REGGE LIMIT AND SCALING IN A DUAL MODEL OF CURRENTS

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A B S T R A C T

We present a model of off-shell dual amplitudes satisfying duality, factorization, absence of exotics, having no unphysical singularities and reproducing the N point function of the conventional dual model on the mass shell. These off-shell amplitudes can be identified with currents, however, the channel asymptotic behaviour of the Compton off-shell amplitude yields only moving poles but no fixed poles. Further, the Bjorken limit at $t = 0$ yields scaling behaviour, but with anomalous dimension, at the expense of having Gaussian form factors. This suggests that a pure current-hadron bootstrap scheme based on the quarkless conventional dual model describes a too "soft" structure to be able to incorporate fixed poles, canonical scaling and power-behaved form factors.

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1. - INTRODUCTION

A long standing problem has been to define off-shell extrapolations of the dual resonance model consistent with its basic properties. Such off-shell extrapolations are needed for purposes internal to the dual model such as the study of the renormalization of planar loop graphs through a method of analytic continuation ^{1),2)}, or for solving the problem of the mixing of states of different masses, or for eventually introducing a Higgs-type mechanism to generate a model with a massive ρ meson ³⁾. Another problem of obvious interest is the coupling of leptons to the hadronic particles in the framework of the dual model. Since duality seems to be a concept valid also in the deep inelastic region ⁴⁾, one may hope that the off-shell extrapolations of the dual model would be a natural basis to construct current amplitudes satisfying Regge behaviour, duality and scaling.

As a first step towards the realization of such a program, one seeks to construct a model within the framework of the conventional dual model satisfying the following requirements.

- 1) The model describes a set of dual amplitudes for the scattering of (scalar/vector/tensor) currents on scalar ground state tachyons such that these amplitudes have only poles in the Mandelstam variables of allowed channels and in the external masses. No cuts or mass-dependent poles are allowed in these amplitudes.
- 2) These amplitudes can be written in a factorized form, within the original space spanned by the oscillators $a_{\mu}^{(n)}$ which factorize the on-shell amplitudes. The poles in the external masses should be generated by conventional propagators so that when all currents are put on the mass shell, the amplitude develops a multiple pole whose residue is a hadronic on-shell amplitude. In that case we have a hadron-current bootstrap scheme ⁵⁾.
- 3) Appropriate conservation laws should be obeyed by vector/tensor... amplitudes.
- 4) Two-kinds of amplitudes can be considered : non-planar or planar-current amplitudes. Non-planar currents do not carry any charge or isospin and their residue on the mass shell is a sum of the Veneziano terms obtained by putting the current legs anywhere on the graph. Planar currents carry charges like isospin, Chan-Paton factors are associated with each current leg, and their residue on the mass shell is a single Veneziano term. Both are a priori physical since amplitudes for graviton scattering from hadrons ⁶⁾ may well be described by non-planar amplitudes while electromagnetic amplitudes have to be planar.

- 5) Regge asymptotic behaviour should be maintained off the mass shell for fixed values of the external masses.

These postulates define amplitudes which are suitable off-shell extrapolations of the conventional dual model. In terms of these amplitudes one can eventually construct currents. In the simplest possible hadron bootstrap scheme each of these scalar/vector... amplitude is identified with a current. This identification may, however, not be true, for instance the electromagnetic currents may be a superposition of a vector off-shell amplitude satisfying a kind of general vector dominance and of amplitudes without poles in the external masses. In the following this distinction will be often forgotten, since we will be mainly dealing with scalar currents/off-shell amplitudes.

Although this seems to be a loose framework failing to incorporate such essential requirements as causality, current algebra or scaling, already to themselves these conditions seem to be very stringent, and have been very hard to meet. For instance, some models avoid unphysical singularities but increase the hadronic spectrum ⁷⁾, others meet factorization correctly but have unphysical singularities, others have exotics due to non-planarity [see Ref. 8].

In Refs. 2)-9), 10) a general framework was proposed to solve simultaneously the conditions (1)-(5). This framework is based on the factorization proved in Ref. 2) of the Pomeron singularity in the Virasoro case $\alpha(0) = 1$. When $D = 26$, the Pomeron singularity becomes a sector of factorizable resonances and the non-planar orientable one-loop graph can be equally regarded as a graph where direct transitions occur between the "Reggeonic" sector and the "Pomeronic" sector (which seems closely related if not identical to the Shapiro-Virasoro model). Since Pomeronic and Reggeonic resonances are not on the same mass shell, the expansion of this graph in terms of such transitions provides us automatically with an infinite set of one-current amplitudes of any spin, satisfying the above conditions. These amplitudes are non-planar, however, by restricting the integration region of the Koba-Nielsen variables one can also generate a set of planar amplitudes satisfying the same basic criteria. Then using factorization multicurrent amplitudes are constructed ¹⁰⁾. ~~Although the mathematical sophistication of the model may seem exaggerate one may think that such a sophistication is~~ needed since earlier attempts have failed.

This framework can also be applied to the dual pion model in the one-current case and yields a conserved vector current and an axial vector current satisfying a PCAC type of relation ¹¹⁾.

Negative norm states are automatically decoupled from one-current amplitudes and should be also therefore eliminated from multicurrent amplitudes ; however, this problem has not yet been solved due to its great difficulty. Nothing a priori prevents off-shell amplitudes to propagate ghosts, however, this would be a bad property for physical currents.

Since non-planar currents have already been studied extensively ^{9),10)} we concentrate on the planar off-shell amplitudes. As the mathematical derivation of these from factorization has already been presented ¹⁰⁾ we concentrate here on the physical properties of these amplitudes and mathematical details are contained in the Appendices. The presentation of these amplitudes is self-contained so that the reader does not need to be aware of the previous papers concerning this subject. Only the simplest of these planar amplitudes are studied, i.e., amplitudes involving scalar currents.

In the second part we exhibit scalar one-current amplitudes written in a Möbius-invariant way, and discuss their integration region. We recall the results obtained in the Regge limit. We exhibit also a conserved vector amplitude which could be a candidate for the electromagnetic current. In the case of the electromagnetic form factor, the ground state both in the conventional and in the Neveu-Schwarz model exhibits only the ρ pole, while higher ρ recurrences couple to four or more ground states. This may explain why ρ recurrences do not appear in two pions but eventually do in four-pion channels ¹²⁾ At that stage, the one-current amplitudes have an arbitrary normalization factor $\lambda_0^{-\alpha(q^2)}$ multiplying them so that exponentially decreasing form factors are expected unless λ_0 can be set equal to 1.

In the third part, we study two-current amplitudes in the scalar and planar cases. The integration region is examined in detail, and it is shown how planarity and the absence of unphysical singularities for fixed masses is guaranteed. Then we show in the fourth part that the two-current amplitudes obey Regge behaviour. An unconventional type of Regge emerges, involving both a conventional term $s^{\alpha(t)}$ and a non-conventional term $s^{\tilde{\alpha}(t)}$ where $\tilde{\alpha}(t)$ is a companion trajectory, of slope $\alpha'/2$ whose intercept depends on the external masses. Remarkably enough, one can show that this companion trajectory

has a vanishing residue at each integer so that it never materializes into resonances, in consistency with condition (1). On shell the residue of the companion term vanishes so that conventional Regge behaviour is reinstated. In the case of one-current amplitudes the model was compared to photoproduction data ¹³⁾ and a reasonable description of the energy dependence at large momentum transfer was obtained. No fixed pole behaviour is found in two-current amplitudes which makes it dubious that current algebra can be satisfied in such a model, although one may object that since we are dealing only with scalar currents the problem is not relevant.

In our model, the mathematical origin of the companion trajectory $\tilde{\alpha}(t)$ is very simple : our planar amplitudes have the same integrand as non-planar amplitudes, but a restricted integration region. Since non-planar amplitudes are Regge behaved, with conventional Regge trajectories, the new boundaries introduced by planarization must result either in additional exponential behaviour, or in additional Regge trajectories, or in additional fixed poles. Our scheme excludes exponential behaviour and yields an additional Regge piece, but no fixed poles.

Examining then the Bjorken limit of the two-current amplitude, we notice that a power behaviour in q^2 can only be achieved by fixing λ_0 to a defined value different from 1 so that indeed we have exponentially falling form factors. This situation is the same as has been found in the relativistic quark model of Feynman, Kislinger and Ravndal ¹⁴⁾ and seems to be due in both cases to the softness of the harmonic forces at short distances. It was also suggested ¹⁵⁾ that field theoretic unitarity corrections may alter this asymptotic behaviour and we will take this attitude. In this model, which looks very close to the Nielsen-Susskind-Kraemer approach ¹⁶⁾, the exponential behaviour of the form factor is not put in as a new fundamental constant introducing a minimal distance between partons but rather is fixed by factorization.

Although the model does not look realistic for large q^2 one can still investigate the deep-inelastic limit of the off-shell Compton amplitude. One finds that scaling does occur, however, with a non-canonical dimension depending on the t channel, i.e., the amplitudes behave like $(-q^2)^{\alpha(t)-\frac{3}{2}}$. So the scale dimension at $t = 0$ is related to the Regge intercept of the t channel, a situation met already in some models ¹⁷⁾ and which stems from the fact that the t channel contains only conventional resonances. The scaling

function $(\omega_0 - \omega)^{\alpha(t)}$ although non trivial is much too simple to account for experiment. All these limits are smooth and satisfy the commutativity hypothesis of Brandt¹⁸⁾. Conventional light cone dominance, however, does not hold in this model, due to the t dependence of the scale dimension.

Finally we check that in a two-current amplitude, if one of the current masses is held fixed, and the other goes to infinity the same exponential behaviour, $\lambda_0^{-\alpha(q_1^2)}$ is obtained as in the case of the one-current amplitude. λ_0 therefore really characterizes the form factor behaviour of these amplitudes, irrespective of the number of on-shell or off-shell legs.

Hence our model of off-shell amplitudes has the paradoxical properties of being compatible with the conventional dual resonance model, but to fail in producing fixed poles, canonical light cone behaviour and power-behaved form factors. We feel that although the ghost problem has not been fully solved in our model, the trouble does not lie in our construction of off-shell amplitudes, but rather on the hadron-current scheme based on the conventional model. Firstly, there is the problem of the relation between currents and off-shell amplitudes. In a hadronic model where satisfactory off-shell amplitudes have been derived, one gets an infinite number of off-shell amplitudes and currents may be an infinite superposition of these. Then of course even if each off-shell amplitude has the above features, it may be that the sum behaves differently. Secondly, it is not obvious either that all currents must proceed directly through hadrons, like in a generalized vector dominance model. We know for sure that there is a piece of the currents that we would like to be resonance dominated, but it is not obvious that all have to be resonance dominated. So there may be an additional piece in the currents which would have no singularities in the external masses and eventually would provide scaling and fixed pole singularities. However, to introduce such a piece, we are lacking real "quarks" in the conventional dual model which consists only of a "gluonic" string¹⁹⁾ with free ends, and no quarks (only mathematical quarks are introduced via Chan-Paton factors). So it is not obvious whether such a two-component theory of currents would even work in the framework of the conventional dual model. Indeed, the absence of any point-like constituents reflects itself in the failure of the conventional dual model even on the mass shell to predict the phenomenon of the large momentum transfers observed²⁰⁾ at the ISR (at least, of course, in the tree approximation). If indeed there is a connection²¹⁾ between large momentum transfers and scaling in electromagnetic processes, as has been

recently suggested our result simply means that a current-hadron bootstrap scheme based on the conventional, quarkless, dual model describes a too "soft" structure to be compatible with experiments.

2. - ONE-CURRENT AMPLITUDES

The N point function of the conventional dual resonance model for the scattering of N scalar ground states has in the $\alpha(0) = 1$ case the following integral representation :

$$B^{(N)}(p_1, \dots, p_N) = \int \prod_{i=1}^N dy_i / dV_{abc} \prod_{1 \leq i < j \leq N} (y_j - y_i)^{-p_i \cdot p_j} \quad (2.1)$$

The slope of the Regge trajectory $\alpha(s) = 1 + \frac{s}{2}$ has been normalized to $\frac{1}{2}$.

The y_i are real variables ordered on the real axis :

$$y_1 \leq y_2 \leq \dots \leq y_N$$

Three of these are arbitrarily fixed and one must divide by the corresponding integration measure $dV_{abc} = (dy_a dy_b dy_c) / ((y_a - y_b)(y_b - y_c)(y_c - y_a))$. In this form the invariance of the integrand under Möbius transformations $y_i \rightarrow y'_i = (\alpha y_i + \beta) / (\gamma y_i + \delta)$ of the $SL(2, R)$ group is easy to check.

