement is perfect.

The K-transform relationship in Eqs. (A.1-2) holds not only for $\tau(M^2)$ but also for other quantities such as, for example, the single-particle distribution or the generating function of the multiplicity distribution. Therefore the same methods as above can also be used to determine their asymptotic behaviour. For instance, the asymptotic behaviour of $\tau(r^2)\psi(r^2; h)$ is obtained if instead of putting $\phi(k)$ equal to 1 we put $\phi(k) = h$. Let us in that case denote the function corresponding to $Z(\beta)$ by $Z_\psi(\beta, h)$:

^{*)} Their Table I, Model 1.

$$Z_{\psi}(\beta, h) = G(h \cdot z) \quad (\text{A.25})$$

Now we have Eqs. (A.1-2) for $Z_{\psi}(\beta, h)$ and $\tau(M^2)\psi(M^2, h)$ instead of $Z(\beta)$ and $\tau(M^2)$. The only difference in the asymptotic behaviour of $\tau(M^2)\psi(M^2, h)$ compared to $\tau(M^2)$ is that, as is shown by Eq. (A.25), the singularities of $Z_{\psi}(\beta, h)$ are shifted compared to $Z(\beta)$. If $G(z)$ has a singularity at Z_0 , then $Z(\beta)$ has it at β_0 given by (A.6) and $Z_{\psi}(\beta, h)$ at $\beta_0(h)$ determined by

$$\frac{\beta_0(h)}{\beta_0} = h \cdot \frac{K_1(m\beta_0(h))}{K_1(m\beta_0)} \quad (\text{A.26})$$

The resulting asymptotic form $\psi_{\infty}(M^2, h)$ of $\psi(M^2, h)$ was investigated in Section 5 of Ref. [16].

APPENDIX B

ASYMPTOTIC BEHAVIOUR IN A GENERAL STATISTICAL BOOTSTRAP MODEL

It is expected on general grounds that if the decaying fireball has a large mass then the pion distribution tends to a simple asymptotic form. The reason for that is the low average kinetic energy of the emitted pions which makes the recoil effects asymptotically negligible. This is a quite general property of statistical bootstrap models, as we shall show in this appendix.

We call a "general statistical bootstrap model" a model with the following properties:

- i) the reciprocity postulate holds; i.e. the decay pattern is the exact counterpart of the composition of the fireball;
- ii) the partition functional $Z[b, \phi(k)]$ is related to the one-pion partition functional $z[b, \phi(k)]$ through a function $Z = G(z)$ of which we only know that
- iii) there exists a limiting temperature T_0 ; i.e. the function $G(z)$ has a singularity at $z_0 = z(T_0, m)$.

In general, instead of Eq. (2.15), now $G(z)$ satisfies an implicit function-equation of the form

$$G(z) = F(z, G(z)) \quad (B.1)$$

with some given F .

The determination of the asymptotic behaviour of the inclusive distributions defined in Eq. (2.5) goes as follows. The Laplace transform of $\tau(r^2)D_n(r; k_1, \dots, k_n)$ is

$$\begin{aligned} \Omega_n(b; k_1, \dots, k_m) &:= \int d^4r e^{-br} B \tau(r^2) D_m(r; k_1, \dots, k_m) = \\ &= \frac{\delta^m}{\delta \phi(k_1) \dots \delta \phi(k_m)} Z[b; \phi(k)]_{\phi=1} \end{aligned} \quad (B.2)$$

From our assumption (ii) and the fact that $z[b, \phi]$ in Eq. (2.13) is a linear functional, it follows that

$$\Omega_m(t; k_1, \dots, k_m) = \frac{d^m}{dz^m} G(z(T, m)) \cdot \prod_{i=1}^m \delta_0(k_i^2 - m^2) e^{-bk_i} \quad (B.3)$$

The argument $z(T, m)$ of $G(z)$ is defined in Eq. (2.21).

Let us first look at the asymptotic behaviour of the integral of the n -particle inclusive distribution $D_n(r^2)$ [see Eq. (2.8)]. Integrating Eq. (B.2) over $k_1, \dots, k_n$ we obtain with $\beta \equiv 1/T$ the invariant function

$$\begin{aligned} v_m(\beta) &= \int d^4k_1 \dots d^4k_m \Omega_m(t; k_1, \dots, k_m) = \\ &= \int d^4z e^{-bz} B\tau(z^2) D_m(z^2) = [z(T, m)]^n \cdot \frac{d^n G(z)}{dz^n} \end{aligned} \quad (B.4)$$

