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RECALCULATION OF THE UNITARY SINGLE PLANAR DUAL LOOP
IN THE CRITICAL DIMENSION OF SPACE-TIME

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A B S T R A C T

The single planar dual loop amplitudes are recalculated in the critical dimension of space-time paying particular attention to the unitarity property by ensuring that the only states propagating internally in the loop are those needed to factorize tree diagram residues in a positive definite way. The two new technical features which make this possible are

- 1) our newly discovered physical state projection operator valid if the space-time dimension takes the critical value;
- 2) the use of Feynman's tree theorem whereby it is sufficient to have the above projection operator on the mass shell.

The final result agrees with a previous conjecture in that it differs from the original calculations with unprojected propagators by two inverse power of a certain partition function. The results apply to both the ordinary dual model and the Neveu-Schwarz model.

1. INTRODUCTION

The singularities of the dual model amplitudes are just poles with factorizing residues and this suggests that we have the tree diagrams of a new diagrammatic theory. The loop diagrams needed to supplement the tree diagrams have been constructed by a sewing procedure ¹⁾ but until now it has not been possible to examine the unitarity properties of these loops, the reason being that only recently has the structure of the spectrum of physical states been at all understood ²⁾. In a certain critical number of space-time dimensions (26 for the conventional model, 10 for the Neveu-Schwarz model) the physical spectrum of the theory is given by a positive definite Fock space of $(d-2)$ dimensional oscillators rather than the original d dimensional indefinite metric Fock space with which the amplitudes were factorized first ³⁾.

In a previous paper ⁴⁾ we constructed the projection operator onto the physical subspace from the full space for this critical dimension and we now propose to use it in order to recalculate the single planar dual loops. These are the simplest loops to consider.

We would also like to recalculate the more complicated loops and in particular the non-planar loops with Pomeron singularities, but new problems arise which we have not yet solved. This would be important to understand in view of the conjecture that the Pomeron graph can be unitary only for the critical dimension of space-time ⁵⁾. This is in fact the main reason why we consider the loop graphs only for this critical dimension. (For dimensions less than the critical one loop calculations based on quite different methods have already been made ⁶⁾.)

The method of loop construction is that due to Feynman ⁷⁾ in which a single loop is expressed in terms of sums of tree graphs with the external states physical and on the mass shell. Section 2 is devoted to a résumé of Feynman's argument since it does not seem to be readily available in the literature. According to this argument the physical state projection operator needs only be known on the mass shell in order for the perturbative unitarity of the loop so constructed to be guaranteed.

In Section 3 we present our projection operator, which is most conveniently expressed in terms of the "Virasoro gauge operators" which generate the symmetries of the couplings. We then proceed to eliminate this projector from a typical loop expression by a procedure of moving the

gauge operators around the loop. Those with positive index are moved to the right and those with negative index to the left around the loop. Each of the two manoeuvres gives rise to an identical correction factor, namely an inverse power of a certain partition function. Apart from those correction factors the result is identical to that calculated by omitting the projection operator (as was done originally ¹⁾). This is shown in Section 4 where we put together the results from Sections 2 and 3. This result has been conjectured before ²⁾, but to our knowledge, never proved except in so far as it has recently been obtained by functional integration methods based on the rubber string picture ⁸⁾. Since our method is based on the operator formalism we think that the connection between unitarity and the correction factors is made much clearer. Also the role of the critical dimension is quite clear.

In Section 5 we will use the same methods to construct the single planar loops in the Neveu-Schwarz model ⁹⁾ (for which there are no functional integration methods at the present time).

2. FEYNMAN'S TREE THEOREM

This was first presented by Feynman ⁷⁾ in connection with his study of diagrammatic rules for Yang-Mills theories, but since it is not readily accessible in the literature we shall now give a resumé with a view to the dual model applications. Let us define the retarded ($\Delta_R(x)$) Feynman ($\Delta_F(x)$) and Δ_+ Green's functions by

$$\Delta_i(x) = \frac{-1}{(2\pi)^4} \int_{C_i} d^4k \frac{e^{ik \cdot x}}{k^2 - m^2} \quad (2.1)$$

where the contours C_i in the k^0 plane are orientated with respect to the poles of the integrand as shown in Fig. 2.1a.