The one-current scalar amplitude obtained in Ref. 9) from a factorized formalism can be written in an analogous form

$$T^{(N+1)}(p_1, \dots, p_N, q) = \int \frac{d^2 z}{\lambda_0^2 \left| \frac{z - z^*}{2} \right|^2} \prod_{i=1}^N \frac{dy_i}{dV_{abc}} \prod_{1 \leq i < j \leq N} (y_j - y_i)^{-p_i \cdot p_j} \prod_{i=1}^N |y_i - z|^{-q \cdot p_i} \left| \frac{\lambda_0}{2} (z - z^*) \right|^{-q^2/2} \quad (2.2)$$

z is a complex variable attached to the off-shell leg of momentum q . Poles in the external mass $\alpha(q^2)$ appear when z tends to the real axis. Note that the integration measure is identical to the one derived by Nielsen,

Susskind and Kraemer since if we set $z = \exp \lambda + i\varphi$ $(d^2 z)/(\lambda_0^2 |z-z^*|^2) = (d\lambda/\lambda_0^2)(d\varphi/\sin^2 \varphi)$ and this last quantity in their model is interpreted as the density of propagators inside the strip. It is finite inside the strip and blows up at the edges $[\varphi = 0 \text{ and } \varphi = \pi]$. Hence on-shell particles which correspond to points on the real axis "feel" a different density of propagators as off-shell currents. In Ref. 9) λ_0 was chosen to be 1, but we keep it arbitrary for reasons which will be explained later.

This expression for one-current amplitudes coincides also with those proposed by Drummond and Rebbi ⁷⁾ apart from the integration region (Γ) which gives to our amplitude a planar character.

If we did not restrict the integration region and z was allowed to vary in all the complex plane, z could approach any of the y_i variables and hence we would get poles in any of the $(q+p_i)^2$ variables. This would lead to exotics if we introduced a Chan-Paton factor for the current so that the current has to be neutral. The simplest possible restriction that one can introduce in order to be compatible with Möbius invariance must be expressed in terms of circles and of orthogonality conditions. The restriction found in Ref. 9) is indeed of that type : consider the circle orthogonal to the real axis going through the external points y_1, y_N . Then we restrict the integration region on z so that z has to be outside this circle (see Fig. 1). It is obvious then that the amplitude is planar since only the factors $|z-y_1|^{-q p_1}, |z-y_N|^{-q p_N}$ can blow up so that we may say that the current leg q is between p_1 and p_N . In the language of Ref. 9), one can also say that the angle defined by the relation

$$e^{2i\theta} = \frac{z-y_1}{z^*-y_1} \frac{z^*-y_N}{z-y_N} \quad (2.3)$$

which is Möbius invariant under $SL(2, R)$, has to vary only between 0 and $\pi/2$. The peculiar feature of this planarization procedure is to be compatible with factorization. dV_{abc} is an invariant volume element obtained by fixing three points. These three points can be either three real variables, in which case it has the previous form. Or one can fix z and one point on the real axis in which case $dV_{abc} = (dz dz^* dy_a)/((z-y_a)(y_a-z^*)(z^*-z))$. One can also fix two points on the real axis and impose a condition between $\text{Re } z$ and $\text{Im } z$.

To obtain the integral representation of Ref. 9) one maps the real axis into the unit circle, and z is fixed to 0, $y_1 = 1$. One gets :

$$T^{(N+1)} = \lambda_0^{-\alpha(q^2)} \int_0^{\pi/2} \prod_{i=2}^N d\theta_i \prod_{1 \leq i < j \leq N} \sin(\theta_j - \theta_i)^{-P_i \cdot P_j} \quad (2.4)$$

$$\theta_1 = 0 \leq \theta_2 \leq \dots \leq \theta_N \leq \pi/2$$

By a similar procedure, it has been possible to obtain also a conserved vector amplitude, and also higher tensor amplitudes. If we denote by $d\omega$ the integration measure of the scalar current

$$T^{(N+1)} = \int_{(\Gamma)} d\omega$$

the currents with spin indices one obtains are always of the form :

$$T_{\mu_1 \dots \mu_r}^{(N+1)} = \int_{(\Gamma)} d\omega \mathcal{V}_{\mu_1 \dots \mu_r}(z, y_1, \dots, y_N)$$

For instance the vector current is given by :

$$T_{\mu}^{(N+1)} = \int_{(\Gamma)} d\omega \mathcal{V}_{\mu}$$

where

$$\mathcal{V}_{\mu} = \sum_{i=1}^N P_{i,\mu} \operatorname{Im} \left[\frac{y_N - z^*}{y_N - z} \frac{y_i - z}{y_i - z^*} + \frac{y_1 - z^*}{y_1 - z} \frac{y_i - z}{y_i - z^*} \right]$$

This vector current is likely to be associated with electromagnetism since it comes out conserved : $q^{\mu} T_{\mu}^{(N+1)} = 0$. For more details the reader is referred to Ref. 9). We now list the results which have been derived in the study of one-current amplitudes.

These amplitudes have only planar singularities. The scalar amplitude has poles in $\alpha(q^2)$ for $\alpha(q^2) = 0, 2, 4, \dots$, the vector amplitude for $\alpha(q^2) = 1, 3, 5, \dots$. When $\alpha(q^2)$ goes to zero

$$T^{(N+1)} \simeq \frac{1}{\alpha(q^2)} B^{(N+1)}(p_1, \dots, p_N, q)$$

while when $\alpha(q^2) \rightarrow 1$

$$T_\mu^{(N+1)} \simeq \frac{1}{\alpha(q^2)-1} V_\mu^{(N+1)}(p_1, \dots, p_N, q)$$

where V_μ is the ρ scattering amplitude on N tachyons.

These amplitudes are factorizable with the ordinary set of oscillators $a_\mu^{(n)}$ (no doubling of the number of oscillators is needed) and incorporate no ghost states since the tree on one side of the currents automatically decouples them.

If we consider the four-point function for fixed q^2 and t and let $s \rightarrow -\infty$ we obtain :

$$T^{(4)}(s, t, q^2) \underset{s \rightarrow -\infty}{\simeq} \lambda_0^{-\alpha(q^2)} \left\{ [-\alpha(s)]^{\alpha(t)} \Gamma(-\alpha(t)) B(-\frac{1}{2}\alpha(q^2); \alpha(t) - \tilde{\alpha}(t)) \right. \\ \left. + [-\alpha(s)]^{\tilde{\alpha}(t)} \Gamma(\tilde{\alpha}(t) - \alpha(t)) 2^{[\tilde{\alpha}(t) - \alpha(t)]} \right\} \quad (2.5)$$

where $\tilde{\alpha}(t) = (\alpha(t) - 1/2)$. The first term dominates for $\alpha(t) > -1$ and corresponds to the normal Regge behaviour. When $\alpha(q^2) = 0, 2, \dots$ its residue blows up so that it contributes to the hadronic amplitude. On the other hand the second term which dominates for $\alpha(t) < -1$ has no poles in $\alpha(q^2)$ so that it is a purely off-shell effect. This companion trajectory does not materialize in particles since the $(-\alpha(s))^{\tilde{\alpha}(t)}$ is not accompanied with a $\Gamma(-\tilde{\alpha}(t))$ factor but rather with the factor $\Gamma(\tilde{\alpha}(t) - \alpha(t))$ which indicates that a blow up occurs when the two trajectories $\alpha(t)$ and $\tilde{\alpha}(t)$ cross each other. In fact the two terms blow up and the residue cancels so that there is no singularity at $\alpha(t) = -1$, in conformity with the requirement (1). We will see this companion trajectory reappears in the two-current amplitude with a moving intercept. This behaviour is similar to

the one observed in the Pomeron loop graph except that the Pomeron term $s^{\alpha_p(t)}$ has a $\Gamma(-\alpha_p(t))$ associated with it and does materialize into particles. The comparison with experiment of this behaviour has been attempted and has given reasonable results ¹³⁾.

We now turn our attention to the form factors. The scalar form factor of the ground state

$$T^{(3)}(q^2) = \lambda_0^{-\alpha(q^2)} B(1/2; -\alpha(q^2)/2)$$

behaves when $q^2 \rightarrow -\infty$ like :

$$T^{(3)}(q^2) \sim \lambda_0^{-\alpha(q^2)} (-q^2)^{-1/2}$$

The electromagnetic form factor itself is very simple :

$$T_\mu^3(q^2, p_1, p_2) = \frac{(p_1 - p_2)_\mu}{\alpha(q^2) - 1} \lambda_0^{-\alpha(q^2)} \quad (2.6)$$

Contrary to the scalar case, this form factor exhibits only one pole.

In the Neveu-Schwarz model, a similar phenomenon occurs, and the pion form factor exhibits also only the ρ pole, so that this may explain why the ρ' does not couple to 2π channels but only to 4π , 6π , etc., channels ¹²⁾.

It would be tempting to set the arbitrary constant $\lambda_0 = 1$ so that the pion form factor would exhibit pure pole dominance, while the scalar form factor would decrease like $(-q^2)^{-1/2}$. However, this choice of $\lambda_0 = 1$ is not consistent with the study of the form factors of higher excited states. If we write down an operator which reproduces the scalar current between tree graphs we can evaluate its matrix elements for instance between two states of spin J_1, J_2 lying on the leading trajectory. This operator has the following form :

$$S(a, a^+, q) = \lambda_0^{-\alpha(q^2)} \int_0^{\pi/2} d\theta \sin \theta^{-1-\alpha(p^2)} \exp \sum_{n=1}^{\infty} q \cdot a_n^+ \cos n\theta \exp \sum_{n=1}^{\infty} q \cdot a_n \cos n\theta \quad (2.7)$$

Hence :

$$\begin{aligned} & \epsilon_{\mu_1 \dots \mu_{J_1}} \langle 0 | a_{\mu_1}^{(1)} \dots a_{\mu_{J_1}}^{(1)} S a_{\nu_1}^{+(1)} \dots a_{\nu_{J_2}}^{+(1)} | 0 \rangle \epsilon_{\nu_1 \dots \nu_{J_2}} \\ &= \epsilon_{\mu_1 \dots \mu_{J_1}} \epsilon_{\nu_1 \dots \nu_{J_2}} q^{\mu_1} \dots q^{\mu_{J_1}} q^{\nu_1} \dots q^{\nu_{J_2}} F_{J_1 J_2}(q^2) \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} F_{J_1 J_2}(q^2) &= \lambda_0^{-\alpha(q^2)} \int_0^{\pi/2} d\theta \sin \theta^{-1-\alpha(q^2)} (\cos \theta)^{J_1+J_2} \\ F_{J_1 J_2}(q^2) &\underset{q^2 \rightarrow -\infty}{\sim} \lambda_0^{-\alpha(q^2)} (-q^2)^{-\frac{1}{2} - (J_1+J_2)/2} \end{aligned} \quad (2.9)$$

If $\lambda_0 = 1$, this is contradictory with the results obtained from the study of the Bethe-Salpeter ladder model which predicts a faster fall off with J_1, J_2 in $(-q^2)^{-J_1-J_2-22}$.

The situation is even worse when one considers states built from the oscillator a_2 since $\cos 2\theta$ does not vanish at $\theta = \pi/2$ which is the point contributing to the asymptotic behaviour. For instance

$$\langle 0 | S (a_2 \cdot a_2)^N | 0 \rangle \underset{q^2 \rightarrow -\infty}{\sim} (-q^2)^{-\frac{1}{2} + N} \lambda_0^{-\alpha(q^2)}$$

These states will give a contribution to the deep inelastic region which will grow up for negative q^2 , if λ_0 is set equal to one, as shown by Dorren and Drummond in a more direct way ¹³⁾. Hence we are led to choose $\lambda_0 < 1$ so that the form factors decrease exponentially fast for negative q^2 . This is not unreasonable from the theoretical point of view since in the Bethe-Salpeter ladder model form factors which decrease faster than any power of q^2 are found in the limit of infinitely composed particles ²²⁾, which correspond to the bootstrap situation of the dual model where each particle is a resonance in an N particle channel.

Since the dual model is believed to be consistent with experiment for large s behaviour, one may wonder if unitarization will not restore the bad situation in the q^2 behaviour. Such arguments have also been advanced in the framework of the relativistic quark oscillator model ¹⁵⁾. Here we may argue that the introduction of cuts may change an exponential behaviour into a power behaviour and we can even give an example of such occurrence in the framework of dual models.

The Pomeron graph can be written for $D = 26$ in the form :

$$F = \left(\frac{\pi}{2}\right)^{q^2} \sum_{N=0}^{\infty} \frac{R_N}{2N - \alpha_p(q^2)} \quad (2.10)$$

exhibiting the Pomeron poles. The residues are factorized, but each Lorentz tensor T_α still depends on q^2 :

$$R_N = \sum_{\alpha=d(N)}^{\alpha=d(N)} T_\alpha(p_i, q^2) \cdot T_\alpha(p_f', q^2) \quad (2.11)$$

One can check that each T_α is power-behaved in q^2 so that each term in the sum behaves like $(\pi/2)q^2$. Therefore one may believe that the sum itself behaves exponentially for large q^2 . However, this is not true as shown in Ref. 23), and this graph is power-behaved. The reason for this is that the sum is not uniformly convergent since the graphs exhibit also two-particle discontinuities. So one sees an explicit example on which exponential behaviours are cancelled by cuts, and one may hope that eventually the same will happen to the form factor through unitarization.