Carrying out the Laplace transform in the rest frame of the fireball, and introducing a factor $1 = \int dM^2 \delta(r^2 - M^2)$ we obtain with $G(z) = Z(T)$

$$\begin{aligned} v_m(\beta) &= [z(\beta, m)/(dz/d\beta)]^n \cdot \frac{d^n Z(\beta)}{d\beta^n} = \\ &= 4\pi \int_m^\infty dM B\tau(M^2) D_m(M^2) \left(\frac{M}{\beta}\right)^{3/2} \sqrt{M\beta} K_1(M\beta) \end{aligned} \quad (B.5)$$

From this equation $\tau(M^2) D_n(M^2)$ can be expressed doing the inverse K transform as in Eq. (A.2). The asymptotic behaviour of $\tau(M^2) D_n(M^2)$ comes from the singularities of $v_n(\beta)$ in the β -plane. These are, however, at the same positions as the singularities of $Z(\beta)$. Moreover, the leading asymptotic behaviour comes from the term, obtained by partial integration, in which $(-d/d\beta)$ acts on $I_1(M\beta)$ in the integral. This is because such a differentiation produces an extra factor $(-M)$ multiplied by some combination of Bessel functions $I_j(M\beta)$ which, however, for any j behave asymptotically in the same way as $I_1(M\beta)$. Therefore the asymptotic behaviour of $D_n(M^2)$ will be

$$D_n(M^2) \approx M^n \left\{ \frac{z(\beta_0, m)}{(-dz/d\beta)_{\beta_0}} \right\}^n = \left(\frac{M}{m} \right)^n \left\{ \frac{K_1(m\beta_0)}{K_2(m\beta_0)} \right\}^n \quad (B.6)$$

Turning back to Eq. (B.3), it follows from it that if the δ -functions in k_j^2 are separated out from $D_n(r; k_1, \dots, k_n)$,

$$D_n(r; k_1, \dots, k_n) = : D_n^{\text{inv}} \delta_0(m^2 - k_1^2) \dots \delta_0(m^2 - k_n^2) \quad (B.7)$$

then D_n^{inv} is an invariant function of r and $k_1 + \dots + k_n$, hence of r^2 and $(r - k_1, \dots, k_n)^2$ only. Moreover, $\tau(r^2) D_n^{\text{inv}}$ is a function of the variable $(r - k_1, \dots, k_n)^2$ alone

$$\tau(r^2) D_n^{\text{inv}} = : D_n^{(\tau)} [(r - k_1 - \dots - k_n)^2] \quad (B.8)$$

On the basis of Eqs. (B.2, 3) the Laplace transform of $D_n^{(\tau)}$ is given by

$$B \int d^4r e^{-br} D_n^{(\tau)}(r^2) = \frac{d^n G(z)}{dz^n} = \frac{v_m(\beta)}{[z(\beta, m)]^n} \quad (B.9)$$

Introducing the notation M'^2 for $(r - k_1, \dots, k_n)^2$, the asymptotic behaviour of the invariant n -particle inclusive cross-section $D_n^{\text{inv}}(M^2, M'^2)$ can be obtained by the now standard method of K transforms:

$$D_n^{\text{inv}}(M^2, M'^2) \approx e^{-\beta_0(M - M')} D_n(M^2) \cdot \left[2\pi m T_0 K_1\left(\frac{m}{T_0}\right) \right]^{-n} \quad (B.10)$$

This relation is true if $M, M' \rightarrow \infty$ and at the same time $M/M' \rightarrow 1$ [of course, this latter statement does not imply that also $(M - M') \rightarrow 0$]. For fixed (large) M , Eq. (B.10) shows the typical exponential drop as a function of $(M - M')$. Note that the variable M' is the missing-mass belonging to the unobserved pions in the final state, therefore it is always between $(M - nm)$ and m .

The present analysis only supposed that $G(z)$ had a singularity equivalent to a limiting temperature. It not only contains the full non-linear and the linearized bootstrap discussed in Sections 2 and 3, but in fact many more models, as for instance the truncated bootstrap model of Chiu [22].

INFINITE SERIES REPRESENTATION OF THE FUNCTION $G(z)$

The function $G(z)$ defined by the implicit relation Eq. (2.15) plays an important role, as the solution of the bootstrap equation Eq. (2.11) is given in terms of it. For instance, the exclusive probabilities in Eq. (2.19) are determined by the coefficients g_ℓ in the infinite series expansion $G(z) = \sum_{\ell=1}^{\infty} g_\ell z^\ell$. In this appendix infinite series representations are derived for $G(z)$ and g_ℓ .

