CM-P00060413

Archives

Ref.TH.1662-CERN

A SIMPLE PHYSICAL INTERPRETATION OF THE CRITICAL
DIMENSION OF SPACE-TIME IN DUAL MODELS

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ABSTRACT

The energy due to zero point fluctuations of the dual string is calculated and shown to be divergent. We make it finite by introducing a cut-off. The result is non-covariant but invariant under reparametrization, so one has to modify the classical Lagrangian so as to get a covariant result. It is then shown that this leads to the well-known relation relating the mass squared of the lowest state to the number of dimensions of space-time. This result is independent of the cut-off and shows clearly the physical significance of the critical dimension of space-time. Similar considerations for the Neveu-Schwarz model are also discussed.

A fundamental property of dual models which has been evident during the last year is the dependence of the models on the dimension of space-time. It was first conjectured ¹⁾ that the Pomeron-like singularity, which is obtained from a specific loop graph, is unitary only in 26 dimensions (25 space + 1 time) for the ordinary dual model. By studying the spectrum of the dual models, it was then found ²⁾ that if the dimension $d \leq 26$ (10 for the Neveu-Schwarz model), there are no negative norm states ("ghost" states) coupling in the Born terms. Furthermore, in the critical dimension (26 and 10, respectively) the number of coupling dimensions is only $d-2$, the degrees of freedom being described by transverse degrees only. The dependence of space-time was clarified considerably in the fundamental paper on the quantum mechanics of a relativistic string by Goddard, Goldstone, Rebbi and Thorn ³⁾. They showed that when quantizing the string, this was possible only if $d=26$ (at least if one had chosen a specific gauge). Then the spectrum of this string is exactly the same as that of the ordinary dual model. They also obtained a relation determining the mass of the lowest state from the dimension of space-time ⁴⁾.

We shall in this letter concentrate on the latter question, giving a hopefully simpler and more physical explanation of the mentioned relation. The reasoning is naturally similar to the one by Goddard et al., but we want to stress the physics behind it. By doing that, it will be rather evident how to proceed in determining similar relations in other dual models and we end this note by giving a discussion for the Neveu-Schwarz model. A probably far-fetched hope could be that in this way one could understand how a realistic hadron string would behave.

The way we are interpreting the mass of the lowest state is to attribute it to the energy due to zero-point fluctuations of the dual string. This will be shown to be a divergent contribution to the mass squared. However, this contribution can be seen to be non-covariant. To restore Lorentz covariance, it is thus necessary to introduce a non-covariant modification of the classical Lagrangian for the dual string. A covariant finite mass squared will then be found left over which fulfils

$$m_0^2 = - \frac{d-2}{24} \frac{1}{\alpha'}, \quad (1)$$

where α' is the universal slope.

We take the classical ground state that we consider as a rotating string in the limit when it shrinks to a point. This ground state can be seen to be massless. This is implicit in the formulae of Ref. 3). It can,

for example, be seen if one chooses an "orthonormal" parametrization such that

$$\frac{\partial X}{\partial \sigma} \cdot \frac{\partial X}{\partial \tau} = 0 \quad (2)$$

$$\left(\frac{\partial X}{\partial \sigma}\right)^2 + \left(\frac{\partial X}{\partial \tau}\right)^2 = 0 \quad (3)$$

Then the equation of motion is

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) X^\mu(\sigma, \tau) = 0 \quad (4)$$

σ and τ are the parameters of the string. If we consider a solution which is independent of σ then (4) shows that

$$X^\mu = q^\mu + p^\mu \tau \quad (5)$$

Equation (3) is then $(\partial X / \partial \tau)^2 = 0$ which means that $p^2 = 0$ and hence the particle is massless. One can now check that this solution also fulfils the boundary conditions.

The ground state of the quantum mechanical string must have an energy due to zero-point fluctuations of the harmonic oscillators into which the dynamical variables of the string can be expanded. We shall show that this zero point energy can be written as

$$E_{\text{zero}} = d_{\text{eff}} \sum_{n=1}^{\infty} \frac{1}{2} \omega_n = d_{\text{eff}} \sum_{n=1}^{\infty} \frac{1}{2} \frac{1}{\alpha'} \frac{n}{2E} \quad (6)$$

which diverges. d_{eff} is the number of phonon polarizations. Hence for a string with only transverse modes $d_{\text{eff}} = d-2$. E is the total energy of the string.