We end this review of the properties of the one-current amplitudes by mentioning the result obtained in the Bjorken limit of $T^{(4)}(s, t, q^2)$ with one mass fixed, when $q^2 \rightarrow -\infty$ with $s/q^2 = 1 - \omega$ fixed, t being also fixed. One gets ¹³⁾

$$T^{(4)}(s, t, q^2) \simeq \lambda_0^{-\alpha(q^2)} \left(-\alpha(q^2)/2\right)^{\tilde{\alpha}(t)} F(t, \omega)$$

$$F(t, \omega) = \frac{\sqrt{\pi}}{4} \Gamma\left(-\frac{\alpha(t)}{2}\right) [\omega(1-\omega)]^{\alpha(t)/2} - \frac{1}{2} \Gamma\left(\frac{1-\alpha(t)}{2}\right) (1-\omega)^{\alpha(t)} {}_2F_1\left(\frac{1-\alpha(t)}{2}; 1; \frac{3}{2}; 1-\omega\right) \quad (2.12)$$

Notice that apart from the exponentially falling form factor one gets scaling with a t dependent scale dimension. Scaling in s/q^2 is not trivial in this model since the presence even at high energy of the fundamental constant α' would have in principle allowed scaling in arbitrary ratios $\alpha(s) \cdot \alpha(q^2)^{-r}$. We will see that in the deep inelastic region of two-current amplitudes the exponential is cancelled so that one is left with a power behaviour in q^2 multiplied by a scaling function. Due to the very quick approach of the function to its Stirling approximation, it is possible to show that the asymptotic form is valid very quickly, for $q^2 \simeq -1 \text{ GeV}^2$, which may be an explanation for precocious asymptopia.

3. - TWO-CURRENT AMPLITUDES

1) - Integral representation

The amplitude $T^{(N+2)}$ for the scattering of two-scalar currents of momenta q_1, q_2 on N scalar ground states of momenta $p_1, \dots, p_N$ has the following integral representation :

$$T^{(N+2)} = \int \prod_{\alpha=1,2} \frac{d^2 z_\alpha}{\left(\frac{\sum_{\alpha} |z_\alpha - z_\alpha^*|^2}{2}\right)^2} \prod_{i=1}^N \frac{dy_i}{dV_{abc}} \mu(x^2) \prod_{i,j} (y_j - y_i)^{-p_i \cdot p_j} \prod_{\substack{i=1,N \\ \alpha=1,2}} |y_i - z_\alpha|^{-q_\alpha \cdot p_i} \\ \prod_{\alpha=1,2} \left| \frac{\sum_{\alpha} (z_\alpha - z_\alpha^*)}{2} \alpha^{\frac{1}{4}}(x^2) \right|^{-\frac{q_\alpha^2}{2}} |(z_1 - z_2)(z_1 - z_2^*)|^{-\frac{q_1 \cdot q_2}{2}} \quad (3.1)$$

This result has been obtained from factorization of one-current amplitudes. It is an almost direct extension from Eq. (2.2) in the sense that now we have two complex variables z_1 and z_2 and the terms involving two legs have still the interpretation of being related to the electrostatic potentials between two points. Non-trivial terms occur, however, for the integration measure $[\mu(x^2)]$ and for the self-energy terms $[a(x^2)]$. This is not too surprising since one could expect that the self-energy term would be determined by a limiting procedure giving a non-trivial result. The reason

why this happens at the two-current level instead of being present at the first is that with two complex variables we can form the following $SL(2, R)$ invariant quantity :

$$\frac{z_1 - z_2}{z_1 - z_2^*} \frac{z_1^* - z_2^*}{z_1^* - z_2}$$

This cross ratio is real so that we can define from it a real variable x through the relation

$$\left(\frac{1-x}{1+x} \right)^2 = \frac{z_1 - z_2}{z_1 - z_2^*} \frac{z_1^* - z_2^*}{z_1^* - z_2} \quad 0 \leq x \leq 1 \quad (3.2)$$

Note that when any of the variables z_1 or z_2 becomes real x reaches 0. This is why for one-current amplitudes no such dependence of the integrand occurs. In this model $\mu(x^2)$, $a(x^2)$ can be in principle computed, but are very complicated functions. Fortunately we will never need their expression explicitly to compute asymptotic behaviours and we will just assume that they are slowly varying functions of x in the integration region. We will just use the normalization condition that $a(0) = \mu(0) = 1$. In what follows we will try to give derivations of asymptotic behaviours which will be as model-independent as possible, i.e., we will assume the form of Eq. (3.1) to be true, and the shape of the integration region to be exact, because the exact expressions of the various functions involved in the integrand will depend on the elimination of ghost states which we have not done.

2) - Integration region

If we were integrating over the whole complex plane without restriction, the resulting amplitude would have a non-planar character. It seems that there would then be a problem when the two currents z_1, z_2 collide because of the vanishing of $|z_1 - z_2|^{(-q_1 q_2)/2}$. In fact when $z_1 = z_2$ x is automatically equal to 1 and the function $a(x)$ vanishes like $(1-x)^2$ also. In the t channel one gets poles in the variable $\alpha_p(q_1 + q_2)^2 = \alpha_p(0) + (\alpha'/2)(q_1 + q_2)^2$, i.e., poles due to the Pomeron trajectory going through integers.

In the case of planar currents we are considering, the integration region which emerges from factorization is very much restricted, and there are three types of restrictions.

- a) The invariant x no longer varies between 0 and 1 but between 0 and x_0 where x_0 is determined from factorization, and is smaller than 1. A very rough estimate yields $x_0 \simeq 0.8$ but this value will not be needed. Since x_0 never reaches 1, z_1 and z_2 can never touch each other away from the real axis : the minimal distance between a "parton" of co-ordinate z and its nearest neighbour is $d = (1-x_0/2x_0)|\text{Im } z|$ so that the minimal distance between two partons interacting with the current is finite inside the strip and infinitesimal at the edges. The condition $x < x_0$ does not prohibit two points of going toward each other as long as they simultaneously tend to the real axis. In this planar amplitude the Pomeron will never appear. A model where such a cut-off is introduced "by hand" has also been studied by Kobuskin ²⁴).
- b) The real variables $y_1, \dots, y_N$ are ordered : $y_1 < y_2 < \dots < y_N$ and vary in regions which depend upon z_1 and z_2 . These regions are determined as follows. Consider a point A on the real axis ; the circle going through z_1, z_2, A intersects the real axis in a point B. Consider then the circle (Δ) going through z_1, z_2 orthogonal to the real axis, and the two circles C_1, C_2 orthogonal to (Δ) passing through z_1, A and z_2, B . They intersect the real axis in two more points D, C. Through this construction once A is given, B, C, D are also given and they determine a partition of the real axis into four segments, two of which are allowed regions of integration for the Koba-Nielsen variables y_i . The segments AD and BC are forbidden regions and the regions AB and DC are allowed (see Fig. 2). The line with the arrow can be considered as the quark line running through the complex plane, along which the variables are ordered (for instance on the figure : $y_1 < y_2 < z_1 < y_4 < y_5 < z_2 < y_6 < y_7$, etc). In order for this picture to be Möbius invariant, it is only necessary that the points A (or B) be given in terms of z_1, z_2 and of a third parameter depending upon z_1, z_2 which, having to be Möbius invariant itself, can only be the parameter x .

One obtains a simpler picture if we fix for instance $z_2 = i$, $z_1 = ix$. Equation (3.2) is then satisfied and we obtain Fig. 3 where the points C, D, A, B have the co-ordinates $-(x/g(x))$, $-g(x)$, $g(x)$, $x/g(x)$. $g(x)$ is a function of x only which determines the integration region :

$$-x/g(x) \leq y_i \leq -g(x) \quad (I) \quad \text{or} \quad g(x) \leq y_i \leq x/g(x) \quad (II)$$

If all y_i belong to one region only then we have adjacent currents, if not we have non-adjacent currents.

A basic requirement on $g(x)$ comes from the fact that the current q_1 has to make a pole with the adjacent variable y_1 . This implies that the factors :

$$|y_1 - z_1|^{-q_1 \cdot p_1} = (y_1^2 + x^2)^{-\frac{q_1 \cdot p_1}{2}}$$

and :

$$\left| \frac{z_1 - z_1^*}{2} \right|^{-q_1^2/2} = x^{-q_1^2/2}$$

can vanish at the same rate, so that y_1 can go to zero faster than x . If this is the case

$$(y_1^2 + x^2)^{-q_1 \cdot p_1/2} x^{-q_1^2/2} \underset{\substack{x \rightarrow 0 \\ y_1/x \rightarrow 0}}{\sim} x^{-\alpha(q_1 + p_1)^2}$$

So the limit where $x \rightarrow 0$ and y_1 goes to zero faster than x yields the poles in $\alpha [(q_1 + p_1)^2]$. Since $0 < g(x) < y_1$, it is necessary that $g(x)$ vanishes quicker than x with $x \rightarrow 0$: $g(x) \underset{x \rightarrow 0}{\sim} x^\beta$ $\beta > 1$. β has to be an integer so that the poles be integer spaced.

In our model $g(x) \underset{x \rightarrow 0}{\sim} g_0 x^2$, with $g_0 = 2/\pi$. When $x \rightarrow x_0$, one can show from the explicit expression of g that $(x/g(x)) - g(x)$ goes to zero and the Koba-Nielsen variables all collapse in a point. One can show that this is a necessary condition for scaling, otherwise the deep inelastic limit would yield a trivial result of the type $\text{Const.} \times s^{\alpha(t) \cdot (q^2)^{-n}}$. Explicitly $(x/g(x)) - g(x) \simeq \sqrt{x_0 - x}$ when $x \rightarrow x_0$.

c) A third and last constraint stems from the need of recovering, when $x \rightarrow 0$ for fixed variables y_i , the one-current amplitude as a residue. This is not obvious since the extreme points of the integration being $-(x/g(x))$ and $+(x/g(x))$ the angle under which the variable z_2 sees this interval tends to π when $x \rightarrow 0$ while it should remain smaller than $\pi/2$ in the $x = 0$ limit. One finds indeed that such restrictions come from factorization to meet that condition. For variables separated by one current, we find the following conditions :

$$y_2 \in [-x/g(x), -g(x)] \quad y_1 \in [g(x), x/g(x)]$$

$$f_1(y_2, x) \leq y_1 \leq f_2(y_2, x) \quad (3.3)$$

where f_1, f_2 are two functions of x such that :

$$\begin{aligned} f_1 \left(-\frac{x}{g(x)}, x \right) &= f_2 \left(-\frac{x}{g(x)}, x \right) = g(x) \\ f_1 \left(-g(x), x \right) &= f_2 \left(-g(x), x \right) = \frac{x}{g(x)} \end{aligned} \quad (3.4)$$

For fixed x , this gives for y_1, y_2 the integration region described on Fig. 4. When $x \rightarrow 0$ we must recover the condition that the angle under which z_2 sees the interval y_1, y_2 is smaller or equal to $\pi/2$. This implies $y_1 y_2 \geq -1$. Hence when x goes to zero :

$$\begin{aligned} f_1(y_2, x) &\rightarrow 0 \\ f_2(y_2, x) &\rightarrow -1/y_2 \end{aligned}$$

There is also finally a restriction for variables belonging to the same region : for instance, if $y_1, y_2 \in [g(x), (x/g(x))]$, then one must have :

$$g(x) \leq y_1 \leq y_2 \leq h(y_1, x) \leq \frac{x}{g(x)} \quad (3.5)$$

where h is such that

$$\begin{aligned} h(g(x), x) &= g(x) \\ h\left(\frac{x}{g(x)}, x\right) &= \frac{x}{g(x)} \end{aligned} \quad (3.6)$$

(See Fig. 5).

When $x \rightarrow 0$, since the angle under which z_2 sees $y_1 y_2$ is allowed to vary between 0 and $\pi/2$, one gets $h(y_1, x) \xrightarrow{x \rightarrow 0} +\infty$.

When $x \rightarrow x_0$, we see that these integration regions shrink to zero. It is necessary in order to compute the behaviour of the amplitude in the deep inelastic limit to know exactly how they shrink to zero. One can show that $h(y_1, x) - y_1$ shrinks to zero like $(x_0 - x)$, i.e., faster than $(x/g(x)) - g(x)$ which shrinks to zero like $(x_0 - x)^{\frac{1}{2}}$. If we introduce a variable

$$y'_1 = \frac{y_1 - g(x)}{(x/g(x)) - g(x)}$$

this variable is such that $0 \leq y'_1 \leq 1$.

Then

$$h(y_1, x) - y_1 \underset{x \rightarrow x_0}{\simeq} \varphi(y_1') (x - x_0) \quad (3.7)$$

$\varphi(y_1')$ because of Eq. (3.6) vanishes at zero and 1, and one can show that one has precisely the behaviour :

$$h(y_1, x) - y_1 \underset{x \rightarrow x_0}{\simeq} C y_1' (1 - y_1') (x - x_0) \quad (3.8)$$

where C is a constant. This will be important in the deep inelastic limit.