The inverse function $z(G)$ of $G(z)$ is given by Eq. (2.15)

$$z(G) = (1+\alpha)G + \alpha(1 - e^G) \quad (C.1)$$

The problem is to invert this relation in order to obtain G as a function of z . It is well known [29] that the inverse function can be expressed by a Cauchy integral:

$$G(z) = \frac{1}{2\pi i} \int_{L_G} \frac{g z'(g)}{z(g) - z} dg \quad (C.2)$$

The function $z'(G)$ is the derivative of $z(G)$, and the integration goes in the G -plane along some curve L_G encircling in a positive sense the point $G(z)$. The coefficients b_k of the series expansion of $G(z)$ about $z = z_1$ can easily be expressed from Eq. (C.2). If

$$G(z) = G_1 + \sum_{k=1}^{\infty} b_k (z - z_1)^k \quad (C.3)$$

then b_k is given by

$$b_k = \frac{1}{2\pi i} \int_{L_1} \frac{g z'(g) dg}{(z(g) - z_1)^{k+1}} = \frac{1}{2\pi i k} \int_{L_1} \frac{dg}{(z(g) - z_1)^k} \quad (C.4)$$

The curve L_1 encircles positively the point $G_1 = G(z_1)$ within the analyticity domain about G_1 . Calculating the residuum in the second expression gives

$$b_k = \frac{1}{k!} \left\{ \frac{d^{k-1}}{dG^{k-1}} \left[\frac{G - G_1}{z(G) - z_1} \right]^k \right\}_{G=G_1} \quad (C.5)$$

Instead of the immediate application of this formula to the function $z(G)$ above, it is better to introduce an intermediate step in order to facilitate the calculation. Let us define the function $y(z)$ by

$$y(z) := \alpha e^{G(z)} \quad (C.6)$$

From Eqs. (2.15) $\equiv$ (C.1) it follows then that

$$G(z) = \frac{z - \alpha + y(z)}{1 + \alpha} \quad (C.7)$$

$$y e^{-\frac{y}{1+\alpha}} = \alpha e^{\frac{z - \alpha}{1+\alpha}} =: w(y)$$

The formula (C.5) is well suited to inverting the last relation between y and w , that is to find the series representation of $y(w)$

$$y(w) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{k}{1+\alpha} \right)^{k-1} w^k \quad (C.8)$$

This gives then for $G(z)$

$$G(z) = \frac{z}{1+\alpha} + \frac{\alpha}{1+\alpha} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\alpha k}{1+\alpha} \right)^{k-1} e^{\frac{z - \alpha}{1+\alpha} \cdot k} \quad (C.9)$$

It is easy to show that the domain of convergence of this series is the half plane $\text{Re } z < z_0$ if z_0 is given by Eq. (A.14). From Eq. (C.9) the coefficients g_k turn out to be

$$g_1 = 1$$

$$g_\ell = \frac{\alpha^{1-\ell}}{\ell!(1+\alpha)} \sum_{k=1}^{\infty} \frac{1}{k!} e^{-\frac{\alpha k}{1+\alpha}} \left(\frac{\alpha k}{1+\alpha}\right)^{k+\ell-1} \quad (C.10)$$

The infinite sum over k in this expression converges for all $0 \leq \alpha < \infty$. For large ℓ we can evaluate g_ℓ by approximating the sum by an integral and using the Stirling formula; the result agrees with that of Yellin [12]

$$g_\ell \xrightarrow{\text{large } \ell} z_0^{-\ell} \ell^{-3/2} \sqrt{\frac{z_0}{2\pi(1+\alpha)}} \quad (C.11)$$

$$z_0 = (1+\alpha) \ln\left(1 + \frac{1}{\alpha}\right) - 1$$

REFERENCES

- 1) R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965).
- 2) R. Hagedorn, Nuclear Phys. B24, 93 (1970).
- 3) R. Hagedorn, CERN Lecture Notes, CERN 71-12 (1971).
- 4) E. Beth and G.E. Uhlenbeck, Physica 4, 915 (1937).
- 5) S.Z. Belenkij, Nuclear Phys. 2, 259 (1956).
- 6) R. Hagedorn and J. Ranft, Nuovo Cimento Suppl. 6, 169 (1968).
- 7) S. Frautschi, Phys. Rev. D3, 2821 (1971).
- 8) W. Nahm, Nuclear Phys. B45, 525 (1972).
- 9) S. Frautschi and C.J. Hamer, Phys. Rev. D4, 2125 (1971).
- 10) S.C. Frautschi and C.J. Hamer, Nuovo Cimento 13 A, 645 (1973).
- 11) C.J. Hamer, Nuovo Cimento 12 A, 162 (1972).
- 12) J. Yellin, An explicit solution of the statistical bootstrap, CERN preprint TH-1513 (1972).
- 13) I. Montvay, Phys. Letters B42, 466 (1972).
- 14) i) M. Banner, G. Bassompierre, M. Croissiaux, J. Gresser, J.L. Hamel, R. Morand, J.P. Pansart, M. Riedinger, M. Schneegans, A.V. Stirling, J. Teiger, H. Zacccone and J. Zsembery, Contribution to the 16th Int. Conf. on High-Energy Physics, Batavia, 1972, paper No. 478.