To get this expression (6), we have used the physical properties of the classical dual string pointed out by Goddard et al. The energy density along the string is $\gamma_{\perp} / 2\pi\alpha'$ [see Eq. (9a') of Ref. 3], where $\gamma_{\perp} = (1-v_{\perp}^2)^{-\frac{1}{2}}$ and $v_{\perp}$ is the transverse velocity. (We have chosen $c = \hbar = 1$.) By considering Lorentz invariance under boosts along the string, one can show that the speed of sound is equal to the speed of light when the string is at rest. From Lorentz invariance, it then follows that the speed of sound is $1/\gamma_{\perp}$.

We now introduce a parametrization of the string such that the energy density along the string is unity. With this choice of parameters the speed of sound is $v_s = 1/2\pi\alpha'$, which is thus the energy passed by the phonon per unit time. It is now evident that the wavelengths of the standing modes of the phonons are

$$\lambda_n = \frac{2E}{n} \quad (7)$$

corresponding to frequencies

$$\omega_n = \frac{2\pi}{\lambda_n} v_s = \frac{1}{\alpha'} \frac{n}{2E} \quad (8)$$

and hence (6) follows.

To make expression (6) finite, we have to introduce a cut-off of the high frequencies. A cut-off corresponding to an atomic structure of the string, where each atom has the same energy and thus the inter-atomic distance is constant in our parametrization, gives the zero point energy

$$E_{\text{zero}} = \text{def} \sum_{n=1}^{\infty} \frac{1}{2} \frac{1}{\alpha'} \frac{n}{2E} f\left(\frac{n}{E}\right) \quad (9)$$

where f is the cut-off function being unity for values of n/E not too high and $f \rightarrow 0$ when $n/E \rightarrow \infty$ fast enough for E_{zero} to be finite. This corresponds to a smooth cut-off at a given frequency measured from the frame of reference in consideration. We calculate expression (9) by comparing it with the corresponding integral

$$\begin{aligned} E_{\text{zero}} &= \text{def} \int_0^{\infty} dx \frac{1}{2\alpha'} \frac{x}{2E} f\left(\frac{x}{E}\right) - \text{def} \int_0^{1/2} dx \frac{1}{2\alpha'} \frac{x}{2E} f\left(\frac{x}{E}\right) \\ &+ \text{def} \left(\sum_{n=1}^{\infty} \left(\frac{1}{2\alpha'} \frac{n}{2E} f\left(\frac{n}{E}\right) - \int_{n-1/2}^{n+1/2} dx \frac{1}{2\alpha'} \frac{x}{2E} f\left(\frac{x}{E}\right) \right) \right) = \\ &= \frac{\text{def}}{4\alpha'} E \int_0^{\infty} dy y f(y) - \frac{\text{def}}{16\alpha'} \frac{1}{2E} \\ &- \text{def} \left[\sum_{n=1}^{\infty} \frac{1}{24} \frac{d^2}{dx^2} \left[\frac{x}{2E} \frac{1}{2\alpha'} f\left(\frac{x}{E}\right) \right]_{x=n} \right] + O\left(\frac{1}{E^2}\right) = \\ &= \frac{\text{def}}{4\alpha'} E \int_0^{\infty} dy y f(y) - \frac{\text{def}}{16\alpha'} \frac{1}{2E} - \frac{\text{def}}{24} \left[\frac{d}{dx} \left[\frac{x}{2E} \frac{1}{2\alpha'} f\left(\frac{x}{E}\right) \right] \right]_{1/2}^{\infty} + O\left(\frac{1}{E^2}\right) \\ &= \frac{\text{def}}{4\alpha'} E \int_0^{\infty} dy y f(y) - \frac{\text{def}}{24\alpha'} \frac{1}{2E} + O\left(\frac{1}{E^2}\right) \end{aligned} \quad (10)$$

We will neglect the terms of order $O(1/E^2)$, which means that we consider the string in the infinite momentum frame. The term proportional to E is non-covariant due to the non-covariant cut-off chosen. The zero point energy E_{zero} is the difference between the energy for the quantum mechanical ground state and the corresponding classical one. $E_{\text{zero}} = E - |\vec{p}|$, as the energy of the classical ground state is the same as its momentum, since this state is massless. As we consider the string in the infinite momentum frame $2E = E + |\vec{p}|$. Hence Eq. (6) can be written as

$$E^2 - |\vec{p}|^2 = \frac{d_{\text{eff}}}{4\alpha'} E(E + |\vec{p}|) \int_0^\infty dy y f(y) - \frac{d_{\text{eff}}}{24\alpha'} + O\left(\frac{1}{E}\right) \quad (10')$$

Thus the second term on the right-hand side can be interpreted in the infinite momentum frame as the change in mass squared by $-(d_{\text{eff}}/24\alpha')$ due to quantization and is then covariant.