In the case of two adjacent and two ground states we present in Fig. 6 the three-dimensional volume of integration which summarizes these restrictions.

These three restrictions on the integration region have essentially the following meaning : the restriction $x \leq x_0 < 1$ excludes the Pomeron singularities from the multi-current amplitudes. The existence of the two disconnected integration regions AB and CD allows us to introduce the mathematical quark line formed by the arcs of circles C_1 and C_2 and the segments AB and CD, so that the Koba-Nielsen variables z_α and y_i become ordered, and we can speak of consecutive and non-consecutive currents. Finally the correlations between real Koba-Nielsen variables give the consistency with the one-current amplitudes of Section 2. In the factorized model of Ref. 10) these conditions do not need to be imposed by hand but merely come out from factorization. The reader who is interested by their mathematical origin is referred to Appendices A and B. It is shown also that the functions f_1 , f_2 and h are related to each other, so that when h is known, f_1 and f_2 are also determined. Hence we have essentially only one complicated cut-off function $h(y, x)$.

3) - Duality

The integrand of $T^{(N+2)}$ is easily checked to be invariant under a transformation of $SL(2, R)$. Since the integration region (Γ) has also been defined in a Möbius invariant way, we conclude that the amplitude $T^{(N+2)}$ is dual. The planar duality character is obvious on Fig. 3 because z_1 can make poles only with the two adjacent variables lying on CD or AB.

4) - The two-point function

Before examining the off-shell Compton amplitude, it is of interest to study the simpler case of vacuum-vacuum transition. There are no Koba-Nielsen variable on the real axis, and $q_1 = -q_2 = q$. $T^{(2)}(q^2)$ can be considered as a "propagator" of scalar hadrons

$$T^{(2)}(q^2) = \int_{(r)} \frac{d^2 z_1 d^2 z_2}{\left(\frac{\lambda_0^2}{4} |z_1 - z_1^*| |z_2 - z_2^*|\right)^2} dV_{abc} \mu(x^2) \left[\frac{\lambda_0^4}{4} \frac{|z_1 - z_1^*| |z_2 - z_2^*|}{|z_1 - z_2| |z_1 - z_2^*|} a^{1/2}(x^2) \right]^{-q^2/2} \quad (3.9)$$

Choosing $z_2 = i$, $z_1 = ix$, and dividing by the Möbius invariant

$$dV_{abc} = i d^2 z_2 dz_1 / (z_2 - z_2^*)(z_2^* - z_1)(z_1 - z_2)$$

we obtain :

$$T^{(2)}(q^2) = \int_0^{x_0} \frac{dx}{\lambda_0^4 x^2} (1-x^2) \mu(x^2) \left[\frac{\lambda_0^4 x a^{1/2}(x^2)}{1-x^2} \right]^{-q^2/2} \quad (3.10)$$

μ and a are functions of x^2 , normalized to 1 for $x = 0$ and analytic in x in a circle of radius 1. Further the function

$$f(x) = \frac{\lambda_0^4 x a^{1/2}(x^2)}{1-x^2}$$

is monotonic on the interval $0 < x \leq x_0$, so that making the change of variables $y = f(x)$, one obtains :

$$\begin{aligned}
 T^{(2)}(q^2) &= \int_0^{y_0} dy \tilde{\mu}(y^2) y^{-1-\alpha(q^2)} \\
 &= \sum_{n=0}^{\infty} \frac{c_n y_0^{2n-\alpha(q^2)}}{2n-\alpha(q^2)}
 \end{aligned} \tag{3.11}$$

where $y_0 = \lambda_0^4 x_0 a^{\frac{1}{2}}(x_0^2)/(1-x_0^2)$.

This propagator for scalar particles has thus an infinite sequence of poles at $\alpha(q^2) = 0, 2, 4, \dots$. It has in particular the tachyon pole at $\alpha(q^2) = 0$. Hence causality is anyway violated by the presence of this tachyon. However, for vector and higher spin currents, this tachyon problem will not be present; therefore one can enquire if the form of Eq. (3.11) is consistent with causality [we may for instance throw away the pole at $n=0$ or assume that $\alpha(0) < 0$ for a moment]. So we ask if it is possible to write Eq. (3.11) in the form of a vacuum expectation value of a time-ordered product of two currents :

$$T^{(2)}(q^2) = \int d^4x e^{iqx} \langle 0 | T(J(x), J(0)) | 0 \rangle \tag{3.12}$$

This is obviously possible only if $y_0 = 1$. Fortunately we can meet this condition since the constant λ_0 has so far been kept arbitrary. $a(x)$ is a well-defined function in the framework of Ref. 10), and if ghosts are removed we expect that its form will change, eventually depending on the number of dimensions of space time, but still remaining well defined. So if we set

$$\lambda_0 = \left[(1-x_0^2)/x_0 a^{\frac{1}{2}}(x_0^2) \right]^{\frac{1}{4}} \tag{3.13}$$

this equation determines λ_0 numerically and λ_0 is not a new fundamental constant of the theory. It is expected that $(x_0 a^{\frac{1}{2}}(x_0^2)/(1-x_0^2))$ is bigger than 1 near $x=x_0$ if the ghosts are removed, as we shall show below from examining the deep inelastic limit of the Compton amplitude. Hence λ_0 should be smaller than 1 which yields exponentially falling form factors for negative q^2 , and this reflects the infinite amount of compositeness of the particles of the dual model.

If we satisfy Eq. (3.13), $T^{(2)}$ now takes the form :

$$T^{(2)}(q^2) = \frac{1}{\alpha'} \sum_{n=0}^{\infty} \frac{c_n}{m_n^2 - q^2 - i\epsilon} \quad m_n^2 = \frac{1}{\alpha'} (2n-1) \quad (3.14)$$

where the coefficients c_n are positive if no ghosts are present. Then :

$$T^{(2)}(q^2) = \frac{i}{\alpha'} \int d^4x e^{iqx} \langle 0 | T(J(x), J(0)) | 0 \rangle \quad (3.15)$$

with

$$J(x) = \sum_{n=0}^{\infty} \sqrt{c_n} \varphi_n(x, m_n^2) \quad (3.16)$$

the $\varphi_n(x, m_n^2)$ are free scalar second quantized fields of mass m_n^2 , operating in different Hilbert spaces. These fields are probably reducible since the density level of scalars grows exponentially with large n . So the model satisfies a field current identity as could be expected. It is not obvious that this field current identity holds also for the Compton amplitude. If it did it would mean that one can write the dual model in terms of an infinite set of second quantized field, and this model is a first step in that direction. Note that for large q^2 , $T^{(2)}(q^2)$ behaves like $1/q^2$, i.e., light cone dominance holds for the vacuum matrix element of the time-ordered product of $J(x) J(0)$, with canonical scale dimension. As we shall see, the scale dimension is no longer canonical for the Compton amplitude. If we assume that field current identity still holds, this has to be blamed on the infinite compositeness of the external resonance states, rather than on the product of two currents itself, which is canonically behaved for large q^2 . It is clear that the external states must be complicated functions of the φ_n in order for instance to reproduce the form factor

$$\lambda_0^{-q^2} B(\frac{1}{2}, -\alpha(q^2)/2) = \int d^4x e^{iqx} \langle p | J(x) | p \rangle$$

Once this choice of λ_0 is done, one can show that for the two-current amplitudes into N scalars, if q_1^2 and q_2^2 go simultaneously to infinity we get a power-behaviour also rather than an exponential. However, it is not obvious whether this property extends to more than two-current amplitudes or not. Therefore we regard the above choice for λ_0 as still tentative.

4. - THE OFF-SHELL COMPTON AMPLITUDE

1) - Kinematics and notations

We consider the amplitude of Fig. 7 and define the variables

$$s = (q_1 + p_1)^2 \quad t = (q_1 + q_2)^2 \quad u = (q_2 + p_1)^2 \quad (4.1)$$

we will also use in the case where $q_1^2 = q_2^2$ the notations :

$$q_1^2 = q_2^2 = q^2$$

$$s/q^2 = 1 - \omega \quad (4.2)$$

To momenta q_1, q_2 are attached the complex variables z_1, z_2 and to momenta p_1, p_2 the real variables y_1, y_2 . An amplitude involving currents of different quantum numbers will be the sum of three graphs each of which will be multiplied by the appropriate $SU(2)$ or $SU(3)$ Chan-Paton factor. We have :

the s-t graph : $0 \leq y_1 \leq y_2$ (Fig. 8a)

the u-t graph : $0 \leq y_2 \leq y_1$ (Fig. 8b)

the s-u graph : $y_1 \leq 0 \quad y_2 \geq 0$ (Fig. 8c).

As before, z_2 and $\text{Re } z_1$ have been fixed so that $z_2 = i$, $z_1 = ix$. The above integration regions are further restricted by the conditions described in paragraph 2 of Section 3.

The s-t amplitude has the following form :

$$T^{(4)}(s, t, q_1^2, q_2^2) = \int_0^{x_0} \frac{dx}{\lambda_0^4} x^{-2} \tilde{\mu}(x) \int_{g(x)}^{x/g(x)} dy_1 \int_{y_1}^{h(y_1, x)} dy_2 (y_1 - y_2)^{-2}$$

$$A_s^{-\alpha(s)/2} A_t^{-t/4} B_1^{-q_1^2/4} B_2^{-q_2^2/4} \quad (4.3)$$

where

$$\tilde{\mu}(x) = (1-x) \mu(x)$$

is a regular function of x : $0 \leq x \leq x_0$

$$A_s = \frac{y_1^2 + x^2}{y_1^2 + 1} \frac{y_2^2 + 1}{y_2^2 + x^2} \quad A_t = \frac{(y_1 - y_2)^2 (1 - x^2)}{(y_2^2 + x^2) (1 + y_1^2)}$$

$$B_1 = \lambda_0^2 \frac{1 + y_1^2}{x^2 + y_1^2} \frac{x^2}{1 - x^2} a^{1/2}(x^2) \quad (4.4)$$

$$B_2 = \lambda_0^2 \frac{x^2 + y_2^2}{1 + y_2^2} \frac{a^{1/2}(x^2)}{1 - x^2}$$

λ_0 has been fixed by the condition (3.13).

The u-t and s-u amplitudes are deduced from the above by changing the integration region as indicated, and then using kinematics to exhibit singularities in u, t or s, u. In what follows, we will concentrate on the s-t amplitude and quote the results obtained from the s-u and u-t amplitudes.

Because of the asymmetric choice $z_2 = i$, $z_1 = ix$ the symmetry of $T(s, t, q_1^2, q_2^2)$ in q_1^2, q_2^2 has been lost apparently. However, through the change of variables : $y_1' = x/y_2, y_2' = x/y_1$, one can check that $T(s, t, q_1^2, q_2^2) = T(s, t, q_2^2, q_1^2)$. Similarly, one can show that the s-u amplitude is symmetric in the simultaneous exchange of s with u and q_1^2 with q_2^2 and that the t-u amplitude is symmetric in the exchange of q_1^2 with q_2^2 .

The integration domain of the s-t term in the variables y_1, y_2, x has been represented on Fig. 6. The singularities of $T(s, t, q_1^2, q_2^2)$ arise from the following regions :

Poles in the t channel :	$y_1 = y_2$	$\alpha(t) = 0, 1, \dots$
Poles in the s channel :	$x = 0 \quad y_1 = 0 \quad \frac{y_1}{x}$ fixed	$\alpha(s) = 0, 1, \dots$
Poles in q_1^2 :	$x = 0 \quad y_1$ fixed	$\alpha(q_1^2) = 0, 2, \dots$
Poles in q_2^2 :	$x = 0 \quad y_2 = 0 \quad \frac{y_1}{x}$ fixed	$\alpha(q_2^2) = 0, 2, \dots$

One checks that when $\alpha(q_1^2)$ and $\alpha(q_2^2)$ go to zero

$$T^{(4)}(s, t, q_1^2, q_2^2) \simeq \frac{1}{\alpha(q_1^2) \alpha(q_2^2)} B(-\alpha(s); -\alpha(t)) \quad (4.5)$$

$T^{(4)}$ reproduces on-shell the Veneziano amplitude. We now start looking at the various asymptotic behaviours of T.

2) - s channel asymptotic behaviour

Let us consider the asymptotic behaviour of the (s, t) amplitude for $s \rightarrow \infty$, t, q_1^2, q_2^2 fixed. Since we have no u channel poles, the simplest method is to let $s \rightarrow -\infty$ and to continue analytically the result to any direction in the complex s plane except the real axis. We will refer to this limit as the R limit.

Since

$$A_s = 1 - \frac{(\gamma_2^2 - \gamma_1^2)(1 - x^2)}{(\gamma_1^2 + 1)(\gamma_2^2 + x^2)} \leq 1 \quad (4.6)$$

we see that $A_s^{-\alpha(s)/2}$ decreases exponentially for $s \rightarrow -\infty$ except at the points where $A_s = 1$.