Fig. 2.1a

Fig. 2.1b

Note that $\Delta_R(x)$ is retarded, i.e., it vanishes if $x^0 < 0$, as can be seen by distorting the integration contour into the lower half plane and that $\Delta_+(x)$ picks up only components of momentum on the positive mass shell hyperboloid. The geometry of the above contours indicates that the three Green's functions are linearly related:

$$\Delta_R(x) = \Delta_F(x) + \Delta_+(x) \quad (2.2)$$

In momentum space this reads

$$\frac{1}{k^2 - m^2 - i\epsilon k^0} = \frac{1}{k^2 - m^2 + i\epsilon} + 2\pi i \Theta(k^0) \delta(k^2 - m^2) \quad (2.3)$$

With the graphical notation of Fig. 1b this reads as shown in Fig. 2.2.

$$\longrightarrow = \longleftarrow + \longrightarrow$$

Fig. 2.2

We shall now insert this identity into the integral

$$\int d^4x \Delta_R(x) \Delta_R(-x) \quad (2.4)$$

and so obtain the diagrammatic equation shown in Fig. 2.3 if we forget about

$$\text{Loop with two arrows} = \text{Loop with two opposite arrows} + \text{Loop with two same arrows (top offset)} + \text{Loop with two opposite arrows (bottom offset)} + \text{Loop with two same arrows (both offset)}$$

Fig. 2.3

problems of ultraviolet divergence (which disappear when we consider single loops with three or more internal lines: the self-energy graph is the simplest graph to consider and illustrates all the main points). Now by the retardedness property the integral in (2.4) vanishes, since one or other of the two factors in the integrand is always zero. In momentum space this corresponds to the fact that the integration contour can be distorted to zero in the lower half plane (at least if the integral converges). Notice that on the right-hand side of the equation in Fig. 2.3 one term is the Feynman integral (with Feynman propagators) and all the other terms have at

least one line with a Δ_+ Green's function so that the corresponding line is on the mass shell. Thus we can rewrite Fig. 2.3 as in Fig. 2.4, where

Fig. 2.4

we have cut open the lines on the mass shell and replaced the integration over the positive sheet of the mass shell hyperboloid by a summation sign indicating a phase space integration (which is the same thing). In other words, the Feynman loop (on the left) is expressed in terms of sums over trees with external lines on the mass shell (on the right). This is Feynman's tree theorem for ϕ^3 theory in the case of the self energy single loop.

The generalization to the single loop with N internal lines (see Fig. 2.5 for the labelling) can be written, in momentum space as

$$\int \frac{i d^4 k}{(2\pi)^4} \prod_i \frac{g_i}{q_i^2 - m_i^2 + i\epsilon} = - \sum_{\substack{\text{ways of} \\ \text{cutting}}} \int \frac{i d^4 k}{(2\pi)^4} \prod_{i=1}^N \Delta_i^0(q_i) g_i \quad (2.5)$$

where

$$\Delta_i^0(q_i) = \begin{cases} \frac{1}{q_i^2 - m_i^2 + i\epsilon} & \text{if line } i \text{ is not cut} \\ 2\pi i \theta(q_i^0) \delta(q_i^2 - m_i^2) & \text{if line } i \text{ is cut} \end{cases} \quad (2.6)$$

q_i is the momentum of the i^{th} internal line, measured in the direction of the loop momentum k , g_i is the coupling constant associated with the emission of the i^{th} external particle, and the sum over ways of cutting refers to $(2^N - 1)$ terms corresponding to the possibility of each internal line being cut (put on mass shell) or not, but excludes the possibility of no lines being cut (since this is the left-hand side).

Fig. 2.5

Now one can ask whether the right-hand side of Eq. (2.5) provides a good candidate for the loop amplitude if one did not know of the preceding argument involving retarded functions (or distorted contours). If one could check that it had the correct analyticity properties, i.e. location of Landau singularities and their associated $i\epsilon$ prescription and discontinuities, then it must be correct, up to a possible subtraction constant. Now, in the case $N=2$, the only singularity should be the normal threshold at $s = (m_1 + m_2)^2$, but the individual terms on the right-hand side of (2.5) or Fig. 2.4 are also singular at the pseudothreshold $s = (m_1 - m_2)^2$. Roughly speaking one must check that the discontinuities across this pseudothreshold cancel between the three terms. Actually the situation is more complicated because of conflicting $i\epsilon$ prescriptions, but all the relevant concepts, techniques and theorems are presented in Ref. 10) and we shall simply state that with these the desired results can be checked.

The importance of the tree theorem is revealed when we consider theories such as that of Yang-Mills or the dual models, in which the tree graph expressions are known for external states on the mass shell and "physical" and in which the "physical" states satisfy a bootstrap property: the totality of physical states completely factorizes the residues of the poles in the tree graph amplitudes.

In such theories the propagator commonly used to construct the tree graphs is not unitary (or not projected), in that it propagates a larger spectrum than the "physical spectrum". The deficiency between the two spectra can be gauge transformed away in tree graphs by the use of

certain Ward identities arising from the symmetries of the coupling. In loops this deficiency cannot be gauge transformed away and so the use of the non-unitary (or non-projected) propagators is incorrect and leads to discontinuity formulae with spurious intermediate states in addition to physical ones. Feynman's proposal is to use the equation in Fig. 2.4, (and its generalization to N internal lines) as the definition of the appropriate single loop graph since it will satisfy the desired analyticity properties, provided when one joins legs of trees one sums only over physical states - or equivalently - uses a physical state projection operator.