ii) CERN-Columbia-Rockefeller Collaboration, Contribution to the 16th Int. Conf. on High-Energy Physics, Batavia, 1972 (no number assigned).

iii) B.J. Blumenfeld, F.W. Büsser, L. Camilleri, R.L. Cool, L. Di Lella, G. Gladding, L.M. Ledermann, L. Litt, A. Placci, B.G. Pope, S.L. Segler, A.M. Smith, J.K. Yoh and E. Zavattini (CERN-Columbia-Rockefeller Collaboration) presented at the New York A.P.S. Meeting (29 Jan. 1973).
- 15) K. Imaeda, Nuovo Cimento Letters 1, 290 (1971).
- 16) I. Montvay, Integral equations for the chain decay of fireballs, CERN preprint TH-1572 (1972).
- 17) A.H. Mueller, Phys. Rev. D4, 150 (1971).
L.S. Brown, Phys. Rev. D5, 748 (1972).
K.J. Biebl and J. Wolf, Nuclear Phys. B44, 301 (1972).

- 18) R.C. Hwa, Phys. Rev. Letters 26, 1143 (1971).
M. Jacob and R. Slansky, Phys. Letters 37 B, 408 (1971).
R.K. Adair, Phys. Rev. D5, 1105 (1972).
F. Johns, B. Margolis, W.J. Meggs and R.K. Logan, Phys. Rev. Letters 29, 756 (1972).
- 19) R.D. Carlitz, Phys. Rev. D5, 3231 (1972).
- 20) For a review see:
C. Møller, K. Danske Vidensk. Selsk. Mat.- Fys. Medd. 36, 1 (1967).
- 21) B. Touche, Nuovo Cimento 58, 295 (1968).
- 22) i) C.B. Chiu, A truncated statistical bootstrap equation,
Univ. of Texas preprint, CPT-153 (1972).

ii) C.B. Chiu and J. Johnson, Phys. Letters B42, 475 (1972).
- 23) See for instance (this list is not exhaustive):
i) H. Grote, R. Hagedorn and J. Ranft, Atlas of particle spectra
(CERN, Geneva, 1970).

ii) R. Hagedorn and J. Ranft, Nuclear Phys. B48, 159 (1972).

iii) J. Ranft, Phys. Letters 41 B, 613 (1972).

iv) J. Letessier and A. Tounsi, Nuovo Cimento 11 A, 353 (1972); and
Inclusive thermodynamical spectra in the range from 19 to
1500 GeV/c, Paris Univ. Preprint IPNO/TH 72-6 (1972).
- 24) E.M. Ilgenfritz and J. Kripfganz, Statistical bootstrap of fireball
decay spectra, Univ. of Leipzig (DDR) preprint, KMU-HEP-73-2 (1973).
- 25) A. Erdélyi, Tables of integral transforms (McGraw-Hill, Inc.,
New York, 1954), Vol. II, Chapter X.
- 26) I.S. Gradshteyn and I.M. Ryzhik, Tables of integrals, series and pro-
ducts (Academic Press, Inc., New York and London, 1965), Section 8.45.
- 27) W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and theorems for
the special functions of mathematical physics, 3rd edition (Springer,
Heidelberg-New York, 1966), Section 6.5.
- 28) Same as ref [27], Section 6.8.2.
- 29) H. Behnke and F. Sommer, Theorie der Funktionen einer komplexen
Veränderlichen (Springer, Berlin, 1955).
- 30) R. Hagedorn, J. Reinfelds, C. Vandoni and L. Van Hove: SIGMA, a new
language for interactive array oriented computing, CERN 73-5 (1973).