This implies that

$$(\gamma_2 - \gamma_1)(\gamma_2 + \gamma_1) \frac{(1 - x^2)}{(\gamma_1^2 + 1)(\gamma_2^2 + x^2)} = 0$$

Hence three regions contribute to the asymptotic behaviour :

- i) $y_2 = y_1$ this implies $A_t = 0$ and yields the conventional Regge poles ;
- ii) $y_1, y_2 \rightarrow 0$ this is possible only if $x \rightarrow 0$, and implies :

$$\frac{y_1}{x} \rightarrow 0 \quad \frac{y_2}{x} \rightarrow 0 \quad x \rightarrow 0 \quad \frac{y_1}{y_2} \text{ fixed}$$
- iii) $y_1, y_2 \rightarrow \infty$ again this is possible also only if $x \rightarrow 0$ with

$$y_1 \rightarrow \infty \quad y_2 \rightarrow \infty \quad \frac{y_1}{y_2} \text{ fixed}$$

Alternatively one may say that only $y_1 = y_2$ contributes but that one must take also into account the contribution of the points of the third kind 2) and 3) (i.e., points where the boundary is non analytic as one sees on Fig. 6).

Expanding near $y_1 = y_2$ we obtain readily :

$$T_c^{(4)} \underset{s \rightarrow -\infty}{\simeq} [-\alpha(s)]^{\alpha(t)} T(-\alpha(t)) R(t, q_1^2, q_2^2) \quad (4.7)$$

The "c" stands here to remind that this is the canonical contribution to the asymptotic behaviour of T. The residue function R is given by :

$$R(t, q_1^2, q_2^2) = \int_0^{x_0} \frac{dx}{\lambda_0^4} x^{-2} \tilde{f}(x^2) \int_{g(x)}^{x/g(x)} \frac{dy (1-y^2)}{(1+y^2)(x^2+y^2)} \tilde{A}_t^{-t/4} B_1^{-q_1^2/4} B_2^{-q_2^2/4} \quad (4.8)$$

with

$$\begin{aligned} \tilde{A}_t &= \frac{y^2+x^2}{y^2} \frac{1+y^2}{1-x^2} & B_1 &= \lambda_0^2 \frac{1+y^2}{x^2+y^2} \frac{x^2}{1-x^2} a^{1/2}(x^2) \\ B_2 &= \lambda_0^2 \frac{x^2+y^2}{1+y^2} \frac{a^{1/2}(x^2)}{1-x^2} \end{aligned} \quad (4.9)$$

$R(t, q_1^2, q_2^2)$ has poles in q_1^2 coming from letting $x \rightarrow 0$ for fixed y , and also in q_2^2 by letting $x \rightarrow 0, y \rightarrow 0, y/x$ fixed. It has in particular a double pole at $\alpha(q_1^2) = \alpha(q_2^2) = 0$ with residue 1.

These are, however, not only the singularities of R since we can also let $y \rightarrow 0$ with $y/g(x)$ fixed : in that case both B_2 and $\tilde{A}_t$ vanish so that there is an abnormal singularity in a variable involving both t and q_2^2 . Similarly, if we let $x \rightarrow 0, y \rightarrow \infty$ with $y(g(x)/x)$ fixed B_1 vanishes and $\tilde{A}_t$ blows up and we get a singularity involving t and q_1^2 .

To exhibit explicitly the first of these abnormal singularities we set $y = vg(x)$ and expand in x : $y \simeq vg_0 x^2$.

The first term in x of the expansion yields :

$$R(t, q_1^2, q_2^2) = \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} (g_0^2)^{\frac{\alpha(t)+1}{2}} \int_0^1 dx x^{-1+\alpha(t)-\alpha(q_2^2)+1} \int_0^\infty dv v^{\alpha(t)} \quad (4.10)$$

the integral on x near $x = 0$ blows up when $\alpha(t) \simeq \alpha(q_2^2) - 1$. Near that point :

$$R(t, q_1^2, q_2^2) \simeq \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} g_0^{\alpha(q_1^2)} \frac{1}{\alpha(q_2^2)(\alpha(q_2^2)-\alpha(t)-1)} \quad (4.11)$$

The second singularity ($x \rightarrow 0, y \rightarrow \infty$) gives a similar result with q_2^2 replaced by q_1^2 . So the residue function contains unphysical singularities in the variables $\alpha(t) - \alpha(q_1^2) + 1$. Since the amplitude $T(s, t, q_1^2, q_2^2)$ must not contain any unphysical singularities, it is necessary that these cancel with other unphysical singularities coming from the other contribution to the Regge behaviour which we have not yet examined.

To verify this, we examine the contribution from region ii). So we set $y_1 = v_1 g(x)$, $y_2 = v_2 g(x)$ and expand in x for fixed v_1, v_2 .

One obtains for this non-canonical behaviour :

$$T_{Nc} \underset{s \rightarrow -\infty}{\simeq} [-\alpha(s)]^{\tilde{\alpha}} \Gamma(-\tilde{\alpha}) 2^{-1-\tilde{\alpha}} \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} g_0^{\alpha(q_2^2)} R_1(t, q_2^2) \quad (4.12)$$

with
$$\tilde{\alpha} = \frac{1}{2} [\alpha(t) + \alpha(q_2^2) - 1]$$

$$R_1(t, q_2^2) = \int_1^\infty dv_1 \int_{v_1}^\infty dv_2 (v_2 - v_1)^{-1-\alpha(t)} (v_2^2 - v_1^2)^{\tilde{\alpha}} \quad (4.13)$$

Again this contribution is Regge-behaved, however, one has the very surprising result that the Regge trajectory $\tilde{\alpha}$ has slope $\frac{1}{2}$ in t and a mass-dependent intercept ! It would of course be disastrous if this trajectory materialized into poles since their position would depend on the external masses. So we have to check that R_1 compensates for the possible blow up of $\Gamma(-\tilde{\alpha})$. Indeed R_1 can be computed and one gets :

$$R_1(t, q_2^2) = 2^{\alpha(q_2^2)-1} \frac{\Gamma(-\alpha(q_2^2)) \Gamma(\tilde{\alpha} - \alpha(t))}{\Gamma(-\tilde{\alpha})} \quad (4.14)$$

So the residue function indeed vanishes at all positive integer values of $\tilde{\alpha}$. We can write Eq. (4.12) in the form

$$T_{Nc} \underset{s \rightarrow -\infty}{\simeq} [-\alpha(s)]^{\tilde{\alpha}} \Gamma(\tilde{\alpha} - \alpha(t)) 2^{-1+\tilde{\alpha}-\alpha(t)} g_0^{\alpha(q_2^2)} \Gamma(-\alpha(q_1^2)) \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} \quad (4.15)$$

The contribution of region iii) would yield the same result with q_2^2 replaced by q_1^2 . So we have three trajectories, two non-canonical mass dependent trajectories which do not materialize into resonances, and the canonical trajectory $\alpha(t)$.

Assume for instance that $\alpha(q_2^2) > \alpha(q_1^2)$. Then if $\alpha(t) - \tilde{\alpha}(t) > 0$ Eq. (4.7) gives the asymptotic behaviour, and if $\tilde{\alpha}(t) - \alpha(t) > 0$, Eq. (4.15) gives it. In the vicinity of the crossing point, we see that the non-canonical contribution behaves like :

$$T_{NC} \simeq [-\alpha(s)]^{\tilde{\alpha}} g_0^{\alpha(q_2^2)} \Gamma(-\alpha(q_2^2)) \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} / 2 (\tilde{\alpha} - \alpha(t)) \quad (4.16)$$

On the other hand the canonical term also blows up because $\alpha(t) - \tilde{\alpha}(t) = \frac{1}{2} [\alpha(t) + 1 - \alpha(q_2^2)]$ and when this approaches zero, R blows up as we have seen previously :

$$T_C \simeq [-\alpha(s)]^{\alpha(t)} \Gamma(-\alpha(t)) g_0^{\alpha(q_2^2)} \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} / 2 \alpha(q_2^2) (\tilde{\alpha} - \alpha) \quad (4.17)$$

$$\simeq [-\alpha(s)]^{\alpha(t)} \Gamma(-\alpha(q_2^2)) g_0^{\alpha(q_2^2)} \lambda_0^{-\alpha(q_1^2)-\alpha(q_2^2)} / 2 (\alpha - \tilde{\alpha}) \quad (4.18)$$

$$= - T_{NC}$$

hence $T_C + T_{NC}$ is finite near $\alpha(t) \simeq \tilde{\alpha}$ and the asymptotic behaviour of T does not show up any unphysical singularities. If we put one leg on the mass shell we obtain back the canonical and non-canonical contributions of the one-current planar amplitudes. It seems that the appearance of the "shadow trajectory" $\tilde{\alpha}$ is inevitably tied up with the planar character of the currents we are considering ; indeed, planarization of non-planar, Regge behaved amplitudes must result in the introduction of either exponential behaviours, or "shadow trajectories", or fixed poles.

In the case of deep inelastic electron scattering since $\alpha(t=0) - \tilde{\alpha}(t=0) = \frac{1}{2} [1 - \alpha' q^2]$ and $q^2 < 0$, their contribution to the asymptotic behaviour of νW_2 for large ω will be negligible. However, their presence could in principle be tested in exclusive experiments on μ pair productions. For $q^2 > 0$ the Froissart bound can be violated, and one has to assume that it is reinstated by the exchange of two or more currents.

These results are identical to those obtained by Fidler ²⁵⁾ except for the interpretation of the mass-dependent singularities which appear in the residue of the non-canonical term.

Finally, one must also compute for t fixed, s going to infinity the asymptotic behaviour of the two other graphs, the u - t and the s - u graphs. The u - t graph gives no problem and s is just replaced by $-s$ in the asymptotic behaviour yielding the appropriate signature factor.

The s - u term is more tricky to deal with because both s and u go to infinity at t fixed. On the mass shell,

$$T^{(4)}(s, u) \sim \frac{1}{\alpha(q_1^2) \alpha(q_2^2)} B(-\alpha(s); -\alpha(u)) \quad (4.19)$$

In this case the usual reasoning one applies is that $B(-\alpha(s); -\alpha(u))$ can be written in the form :

$$B(-\alpha(s); -\alpha(u)) = \int_{-\infty}^{+\infty} d\varphi \exp \alpha(s) \varphi (1 + \exp -\varphi)^{-1-\alpha(u)} \quad (4.20)$$

The phase φ varies from $-i\infty$ to $+i\infty$ and hence if $\alpha(s) \rightarrow i\infty$ B decreases faster than any power. Similarly B falls off faster than any power of s in any direction of the complex plane except the real axis.

This reasoning holds also for $T_{s-u}^{(4)}$ itself away from the mass shell and one can show that the phase factor has no critical points of the first kind, and that the boundaries do not contribute either. So that like B , $T_{s-u}^{(4)}$ vanishes faster than any power of s in any direction except the real axis.

3) - t channel asymptotic behaviour for fixed s

Although this limit is not interesting from the experimental point of view, one has also to check that $T^{(4)}(s, t, q_1^2, q_2^2)$ is Regge behaved when t goes to infinity with s fixed. Again one obtains a canonical and a non-canonical piece. The non-canonical trajectory in the s channel is $\tilde{\alpha}(s) = ((\alpha(s)-1)/2)$ as one could expect since the Reggeon in the s channel links two vertices having one current. We quote here the result : if

$$\begin{aligned} \alpha(s) &\geq -1 \\ T(s, t, q_1^2, q_2^2) &\simeq T_c = [-\alpha(t)]^{\alpha(s)} \Gamma(-\alpha(s)) f(s, q_1^2) f(s, q_2^2) \\ f(s, q^2) &= \lambda_0^{-\alpha(q^2)} B(-\frac{1}{2}\alpha(q^2); \alpha(s) - \tilde{\alpha}(s)) \end{aligned} \quad (4.21)$$

f is the Reggeon-current particle coupling obtained already at the one-current level [see Ref. 9]. If $\alpha(s) \leq -1$

$$\begin{aligned} T(s, t, q_1^2, q_2^2) &\simeq T_{nc} = [-\frac{\alpha(t)}{2}]^{\tilde{\alpha}(s)} \Gamma(\tilde{\alpha}(s) - \alpha(s)) [F(s, q_1^2) + F(s, q_2^2)] \\ F(s, q^2) &= \lambda_0^{-\alpha(q^2)} \int_0^\infty dx (1+x^2)^{\frac{\alpha(q^2) - \alpha(s) - 1}{2}} x^{\tilde{\alpha}(s)} (x+2g_0)^{\frac{\alpha(s)}{2} + \frac{3}{4}} \end{aligned} \quad (4.22)$$

T_C and T_{NC} have each a double pole at the point where the trajectories $\alpha(s)$, $\tilde{\alpha}(s)$ cross each other, i.e., $\alpha(s) = -1$. However, one can check that near $\alpha(s) = -1$, $T_C + T_{NC}$ is regular in s .