In checking these analyticity properties one must obtain certain cancellations between the terms of the right-hand side of Fig. 2.4, just as before. Now this is less trivial since the cut lines (when the external lines of a tree graph are joined) bear a projection operator and the uncut lines (which are internal lines of tree graphs) do not. Nevertheless when an uncut line participates in a singularity it is put on the mass shell and then one is free to insert the physical state projection operator. [This follows from the bootstrap property mentioned above and can be proved explicitly by the Ward identities: for the dual models this is proven in Ref. 4).] The snag is that in Yang-Mills theories Ward identities are valid order by order in perturbation theory and not graph by graph so that one is forced to enlarge the class of graphs considered. (This is discussed by Feynman⁷⁾). In dual theories, fortunately, the situation is simpler since the Ward identities are valid graphs by graph and so the equation in Fig. 2.4 and its generalizations are valid as they stand as candidates for loop amplitudes, with only the appropriate re-interpretation of the vertices and propagators,

$$g_i \rightarrow V(p_i)$$

and instead of (2.6) we now have

$$\Delta_i(q_i) = \begin{cases} -\frac{1}{2} \int_0^1 \frac{dx_i}{x_i} x_i^{L_i-1} & \text{if line } i \text{ is uncut} \\ \sum_{n_i=0}^{\infty} [2\pi i \theta(q_i^0) \delta(q_i^2 - 2(n_i - 1))] \left[\oint \frac{dx_i}{2\pi i x_i} x_i^{H-n_i} \right] \mathcal{P} & \text{if line } i \text{ is cut} \end{cases} \quad (2.7)$$

The dual model notations are defined in Refs. 4) and 11). In the last expression the sum is over the various mass shells, indicated by the function in the first square bracket, the second square bracket projects from the full Fock space onto the subspace contributing to the mass shell

labelled with n_i and $\mathcal{P}$ [defined in the next section, see Eq. (3.1)] projects onto the physical subspace at that level. Thus, instead of (2.6) our planar loop amplitude is given by

$$-\sum_{\text{ways of cutting}} \int \frac{i d^4 k}{(2\pi)^4} \text{Tr} (\Delta_1 V_1 \Delta_2 V_2 \dots \Delta_N V_N) \quad (2.8)$$

In the next section (Section 3) we shall evaluate a typical term in the above sum, in which just one line is cut. In Section 4 we shall see how the evaluation of the other terms follows and how the terms can be combined to perform the summation over the ways of cutting.

We shall now mention two deductions that can be made concerning the above tree theorem. If in expressions (2.7) and (2.8) the projection operator $\mathcal{P}$ was everywhere replaced by unity, 1, then expression (2.8) would reduce to the original expression in which no projections were made ¹⁾

$$\int \frac{i d^4 k}{(2\pi)^4} \text{Tr} \left(\frac{1}{L_0 - 1} V_1 \frac{1}{L_0 - 1} V_2 \dots \frac{1}{L_0 - 1} V_N \right)$$

Secondly, if we had proved that $\mathcal{P}$ was also an off-mass shell projection operator [Eq. (2.33) of Ref. 4)] then we could deduce that the loop expression (2.8) is equal to

$$\int \frac{i d^4 k}{(2\pi)^4} \text{Tr} \left(\frac{1}{L_0 - 1} \mathcal{P} V_1 \frac{1}{L_0 - 1} \mathcal{P} V_2 \dots \frac{1}{L_0 - 1} \mathcal{P} V_N \right)$$

In the absence of this result we shall work with (2.8).

3. EVALUATION OF A TYPICAL TRACE

As explained in our previous paper [Ref. 4)] the projection operator at any mass shell level onto the transverse subspace of physical states at that level is given by

$$\mathcal{P} = \oint \frac{dy}{2\pi i y} y^E \quad (3.1)$$

where the integration contour encircles the origin and

$$E = (D_0 - 1)(L_0 - 1) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n) \quad (3.2)$$

L_n is the conventional Virasoro gauge operator [whose commutator with vertices is known and given by Eq. (3.13) later on], and D_n is a new operator defined in Ref. 4). For the purposes of the present loop calculation we need only the following algebraic properties of the above operators:

$$\left. \begin{aligned} [L_n, D_m] &= -(2n+m) D_{n+m} \\ L_n y^E &= y^{E-n} L_n \end{aligned} \right\} \quad (3.3)$$

In particular we do not explicitly use the rather complicated $[\bar{L}, L]$ commutator. It has already been used to derive the second of the above Eqs. (3.3) (which is valid only in 26 dimensions) and that is all we need. We should like to stress that this is the major simplification in our approach. In addition we shall also need the following relation, proved in Ref. 4):

$$(D_0 - 1) \mathcal{T} = 0 = \mathcal{T} (D_0 - 1) \quad (3.4)$$

As mentioned in Ref. 4) the projection operator (3.1) is not Lorentz invariant because the D_n 's depend on a certain lightlike auxiliary vector k^μ . As we shall see this dependence on k will disappear in the loop calculation.