To restore covariance, we have to change the classical action that we started with in a non-covariant way. We have then to keep in mind when changing the action to still keep the invariance under reparametrizations. The covariant classical action is

$$\begin{aligned} S &= -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[\left(\frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau} \right)^2 - \left(\frac{\partial x}{\partial \sigma} \right)^2 \left(\frac{\partial x}{\partial \tau} \right)^2 \right]^{1/2} \\ &= -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[F^{\mu\nu} F_{\mu\nu} \right]^{1/2} \end{aligned} \quad (11)$$

where

$$F^{\mu\nu} = \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} - \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma}$$

The simplest non-covariant modification of (11) which is still rotation and reparametrization invariant is

$$S = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[F^{\mu\nu} F_{\mu\nu} + 2 F^{\mu\nu} F_{\mu 3} a^3 a_\nu \right]^{1/2} \quad (12)$$

where $a_\mu = (\epsilon, 0, 0, 0)$ in the frame in which we made the cut-off. The quantity E_{zero} is a quantum effect and thus small of the order of $\hbar$. Thus

the Lorentz invariance breaking needed in the classical Lagrangian to restore covariance at the quantum mechanical level is also small of the order of $\hbar$. As thus ϵ is small of the order of $\hbar$, expression (12) is unique, if we do not allow for expressions with second derivatives, which would change the theory too drastically.

The extra term in S in (12) can be interpreted as due to a change in the scale of energy and time. In fact, action (12) is obtained from action (11) by the change

$$x^0 \rightarrow (1+\epsilon) x^0 \quad (13)$$

If ϵ is negative, the change of variable (13) means that the time scale in the new classical system has decreased, and hence velocities and energies have increased in comparison with the old classical system. Solving Eq. (10) again means that on the left-hand side we will get something proportional to E which is bigger than E . Now ϵ can be chosen so that the non-covariant term in Eq. (10) goes away.

We have now shown that this theory has a lowest state of mass squared

$$m_0^2 = - \frac{d_{\text{eff}}}{24\alpha'} \quad (14)$$

We note that this result is independent of the cut-off.

The result (14) does not directly tell us about the magnitude of d_{eff} , but only relates it to the ground state mass squared m_0^2 . We know, however, that the string under consideration has only transverse degrees of freedom. Lorentz invariance then tells us that the spin 1 particle on the leading trajectory must have mass 0. This means that

$$m_0^2 = - \frac{1}{\alpha'} = - \frac{d_{\text{eff}}}{24\alpha'} = - \frac{d-2}{24\alpha'} \quad (15)$$

from where it follows that

$$d = 26 \quad (16)$$

It is from a physical point of view very reasonable that the zero point fluctuations force a Lorentz covariance breakdown, since these must necessarily bring about a slow-down of the effective velocity of sound along the string, where the effective velocity of sound is defined by the ratio $\Delta x / \Delta t$ for Δx and Δt large compared to the zero point fluctuating modes.

Zero point fluctuations namely curl the string such that a sound signal has to run a longer way than Δx to pass a distance Δx . If the classical velocity of sound on the string were not larger than that of light, the effective velocity of sound on the quantum mechanical string would be smaller than that of light. Covariance arguments and the form of the dispersion law for phonons on the string $\omega \propto k$ imply, however, that the effective speed of sound on the quantum mechanical string must equal that of light. Thus we have to accept a "renormalization" of the speed of light and start from a classical string Lagrangian, which is non-covariant, because it describes a string with an unrenormalized speed of light larger than the physical speed of light by a cut-off dependent amount.

Also for the Shapiro-Virasoro model ⁵⁾ we can interpret the mass squared of the ground state particle as due to zero point fluctuations. Since the set of harmonic oscillators is just obtained by doubling that of the conventional model, the mass squared becomes also doubled when calculated in terms of the slope for the Virasoro-Shapiro model. That is to say,

$$m_0^2 = - \frac{d_{\text{eff}}}{12\alpha_{\text{s.v.}}}$$

Since the spin 2 particle on the leading trajectory has to be massless, we can conclude once again that $d_{\text{eff}} = 24$.