4) - Deep inelastic limit

We set $q_1^2 = q_2^2 = q^2$ so that now $T^{(4)}$ reads :

$$T^{(4)} = \int_0^{x_0} \frac{dx}{\lambda_0^4} x^{-2} \tilde{\mu}(x^2) \int_{g(x)}^{x/g(x)} dy_1 \int_{y_1}^{h(y_1, x)} dy_2 (y_1 - y_2)^{-2} A_s^{-\frac{\alpha(s)}{2}} A_t^{-\frac{t}{4}} B^{-\frac{q^2}{4}} \quad (4.23)$$

where

$$B = A_s^{-1} \frac{x^2}{x_0^2} \frac{\alpha(x)}{\alpha(x_0^2)} \left(\frac{1-x_0^2}{1-x^2} \right)^2 \quad (4.24)$$

and look for the limit of $T^{(4)}$ when $s \rightarrow \infty, q^2 \rightarrow -\infty$ with $s/q^2 = 1 - \omega$ fixed. We will refer, following Brandt¹⁸⁾, to this limit as the A limit. There are in principle two ways to obtain the result.

1) We can let first $s \rightarrow -\infty$ with $\omega < 1$, and then analytically continue the result to $\omega > 1$. One obtains then a cut in ω and one defines the "smoothed" asymptotic imaginary part as the discontinuity of the asymptotic behaviour along the cut. This procedure is commonly used for the study of trees or of one-loop graphs and one can show that it is rigorous and gives the correct asymptotic behaviour along any ray in the plane except on a ray parallel to the real axis.

2) We can also in principle study directly the imaginary part of T and expand

$$\text{Im } T = \sum_{n=0}^{\infty} G_n(q^2, t) \delta(\alpha(s) - n) \quad (4.25)$$

$q^2 < 0$

Then we can ask whether $\text{Im } T$ converges in the sense of distribution theory in the Bjorken limit. For instance, one may study the convergence of

$$\langle \text{Im } T \rangle = \int_{n-1/2}^{n+1/2} ds \text{Im } T = G_n(q^2, t)$$

[see Ref. 26) for instance], and study the asymptotic behaviour of $G_n(q^2, t)$ when $q^2, n \rightarrow \infty$ with fixed $n/q^2 = 1 - \omega$.

In general the two procedures may give different results. In the case of tree graphs it has been shown ²⁷⁾ that the "smoothed" imaginary part and the non-smoothed imaginary part are equivalent for large s . This does not need to be the case here, since the cut-off on the integration region, as we will see, will influence the scale dimension of the smoothed imaginary behaviour, while it is clear that the imaginary part itself is independent of the integration region and is in principle solely determined by factorization.

It is, however, extremely difficult to work directly with the imaginary part itself and we will present only the computation of the smoothed asymptotic behaviour, hoping that a technique may be found to deal directly with the imaginary part in order to derive the asymptotic behaviour. In order to derive the asymptotic behaviour we use the technique developed by Alessandrini and Amati ²³⁾ and let $q^2 \rightarrow +i\omega$, $s = (1-\omega)q^2$. Lumping together the factor containing q^2 one gets :

$$T^{(4)} = \int_0^{x_0} \frac{dx}{\lambda_0^4} x^{-2} \tilde{\mu}(x) \int_{g(x)}^{x/g(x)} dy_1 \int_{y_1}^{h(y_1, x)} dy_2 (y_1 - y_2)^{-2} A_s^{-\frac{1}{2}} A_t^{-\frac{5}{4}} \exp - \frac{q^2}{4} \varphi \quad (4.26)$$

with

$$\varphi = \omega \ln \frac{y_1^2 + 1}{y_1^2 + x^2} \frac{y_2^2 + x^2}{y_2^2 + 1} + 2 \ln \frac{x}{x_0} \frac{a^h(x)}{a^h(x_0)} \frac{1-x_0^2}{1-x^2} \quad (4.27)$$

letting $q^2 \rightarrow i\omega$ we look for critical points of the first kind, i.e., points inside the integration volume, where φ is stationary :

$$\frac{\partial \varphi}{\partial y_1} = \frac{\partial \varphi}{\partial y_2} = \frac{\partial \varphi}{\partial x} = 0 \quad (4.28)$$

Since $\varphi = -\omega \ln A_s + r(x)$ the two first equations imply that $(\partial A_s / \partial y_1) = (\partial A_s / \partial y_2) = 0$, i.e., the point which contributes to scaling must satisfy the condition of the Regge asymptotic behaviour. So we see how in such a model it is a general fact that Regge behaviour will influence the scaling

limit. There is in fact no solution to Eq. (4.28) inside the integration region as that we look now for points of the second kind, i.e., points on the boundary where φ has a zero normal derivative.

An obvious candidate is the plane $y_1 = y_2$ since on that plane $A_s = 1$, and

$$\varphi = 2 \ln \frac{x}{x_0} \frac{a^h(x)}{a^h(x_0)} \frac{1-x_0^2}{1-x^2}$$

φ is a monotonic increasing function of x , so that $(\partial\varphi/\partial x) = 0$ is never met. However, one must also consider the end-points as possible contribution. $x = 0$ is discarded since $\varphi = \infty$, but $x = x_0$ contributes since φ is finite ($\varphi = 0$).

So the point $y_1 = y_2$ $x = x_0$ contributes to the scaling limit, one can also show that by considering a point on the line (L) (see Fig. 6) and showing that if s is the arc length along (L), $(\partial\varphi/\partial s) = 0$ at $s = s(x_0)$.

The search for other stationary points on the boundary being negative, only the point $x = x_0$, $y_1 = y_2 = \sqrt{x_0}$ contributes.

To compute the contribution of this point, we expand the integrand around this point. One gets :

$$T^{(4)} \simeq \left[\frac{1-x_0}{x_0(1+x_0)} \right]^{-\frac{h}{4}} \frac{\tilde{\mu}(x_0)}{x_0 \lambda_0^4} \int_0^{x_0} dx \int_{g(x)}^{x/g(x)} dy_1 \int_{y_1}^{h(y_1, x)} dy_2 (y_1 - y_2)^{-1-\alpha(t)} \exp - q^2 \frac{\varphi}{4} \quad (4.29)$$

where φ is also expanded :

$$\varphi \simeq \omega \frac{(y_2^2 - y_1^2)(1-x^2)}{(1+y_1^2)(1+y_2^2)} + \delta(x - x_0) \quad (4.30)$$

$$\simeq \omega (y_2 - y_1) \frac{2(1-x_0)}{\sqrt{x_0}(1+x_0)} + \delta(x - x_0) \quad (4.31)$$

where we have introduced the constant

$$\delta = \frac{\partial\varphi}{\partial x} \Big|_{y_1 = y_2; x = x_0} \quad \delta \geq 0 \quad (4.32)$$

In order to take into account the shrinking of the integration region we perform a change of variables so that we work with a fixed integration region. We set :

$$\begin{aligned} y &= \frac{y_1 - g(x)}{\frac{x}{g(x)} - g(x)} & 0 \leq y \leq 1 \\ v &= \frac{y_2 - y_1}{h(y_1, x) - y_1} & 0 \leq v \leq 1 \end{aligned} \quad (4.33)$$

near $x = x_0$ we know from Appendix B that :

$$y_1 = \sqrt{x_0} + \delta y (x_0 - x)^{1/2} \quad \delta > 0 \quad (4.34)$$

and that

$$\begin{aligned} y_2 - y_1 &= v (h(y_1, x) - y_1) \simeq v c y (1-y) (x_0 - x) \\ c &> 0 \end{aligned} \quad (4.35)$$

hence in the term of the new variables v and y , φ reads :

$$\varphi = (x_0 - x) \left[\omega v y (1-y) \frac{2(1-x_0) c}{\sqrt{x_0} (1+x_0)} - \gamma \right] \quad (4.36)$$

$$= (x_0 - x) 4\gamma \left[\frac{\omega}{\omega_0} v y (1-y) - \frac{1}{4} \right] \quad (4.37)$$

where we define the positive constant ω_0 by :

$$\frac{4}{\omega_0} = \frac{2(1-x_0) c}{\sqrt{x_0} (1+x_0) \gamma} \quad (4.38)$$

Performing the change of variables, we get

$$T^{(4)} \simeq C_0 C_1^{-\frac{k}{4}} \int_0^{x_0} dx (x_0 - x)^{\frac{1}{2} - \alpha(t)} \int_0^1 \frac{dv}{v} \int_0^1 dy [v y (1-y)]^{-\alpha(t)} \exp - q^2 \frac{\varphi}{4} \quad (4.39)$$

where we have lumped together uninteresting numerical constants in C_0, C_1 .
Performing the integration for $q^2 \rightarrow -\infty$, we get :

$$T^{(4)} \simeq (-q^2)^{\alpha(t) - 3/2} \Gamma(\frac{3}{2} - \alpha(t)) F(t, \omega) \quad (4.40)$$

with

$$F(t, \omega) = C_0 C_1^{-t/4} \int_0^1 \frac{dv}{v} \int_0^1 dy [vy(1-y)]^{-\alpha(t)} \left\{ \gamma \left[\frac{\omega}{\omega_0} vy(1-y) - \frac{1}{4} \right] \right\}^{\alpha(t) - 3/2} \quad (4.41)$$

So we get scaling behaviour for T which is non trivial, because of the presence even at high energy of the slope α' , which is a dimensional constant. The fact that the scaling power $d = \alpha(t) - \frac{3}{2}$ is different from the expected $d = -1$ means that the light cone does not dominate the Bjorken limit in this model. Of course, as experimentally one always measures the asymptotic behaviour of the imaginary part at $t = 0$ one could have an accidental coincidence between $\alpha(0) - \frac{3}{2}$ and -1 , which here would imply $\alpha(0) = \frac{1}{2}$. Another possibility which cannot be excluded is that the non-smoothed imaginary part, suitably averaged, behaves differently, eventually exhibiting canonical scale dimension. It is quite obvious in our case that the $-\frac{3}{2}$ term is the sum of a canonical term -1 and of a $-\frac{1}{2}$ term which reflects the shrinking of the integration region $[g(x), x/g(x)]$ when $x \rightarrow x_0$ like $(x_0 - x)^{\frac{1}{2}}$. Since the contribution of $x \simeq x_0$ is purely real and does not contribute to the imaginary part, it is possible that the non-smoothed imaginary part behaves differently.

Considering formula (4.40), we see that it has poles for $\alpha(t) = \frac{3}{2} + N$ which do not correspond to the resonances $\alpha(t) = N$. Hence again we have to check that $F(t, \omega)$ vanishes at all the points $\alpha(t) = \frac{3}{2} + N$. One can indeed compute $F(t, \omega)$ and show that :

$$F(t, \omega) = D_0 D_1^{-t/4} \left(1 - \frac{\omega}{\omega_0}\right)^{\alpha(t)} \Gamma(-\alpha(t)) / \Gamma(\frac{3}{2} - \alpha(t)) \quad (4.42)$$

$D_0, \bar{D}$ are new constants. Again we see that the unwanted poles are cancelled and we can rewrite the scaling limit in the form :

$$T^{(4)} \simeq (-q^2)^{\alpha(t) - 3/2} \Gamma(-\alpha(t)) D_0 D_1^{-t/4} \left(1 - \frac{\omega}{\omega_0}\right)^{\alpha(t)} \quad (4.43)$$

To define the smoothed imaginary part of T we now let ω become greater than 1, and $(1 - \omega/\omega_0)^{\alpha(t)}$ develops a cut for $\omega > \omega_0$. Thus we obtain

$$\text{Im } T = W(t, q^2, \omega) \simeq (-q^2)^{\alpha(t) - 3/2} \pi D_0 D_1^{-t/4} \left(\frac{\omega}{\omega_0} - 1\right)^{\alpha(t)} \theta\left(\frac{\omega}{\omega_0} - 1\right) / \Gamma(1 + \alpha(t)) \quad (4.44)$$

For $t = 0$:

$$W(q^2, \omega) \simeq \frac{1}{A} (-q^2)^{\alpha(0) - 3/2} F(\omega) \quad (4.45)$$

$$F(\omega) = \frac{\pi D_0}{\Gamma(1 + \alpha(0))} \left(\frac{\omega}{\omega_0} - 1\right)^{\alpha(0)} \theta\left(\frac{\omega}{\omega_0} - 1\right) \quad (4.46)$$

We have left here $\alpha(0)$ unspecified in order to exhibit the mechanism which builds up the A limit. Clearly the scaling behaviour is entirely due to the exchange of Reggeons in the t channel.