Figure captions

- Fig. 1 : Qualitative behaviour of the fireball partition function $Z(T) \equiv G(z)$ as function of the single-particle partition function $z(T,m)$. Significance of the curves: $\{a_1 a_2\}$ full bootstrap [Eqs. (2.15), (2.23)], square root singularity at z_{0full} ; $\{b_1 b_2\}$ linear bootstrap [Eq. (3.4)], pole at z_{0lin} . In these two cases T_0 is an absolute upper limit of the temperature in contradistinction to ordinary phase transitions, where the temperature can rise beyond the singularity T_0 [curves $\{a_1 c\}$ and $\{a_1 d\}$].
- Fig. 2 : The relation between B/B_0 and T_0/m according to Eq. (2.29) [full bootstrap; full curve] and Eq. (3.6) [linear bootstrap; dotted line]. In both cases $\alpha = 1$ and $B_0 = (4\pi m^2)^{-1}$. We consider the two values of T_0 belonging to $B/B_0 = 1$ to be respectively the "natural limiting temperatures" of the two models. They are used in all further figures.
- Fig. 3 : Effective temperature [Eq. (2.55)] of pions emitted in the first generation as a function of their energy E (measured in the rest frame of the parent fireball of mass M). A Boltzmann spectrum with this variable temperature would produce the first generation pion distribution $d^{(1)}(M,E)$ [Eq. (2.53)]. Compare with Fig. (12), linear bootstrap. Natural limiting temperature $T_0 = 1.098 m$.
- Fig. 4 : Effective temperature averaged over all first generation pions with energy $\leq E$ according to Eq. (2.56). A Boltzmann spectrum with the energy-independent temperature $T_{av}^{(1)}(M, M/2)$ would yield a reasonable over-all approximation to $d^{(1)}(M,E)$. Note that $T_{av}^{(1)}(M,E) > T_0$. Compare with Fig. (13), linear bootstrap. Natural limiting temperature $T_0 = 1.098 m$.
- Fig. 5 : Mass spectrum of the linear bootstrap in the pole approximation [Eq. (3.9)]. — · — · — · — : asymptotic form $\tau_p^\infty(M^2)$. — — — : asymptotic form with first correction term. ————— : pole approximation $\tau_p(M^2)$. $T_0 = 1.383 m$ is the natural limiting temperature of the linear bootstrap.

- Fig. 6 : Mean final pion multiplicity in the linear bootstrap as function of the fireball mass M ; pole approximation [Eq. (3.10)]. The multiplicity is always roughly proportional to M ; the solid curve shows the proportionality "constant" together with its asymptotic limit and its crudest thermodynamical estimate $(m + 3T_0/2)^{-1}$. Natural limiting temperature $T_0 = 1.383$ m.
- Fig. 7 : The approach of the final single-pion distribution (linear bootstrap, pole approximation) to its asymptotic limit as a function of the energy E (fireball rest frame) for several fireball masses M [Eq. (3.11)]. Natural limiting temperature $T_0 = 1.383$ m.
- Fig. 8 : The same as Fig. 7 but in logarithmic scale and normalized to 1 at threshold $E = m$.
- Fig. 9 : Effective temperature [Eq. (3.14); linear bootstrap, pole approximation] for the final pion distribution. A Boltzmann spectrum with this variable temperature would have the shape of the correct distribution $d_p(M, E)$. Note that these final effective temperatures lie all below T_0 in contradistinction to the first generation effective temperatures [see Fig. 12]. Natural limiting temperature $T_0 = 1.383$ m.
- Fig. 10 : Final effective temperature averaged over all pions with energy $\leq E$ [Eq. (3.15); linear bootstrap, pole approximation]. A Boltzmann spectrum with the single temperature $T_{av}(M, M/2)$ would yield a reasonable over-all approximation to $d(M, E)$. Note that $T_{av}(M, E)$ lies below T_0 for all M and E in contradistinction to the first generation average temperature [see Fig. 13]. Natural limiting temperature $T_0 = 1.383$ m.
- Fig. 11 : Final average temperature $T_{av}(M)$ [Eq. (3.16); linear bootstrap, pole approximation] as a function of M (full curve) compared to the canonical temperature T [Eq. (3.17), (3.19)] which would lead to the same mass M (dotted curve). Natural limiting temperature $T_0 = 1.383$ m.

Fig. 12 : First generation effective temperature [linear bootstrap, pole approximation; otherwise same definition as in Fig. 3]. Natural limiting temperature $T_0 = 1.383$ m.

Fig. 13 : First generation average temperature [linear bootstrap, pole approximation; otherwise same definition as in Fig. 4]. Note the difference between Figs. 12, 13 and Figs. 9, 10: while in the first generation effective and average temperatures lie partly above T_0 , both lie below T_0 in the final distribution. Natural limiting temperature $T_0 = 1.383$ m.

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