A similar calculation may be performed for the Neveu-Schwarz model ⁶⁾. We do not know, however, a classical Neveu-Schwarz string, because there are anticommuting fields on this string. We can thus only give an intuitive argument.

We assume that the zero point energy of a Neveu-Schwarz mode can be calculated in an analogous way to that of a harmonic oscillator. By the Neveu-Schwarz mode, we understand a quantum mechanical system that can be in just two states. This can be described by anticommuting operators b_n^μ and $b_n^{\mu\dagger}$. In analogy to the case of the harmonic oscillator where

$$H = \frac{1}{2} (a^\dagger a + a a^\dagger) = a^\dagger a + \tilde{E}_{\text{zero}} \quad (17)$$

we define

$$H_{\text{NS}} = \frac{1}{2} (b^\dagger b - b b^\dagger) = b^\dagger b + \tilde{E}_{\text{zero, NS}} \quad (18)$$

and find

$$\tilde{E}_{\text{zero,NS}} = -\frac{1}{2} \omega \quad (19)$$

where ω is the energy difference between the two levels. We remark that the Neveu-Schwarz zero point energy $\tilde{E}_{\text{zero,NS}}$ is negative, contrary to that of the harmonic oscillator.

In the Neveu-Schwarz model, it is well known that the eigenmodes of the Neveu-Schwarz type have wavelengths

$$\lambda_n = \frac{2E}{n} \quad (20)$$

where n now is half-integer.

We can now sum up the total zero point energy from these modes

$$E_{\text{zero,NS}} = d_{\text{eff}} \sum_{n=1/2}^{\infty} \left(-\frac{1}{2}\right) \frac{1}{\alpha'} \frac{n}{2E} f\left(\frac{n}{E}\right) \quad (21)$$

We calculate this in an analogous way to Eq. (10) and find

$$E_{\text{zero,NS}} = -\frac{d_{\text{eff}}}{4\alpha'} E \int_0^{\infty} dy y f(y) - \frac{d_{\text{eff}}}{48\alpha'} \frac{1}{2E} + O\left(\frac{1}{E^2}\right) \quad (22)$$

In the Neveu-Schwarz model we have also the conventional harmonic oscillator modes which will give the same contribution, Eq. (10), as before to the zero point energy. The total zero point energy is then found to be

$$E_{\text{zero,total}} = -\frac{d_{\text{eff}}}{16\alpha'} \frac{1}{2E} + O\left(\frac{1}{E^2}\right) \quad (23)$$

We note that the non-covariant terms cancel provided we can use the same cut-off function for the two types of modes, so the introduction of a non-covariant "classical" model may be superfluous in this case.

If we assume that the corresponding classical ground state is a massless point-particle as in the other case, which is very reasonable, we can reach the conclusion that the lowest state in the Neveu-Schwarz model fulfils

$$m_0^2, ns = - \frac{d_{eff}}{16\alpha'}, \quad (24)$$

The Neveu-Schwarz model has been considered in two different gauges, the so-called $\mathcal{F}_1$ and $\mathcal{F}_2$ spaces ⁶⁾. In the $\mathcal{F}_1$ space the lowest state has $m^2 = -1/\alpha'$, but can be shown to be decoupled, while in the $\mathcal{F}_2$ space the lowest state has $m^2 = -1/2\alpha'$. If we insert this state into (24) we deduce that $d_{eff} = 8$. If we want to have only transverse modes d_{eff} has to be $d-2$ and then $d=10$, which is the result we want.

Since we have only been dealing with free strings, it is not obvious which of the two states to insert into (24). It is gratifying to the heuristic considerations that it is the lowest state in the theory with interaction that leads to the right critical dimension.

We note that the two contributions to the mass squared are both negative. The reasoning would not have been changed if one had introduced a spin orbit coupling in the Hamiltonian, as this term would be built up by commuting operators. Thus the result that the lowest state is a tachyon is rather general for strings like the two considered. A way out of this dilemma could be to introduce asymmetric boundary conditions. This could be obtained through a spontaneous breakdown or by putting different quarks at the ends (which could be the same thing).

We conclude that we have found a physical interpretation of the ground state mass squared in string models as zero point fluctuations. This means that relations of type (1) and (24) are fundamental for string models, making it thus difficult to escape the dependence on the dimension of space-time for such models.

We thank David Olive for encouraging discussions and are grateful to Z. Koba and A.J. Buras for a discussion on approximating a sum by an integral in another connection (multiplicities). One of us (H.B.N.) wants to thank CERN Fölgeforskning for a grant to visit CERN.

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