The constant ω_0 has to be greater than 1 in order to have a non-increasing behaviour at fixed s , $q^2 \rightarrow -\infty$. One can in principle check this also in our model. In general one expects λ_0 to be strictly greater than one because of the Drell-Yan-West relation²⁸⁾. One expects $F(\omega)$ to behave near threshold ($\omega = 1$) like $(\omega - 1)^p$ where p is linked to the asymptotic decrease of the excitation form factors in $(q^2)^{-d}$ through a relation of the type $p = 2d - 1$. Since in our case the form factors fall exponentially, $d = \infty$ and $p = \infty$. This means that $F(\omega)$ has a zero of infinite order near $\omega = 1$, which implies and is consistent with $\omega_0 > 1$. This threshold behaviour was met also in the parton picture of Nielsen, Susskind and Kraemer¹⁶⁾ who found $\omega_0 = 3$; in our model it is harder to estimate ω_0 but the qualitative picture is the same: the portion of the scaling curve near the threshold $1 \leq \omega \leq \omega_0$ is not adequately described by our model, because of the too fast fall-off of the form factors.

We can also check that in our model the commutativity hypothesis of Brandt is valid: let us perform first the R limit, and then let $q^2 \rightarrow -\infty$. The non-canonical terms are non-dominant because $\tilde{\alpha} \rightarrow -\infty$ also. So we have only to evaluate the large q^2 behaviour of the canonical part

$$T_{R,A} \simeq [-\alpha(s)]^{\alpha(t)} \Gamma(-\alpha(t)) R(t, q^2) \quad (4.47)$$

And one shows that :

$$R(t, q^2) \simeq (q^2)^{-3/2} R_0 R_1^{t/4} \quad (4.48)$$

Hence :

$$T_{R,A} \simeq (-s)^{\alpha(t)} (-q^2)^{-3/2} \quad (4.49)$$

$$\simeq (-q^2)^{\alpha(t)-3/2} (-\omega)^{\alpha(t)} \quad (4.50)$$

where we have dropped constants and t dependent factors. If we perform the limit in the reversed order, we obtain :

$$T_{A,R} \simeq (-q^2)^{\alpha(t)-3/2} (-\omega)^{\alpha(t)} \quad (4.51)$$

So the two limits indeed commute. The large ω limit of the scaling function is dominated by the expected Regge term.

One can also compute the Bjorken limit of the t - u term : this merely changes ω into $-\omega$. The Bjorken limit of the s - u term finally is rather interesting, since it yields an exponentially falling amplitude for all values of ω , except for $\omega = 0$. For $\omega = 0$, the s - u term behaves like a fixed power of q^2 , namely like $(-q^2)^{-5/2}$ multiplied by a $A^{-t/4}$ term where A is a constant. Hence some fixed power behaviours do arise from our restriction on the integration region, however, they are not of the type of those we would like to find from current algebra since they arise in the s - u rather than in the s - t terms.

5) $-q_1^2 \rightarrow \infty, q_2^2$ fixed limit

For one-current amplitudes, λ_0 characterizes the asymptotic behaviour, irrespectively of the number of legs. In the case of the Compton amplitude if we hold q_2^2 fixed and $q_1^2 \rightarrow -\infty$ we find that the amplitude still behaves like $\lambda_0^{-\alpha(q_1^2)}$. Explicitly, we find for the s-t term :

$$T^{(4)} \simeq \lambda_0^{-\alpha(q_1^2) - \alpha(q_2^2)} \left(\frac{\sqrt{-q_1^2}}{2} \right)^{\alpha(s)-1} B\left(\frac{\alpha(s) - \alpha(q_2^2)}{2}, \frac{1}{2}\right) F(s)$$

if

$$\alpha(s) \geq \alpha(q_2^2)$$

$$T^4 \simeq \lambda_0^{-\alpha(q_1^2) - \alpha(q_2^2)} \left(\frac{\sqrt{-q_1^2}}{2} \right)^{\alpha(q_2^2)} B\left(\frac{\alpha(q_2^2) - \alpha(s)}{2}; -\alpha(t)\right) F(q_2^2)$$

if

$$\alpha(q_2^2) \geq \alpha(s)$$

(4.52)

where

$$F(z) = \int_0^\infty dx \, x^{-2-z/2} \int_0^\infty dy \, \exp \left[-x^2 \left(q_0^2 - \frac{1}{3} \right) - 2q_0 xy - y^2 \right]$$

(4.53)

Hence in our model λ_0 describes really the rate of fall-off of a current amplitude when one mass goes to infinity, the other masses being fixed.

CONCLUSION

The amplitudes obtained in the model seem to obey the basic requirements one can expect from hadronic duality : duality, absence of unphysical singularities, factorization, planarity, Regge behaviour. So they can be considered as a natural set of off-shell extrapolations for the dual model. However, they cannot yet pretend to the rank of currents, for several reasons. Firstly, no fixed pole appears in two-current amplitudes, hence current algebra cannot be satisfied. Secondly, in order to have a power-behaviour in the deep inelastic limit, one needs to have Gaussian form factors while experimentally they are power-behaved. Thirdly, even if scaling is achieved at this expense in the deep inelastic region, it breaks down away from $t = 0$ and no light cone expansion ²⁹⁾, hence no light cone algebra ³⁰⁾ can hold. Although these features may be due to our construction of off-shell amplitudes, it might be that they simply reflect the lack of "quarks" or "elementary constituents" in a hadron-current bootstrap scheme based on the conventional dual model.

ACKNOWLEDGEMENTS

We thank Professor D. Amati for many stimulating discussions during the course of this work and for reading the manuscript.

APPENDIX A - INTEGRATION REGION

1. - Before Jacobi transformation

Let us briefly remind how one obtains the integration region. We start with a factorized form in which to each off-shell particle one associates a parabolic homographic transformation P_i depending on two real parameters, its invariant point η_i and its "translation" parameter C_i . The on-shell particles are described by real variables y'_k . In the case of the two-current amplitude as described in Fig. 9 we have the following ordering of variables :

$$\begin{aligned} & \eta_1 < z_{21}^- < y'_1 < y'_2 \dots < y'_N < P_2^{\frac{1}{2}}(y'_{N+1}) \dots < P_2^{\frac{1}{2}}(y'_{N+n}) \\ & < P_2^{\frac{1}{2}} P_1^{\frac{1}{2}}(y'_1) < z_{21}^+ < \eta_2 < z_{21}^- < P_2^{-\frac{1}{2}}(y'_1) < \dots < P_2^{-\frac{1}{2}}(y'_N) < y'_{N+1} < \dots < y'_{N+n} \\ & < P_1^{\frac{1}{2}}(y'_1) < \dots < P_1^{\frac{1}{2}}(y'_N) < P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}(y'_{N+1}) < z_{21}^+ < \eta_1 < \dots \end{aligned} \quad (A.1)$$

where $z_{21}^\pm$ and $z_{12}^\pm$ are the invariant points of $P_2^{\frac{1}{2}} P_1^{\frac{1}{2}}$ and $P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}$.

The integrand is related to the third Abelian integral associated with the group generated by P_1 and P_2 . If the group instead was generated by $P_1^{\frac{1}{2}}, P_2^{\frac{1}{2}}$, as it is the case for non-planar currents, then the points y' and $P_i^{\frac{1}{2}}(y')$ would correspond to different configurations of the amplitude related by non-planar dual transformation and should not be counted twice. This was the origin of the restriction of the variable y' to a fundamental region of $P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}$ in the non-planar case. (We can see that the interval $[z_{21}^-, z_{21}^+]$ is included in a fundamental region of $P_2 P_1$ and the same for $[z_{12}^-, z_{12}^+]$ for $P_1 P_2$.) One can show that the above ordering of variables in the planar case is derived from factorization only and no further restrictions on the integration region are needed.

For the s-t amplitude the integration region is :

$$z_{12}^- < y'_1 < y'_2 < P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}(y'_1) < z_{12}^+ \quad (A.2)$$

For the u-t amplitude :

$$z_{12}^- < y_2' < y_1' < P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}(y_2') < z_{12}^+ \quad (A.3)$$

or also, equivalently :

$$z_{21}^- < P_1^{-\frac{1}{2}}(y_2') < P_1^{-\frac{1}{2}}(y_1') < P_2^{\frac{1}{2}}(y_2') < z_{21}^+$$

For the s-u amplitude :

$$z_{12}^- < y_1' < P_1^{\frac{1}{2}}(y_2') < P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}(y_1') < z_{12}^+ \quad (A.4)$$

We can choose, using the projective invariance, the representation in which

$$\begin{aligned} P_1(z) &= z + 1 \\ P_2(z) &= -\frac{z\rho}{z-\rho} \end{aligned} \quad (A.5)$$

The real parameter ρ is allowed to vary in the region in which the multiplier of $P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}$ varies between zero and one. This limits the range of ρ to $[0, 1/16]$. In that range, $P_2 P_1, P_1 P_2 \dots$ never become parabolic. Using this representation one can study exactly the integration region for the variables y_1', y_2' and ρ as shown in Figs. 10a and 10b for the s-t and s-u terms.

We note that when $P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}$ becomes parabolic, the integration region for the y' shrinks to a point.

2. - After Jacobi transformation

The Jacobi transformation used in Ref. 10) transforms the real variables y_k' into real Koba-Nielsen variables y_k , and the two parameters η_i, C_i associated with each parabolic transformation into a complex variable z_i . Its paramount importance stems from the fact that in terms of the variables z_i, y_k , the integrand has the simple form exhibited above, while in terms of the variables y_1', η_i, C_i it contains infinite products. Using the notations of Ref. 10) the Jacobi transformation is defined by the equations :

$$\frac{1}{2i\pi} \frac{dy (z_j - z_j^*)}{(y - z_j)(y - z_j^*)} = -c_j \sum_{\alpha \in Q_j} \frac{\partial P_\alpha}{\partial y'} \Big|_{y'=\eta_j} \frac{dy'}{(y' - P_\alpha(\eta_j))^2} \quad (\text{A.6})$$

for $j = 1, 2$.

The consistency between these two relations is ensured by the relation :

$$\lim_{\epsilon \rightarrow 0} \frac{(z_1 - z_2)(z_1^* - z_2^*)}{(z_1 - z_2^*)(z_1^* - z_2)} = -4\pi^2 c_1 c_2 \sum_{\alpha \in Q_{12}} \frac{\partial P_\alpha}{\partial y'} \Big|_{y'=\eta_2} \frac{1}{(\eta_1 - P_\alpha(\eta_2))^2} \quad (\text{A.7})$$

y is a monotonous function of y' and is singular for $y' = P_\alpha(\eta_i)$ where P_α is an element of the group generated by P_1, P_2 . Moreover, y' and $P_\alpha(y')$ have the same transformed y . This property allows us, if we want, to use the domain $[P_2^{-\frac{1}{2}}(z_{21}^-), P_2^{-\frac{1}{2}}(z_{21}^+)]$ instead of $[z_{21}^-, z_{21}^+]$. In this way, the domain does not contain the singular point η_2 and the ordering of the y' variables is identical to the one of the y . The domain is also contained in a fundamental region of $P_1 P_2$: $P_1 P_2(P_2^{-1}(z_{21}^-)) = P_1(z_{21}^-) > z_{12}^+$.

Dividing the two equations of (A.6) to eliminate differentials, one gets :

$$\frac{|y - z_1|^2}{|y - z_2|^2} \frac{|z_2 - z_2^*|}{|z_1 - z_1^*|} = \frac{\varphi_2(y')}{\varphi_1(y')} \quad (\text{A.8})$$

Given y' this equation for y has two solutions y and $\bar{y}$ such that $z_1, z_2, y, \bar{y}$ lie on a circle. It is sufficient to choose the solution at one point and the others are determined by continuity.

3. - Properties of the Jacobi transformation

If we transform y' and η_i, C_i through a homographic transformation T , one can show that :

$$dT(y') \varphi'_i(T(y')) = dy' \varphi_i(y') \quad i = 1, 2. \quad (\text{A.9})$$

In general there are no special relations between φ'_i and φ_i . However, we can find some special transformations T such that (A.9) takes a simple form :

a) let us consider the homographic transformation X defined by :

$$\frac{X(y') - \eta_2}{X(y') - \eta_1} = - \frac{y' - \eta_2}{y' - \eta_1} \quad (\text{A.10})$$

then

$$\begin{aligned} X P_2 X^{-1} &= P_2^{-1} \\ X P_1 X^{-1} &= P_1^{-1} \quad X^2 = 1 \end{aligned} \quad (\text{A.11})$$

Then from the definition of $\varphi_i(y')$ one finds :

$$\varphi'_i(y') = - \varphi_i(y') \quad (\text{A.12})$$

the minus sign comes from C_i whereas the other terms are unchanged since the series is generated by P_1^{-1}, P_2^{-1} as well as by P_1 and P_2 .

In particular we have :

$$\frac{\varphi_2(X(y'))}{\varphi_1(X(y'))} = \frac{\varphi_2(y')}{\varphi_1(y')} \quad (\text{A.13})$$

Therefore the transforms of $y', X(y')$ are cocyclic with z_1, z_2 .

b) let us now consider U defined by :

$$U P_2 U^{-1} = P_1$$

$$U P_1 U^{-1} = P_2$$

or

$$U(y') = \frac{(\eta_2 c_1 + \eta_1 c_2) y' - \eta_2^2 c_1 - \eta_1^2 c_2}{y' (c_1 + c_2) - \eta_2 c_1 - \eta_1 c_2} \quad (\text{A.14})$$

$$U^2 = I$$

Then it is clear that

$$\begin{aligned}\varphi_1'(\gamma') &= \varphi_2(\gamma') \\ \varphi_2'(\gamma') &= \varphi_1(\gamma')\end{aligned}\tag{A.15}$$

or

$$\frac{\varphi_1(U(\gamma'))}{\varphi_2(U(\gamma'))} = \frac{\varphi_2(\gamma')}{\varphi_1(\gamma')}\tag{A.16}$$

The transformations U , X , UX and I form a finite Abelian group with four elements. The transformation UX is of special interest because its invariant points are real.

We note that

$$\frac{\varphi_2(UX(\gamma'))}{\varphi_1(UX(\gamma'))} = \frac{\varphi_1(\gamma')}{\varphi_2(\gamma')}\tag{A.17}$$

We can easily see that this means that the transformed points of γ' and $UX(\gamma')$ are conjugate with respect to the circle passing through z_1, z_2 , orthogonal to the real axis.

The invariant points of UX are given by :

$$\begin{aligned}z_0^\pm &= [\eta_2 c_1 - \eta_1 c_2 \pm (\eta_2 - \eta_1)\sqrt{c_1 c_2}] / (c_1 - c_2) \\ z_0^- &= X(z_0^+)\end{aligned}\tag{A.18}$$

Then

$$\frac{\varphi_2(z_0^\pm)}{\varphi_1(z_0^\pm)} = 1\tag{A.19}$$

These points are interesting since we know explicitly their transformed : they are determined by the two circles passing through z_1, z_2 , tangent to the real axis.

4. - Application to the integration region

We choose the configuration where P_1, P_2 are given by Eqs. (A.5). Further, we specify

$$\begin{aligned} Z_1 &= ix \\ Z_2 &= i \end{aligned} \quad (A.20)$$

The variable x is determined in function of ρ through the relation (A.7) :

$$\ln\left(\frac{1-x}{1+x}\right)^2 = -4\pi^2 \rho \sum_{\rho \in Q_{12}} \frac{\partial P_\rho}{\partial y'} \Big|_{y'=0} \quad (A.21)$$

when ρ varies between 0 and $1/16$ x varies between 0 and x_0 . x_0 is a purely mathematical number determined by Eq. (A.21) in the case $\rho = 1/16$.

Let us study the integration region in that representation.

U, X, UX are given by

$$\begin{aligned} U(y') &= -y' \\ X(y') &= -\frac{\rho}{y'}, & UX(y') &= \frac{\rho}{y'} \end{aligned} \quad (A.22)$$

Then the Jacobi transformation has the following properties : if $y', \bar{y}'$ are two points and their Jacobi transforms are $y, \bar{y}$ the following relation holds :

$$\begin{aligned} \bar{y}' + y' &= 0 & \Leftrightarrow & y + \bar{y} = 0 \\ y' \bar{y}' &= -\rho & \Leftrightarrow & y \bar{y} = -x \\ y' \bar{y}' &= \rho & \Leftrightarrow & y \bar{y} = x \end{aligned} \quad (A.23)$$

In particular the points $\pm\sqrt{\rho}$ are transformed into $\pm\sqrt{x}$. Since the points $z_{12}^\pm, z_{21}^\pm$ are related to one another by one of these transformations, we obtain :

$$\begin{aligned} z_{12}^- &\longleftrightarrow g(x) & z_{21}^- &\longleftrightarrow -x/g(x) \\ z_{12}^+ &\longleftrightarrow x/g(x) & z_{21}^+ &\longleftrightarrow -g(x) \end{aligned} \quad (A.24)$$

5. - Expansion of $g(x)$ for small x

$g(x)$ is defined by :

$$\frac{x^2 + g^2(x)}{[1 + g^2(x)]x} = \frac{\varphi_2(z_{12}^-)}{\varphi_1(z_{12}^-)} = \frac{e}{z_{12}^-} \frac{\varphi_1(e/z_{12}^-)}{\varphi_1(z_{12}^-)} \quad (\text{A.25})$$

Expanding $\varphi_1(y')$ in powers of e :

$$\begin{aligned} \varphi_1(y')/e &= \sum_{n=-\infty}^{+\infty} \frac{1}{(n+y')^2} + e^2 \sum_{n \neq 0} \sum_{n=-\infty}^{+\infty} \frac{1}{r^2 s^2 (y'+n)^2} \left(1 - \frac{e}{rs} - \frac{e}{s(y'+n)}\right)^{-2} + \dots \\ &= \frac{\pi^2}{\sin^2 \pi y'} + e^2 \left\{ \frac{\pi^2}{3y'^2} \left(\frac{\pi^2}{\sin^2 \pi y'} - \frac{y'^2}{e^2} \right) + \frac{\pi^4}{9} \left(\frac{\pi^2}{\sin^2 \pi y'} - \frac{1}{y'^2} \right) \right\} + \dots \end{aligned} \quad (\text{A.26})$$

One can also expand x in powers of e :

$$x = \pi^2 e \left(1 - \frac{2\pi^4}{9} e^2 + \dots \right) \quad (\text{A.27})$$

Then we obtain

$$g(x) \simeq \frac{2}{\pi} x^2 \quad (\text{A.28})$$

near $x = 0$.

6. - Further restrictions

We have more restrictions on the integration region coming from (A.2) and (A.4). Defining the following Jacobi transforms :

$$\begin{aligned} y' &\longleftrightarrow y \\ p_1^{\frac{1}{2}} p_2^{\frac{1}{2}}(y') &\longleftrightarrow h(y, x) \\ p_2^{-\frac{1}{2}}(y') &\longleftrightarrow f_1(y, x) \\ p_1^{\frac{1}{2}}(y') &\longleftrightarrow f_2(y, x) \end{aligned}$$

These three functions fortunately are not independent. Using the properties shown in 3. and 4., in the particular representation chosen we have :

$$y'_1 = P_1^{\frac{1}{2}}(y'_2) \iff -\frac{p}{y'_2} = P_2^{-\frac{1}{2}}\left(-\frac{p}{y'_1}\right) \quad (\text{A.30})$$

then

$$y_1 = f_2(y_2, x) \iff -\frac{x}{y_2} = f_1\left(-\frac{x}{y_1}, x\right) \quad (\text{A.31})$$

Hence f_2 and f_1 are related by :

$$f_2(y, x) = -x / f^{-1}\left(-\frac{x}{y}, x\right) \quad (\text{A.32})$$

On the other hand one also has

$$y'_1 = P_1^{\frac{1}{2}}(y'_2) \iff -y'_2 = P_1^{\frac{1}{2}}(-y'_1) \quad (\text{A.33})$$

Hence $y_1 = f_2(y_2, x) \iff -y_2 = f_2(-y_1, x)$. Hence f_2 satisfies

$$f_2^{-1}(y, x) = -f_2(-y, x) \quad (\text{A.34})$$

In the same way

$$f_1^{-1}(y, x) = -f_1(-y, x) \quad (\text{A.35})$$

Using the fact that y' and $P_{1,2}(y')$ have the same Jacobi transform y one gets :

$$f_i(y, x) = f_i^{-1}(y, x) \quad i = 1, 2. \quad (\text{A.36})$$

From the definition one also has that

$$\begin{aligned}h(y, x) &= f_2 [f_1^{-1}(y, x), x] \\&= f_2 [f_1(y, x), x]\end{aligned}\tag{A.37}$$

Hence the properties of these functions are summarized by

$$\begin{aligned}f_i(y, x) &= -f_i(-y, x) = f_i^{-1}(y, x) \quad i = 1, 2 \\f_2(y, x) &= \frac{x}{f_1(\frac{x}{y}, x)} \\h(y, x) &= f_2(f_1(y, x), x)\end{aligned}\tag{A.38}$$

APPENDIX B - STUDY OF THE NEIGHBOURHOOD OF $x = x_0$

In the representation of Eq. (A.5) the invariant points of $P_1^{\frac{1}{2}}, P_2^{\frac{1}{2}}$ are given by

$$z_{1,2}^{\pm} = \frac{1 \pm \sqrt{1-16\rho}}{4} \quad (\text{B.1})$$

The integration region for the $s-t$ term as explained in Appendix A is :

$$z_{1,2}^- < y_1' < y_2' < P_1^{\frac{1}{2}} P_2^{\frac{1}{2}}(y_1') < z_{1,2}^+ \quad (\text{B.2})$$

or

$$g(x) < y_1 < y_2 < h(y, x) < \frac{x}{g(x)} \quad (\text{B.3})$$

When $\rho = \rho_0 = \frac{1}{16}$, $x = x_0$. In that case the integration region before the Jacobi transformation shrinks to zero like $\sqrt{1-16\rho}$. Since ρ_0 and $z_{1,2}^{\pm}$ are not singular points for the Jacobi transformation we conclude that $(x/g(x)) - g(x)$ behaves like $\sqrt{x_0 - x}$, and hence the integration region shrinks to zero also for y_1, y_2 .

Denoting the Jacobi transformation by $y = J(y', \rho)$ and

$$\frac{\partial J}{\partial y'} = J'(y', \rho) \quad \frac{\partial^2 J}{\partial y'^2} = J''(y', \rho) \quad (\text{B.4})$$

$$\frac{\partial J}{\partial \rho} = \dot{J}(y', \rho) \quad (\text{B.5})$$

The corresponding values at the point $\rho = \frac{1}{16}$, $y = \frac{1}{4}$ being denoted by the constants $J_0, J'_0, J''_0, \dot{J}_0$.

We define the variables :

$$y = \frac{y_1 - g(x)}{\frac{x}{g(x)} - g(x)} \quad (\text{B.6})$$

$$v = \frac{y_2 - y_1}{h(y_1, x) - y_1} \quad (\text{B.7})$$

and

$$\bar{y}'_1 = \frac{y'_1 - z_{12}}{z_{12}^+ - z_{12}^-} \quad (\text{B.8})$$

$$y''_2 = \frac{y'_2 - y'_1}{p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} (y'_1) - y'_1} \quad (\text{B.9})$$

The y and v variables are integrated in the square $0 \leq y \leq 1$, $0 \leq v \leq 1$.

We expand now the expression for y_1, y_2 around $\rho = \frac{1}{16}$, $y = \frac{1}{4}$. The result is :

$$y_i = \sqrt{x_0} - \frac{1}{4} \sqrt{1-16\rho} (1-2\bar{y}'_i) J_0' + \left(\rho - \frac{1}{16}\right) \left(J_0 - \frac{1}{2} (1-2\bar{y}'_i) J_0''\right) + O\left(\rho - \frac{1}{16}\right) \quad (\text{B.10})$$

$g(x)$ is obtained by setting in this expansion $\bar{y}'_1 = 0$ and $x/g(x)$ by setting $\bar{y}'_1 = 1$.

Further,

$$\bar{y}'_2 \approx \bar{y}'_1 + 4 y''_2 \bar{y}'_1 (1 - \bar{y}'_1) \sqrt{1-16\rho} + O(1-16\rho) \quad (\text{B.11})$$

and

$$y \approx \bar{y}'_1 + O(\sqrt{1-16\rho}) \quad (\text{B.12})$$

$$v \approx y''_2 + O(\sqrt{1-16\rho}) \quad (\text{B.13})$$

therefore

$$dy_1 dy_2 \approx dy dv (1-16\rho)^{3/2} \bar{y}_1 (1-\bar{y}_1)$$

(B.14)

and

$$\text{Ln} \frac{y_1'^2 + x^2}{y_1'^2 + 1} \frac{y_2'^2 + 1}{y_2'^2 + x^2} = -4 J_0' (1-16\rho) \gamma(1-\gamma) v \frac{1-x_0}{\sqrt{x_0}(1+x_0)} \quad (\text{B.15})$$

One has also the relation between x and ρ :

$$x - x_0 \approx \left(\rho - \frac{1}{16}\right) (\dot{J}_0 + 2 J_0') \approx \sqrt{x_0}. \quad (\text{B.16})$$

These expansions are all what is needed to derive the behaviour in the Bjorken limit.

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FIGURE CAPTIONS

- Figure 1 Integration region of the one-current planar amplitude.
- Figure 2 Restrictions on the integration region of the two-current amplitude. Dashed segments are forbidden. The mathematical "quark line" is indicated by arrows.
- Figure 3 Same as Figure 2 in the special configuration chosen in the paper.
- Figure 4 Integration region (dashed area) for fixed x for the s - t term.
- Figure 5 Integration region (dashed area) for fixed x for the s - u term.
- Figure 6 Integration region of the s - t term in a tri-dimensional representation.
- Figure 7 The off-shell Compton amplitude.
- Figure 8 a) s - t term ;
 b) u - t term ;
 c) s - u term.
- Figure 9 General two-current amplitude.
- Figure 10 Integration region of the s - t (a) and s - u (b) amplitudes in terms of the original variables, y_1', y_2', p .

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG.5

FIG. 6

FIG. 7

8A

8B

8C

FIG. 8

FIG. 9

A

B

FIG. 10