The planar loop amplitude to be studied has lines as in Fig. 2.5. We take as the typical contribution to (2.8) the term in which just internal line 1 (joining external lines 1 and N) is on the mass shell, and so bears the projection operator

$$\oint \frac{dy}{2\pi i y} y^E \oint \frac{dx_1}{2\pi i x_1} x_1^{L_0-1} = \oint \frac{dy}{2\pi i y} y^E \oint \frac{dx_1}{2\pi i x_1} x_1^{H-n_1-1}$$

where its momentum q_1 satisfies the mass shell condition

$$q_1^2 = m_{n_1}^2 = 2(n_1 - 1) \quad (3.5)$$

All the other internal lines bear the conventional propagators

$$\int_0^1 \frac{dx_i}{x_i} x_i^{L_0-1}$$

Thus if we define

$$x_1^{L_0-1} V_1 x_2^{L_0-1} V_2 \dots x_N^{L_0-1} V_N = P(x_1, x_2, \dots, x_N) = P(x) \quad (3.6)$$

we need to evaluate

$$\oint \frac{dy}{2\pi i y} \oint \frac{dx_1}{2\pi i x_1} \prod_{i=2}^N \int_0^1 \frac{dx_i}{x_i} \text{Tr} (y^E P(x)) \quad (3.7)$$

since it is proportional to the loop momentum integrand of the term we want.

In fact we shall show that the integral (3.7) is equal to

$$\oint \frac{dx_1}{2\pi i x_1} \prod_{i=2}^N \int_0^1 \frac{dx_i}{x_i} \prod_{n=1}^{\infty} (1 - x_{1N}^n)^2 \text{Tr} (P(x)) \quad (3.8)$$

where $x_{1N} = x_1 \cdot x_2 \dots x_N$. Thus the only effect of inserting the projection operator (3.1) is to produce the factor $\prod_{n=1}^{\infty} (1 - x_{1N}^n)^2$, which is the promised double inverse power of the partition function. Notice that the apparent dependence of expression (3.7) upon the momentum k^μ appearing in the projection operator has disappeared as it should to ensure that the final amplitude is indeed Lorentz invariant. Notice also that the final integral converges since the variable x_1 is not integrated to 1 and so the singularity of the partition function is not reached. The importance of this is that at this stage all our manipulations will be well defined and rigorous.

Let us now illustrate the method we shall use to evaluate the trace in (3.7) in the following trivial case

$$f(y) = \text{Tr} (y^{a^\dagger a})$$

where

$$a a^\dagger - \epsilon a^\dagger a = 1$$

so that we have a Bose or Fermi oscillator according as ϵ is $+1$ or -1 . We differentiate with respect to y

$$y \frac{\partial f}{\partial y} = \text{Tr} (y^{a^\dagger a} a^\dagger a)$$

and then move the destruction operator around the trace:

$$= \text{Tr} (a y^{a^\dagger a} a^\dagger) = y \text{Tr} (y^{a^\dagger a} a a^\dagger) = y \text{Tr} (y^{a^\dagger a} (\epsilon a^\dagger a + 1)) \quad (3.9)$$

Thus, we obtain the differential equation

$$\frac{1}{f} \frac{\partial f}{\partial y} = \frac{1}{1 - \epsilon y} \quad (3.10)$$

with solutions

$$f(y) = \begin{cases} \frac{\text{const.}}{1-y} & \text{if } \epsilon = 1 \\ \text{const.} (1+y) & \text{if } \epsilon = -1 \end{cases} \quad (3.11)$$

We now apply this idea to the more complicated trace under the integration in (3.7) and again differentiate with respect to y

$$y \frac{\partial}{\partial y} \text{Tr} (y^E P(x)) = x_1 \frac{\partial}{\partial x_1} \text{Tr} (y^E (D_0 - 1) P(x)) + \sum_{n=1}^{\infty} \text{Tr} (y^E (D_{-n} L_n + L_{-n} D_n) P(x)) \quad (3.12)$$

In the first term on the right-hand side we have eliminated a factor $L_0 - 1$ in favour of a differentiation, a manoeuvre we shall often repeat. We shall now imitate the procedure in our simple example by considering the term with $D_{-n} L_n$ and obtaining an equation for it by moving the L_n to the right around the trace (or loop). Now we see the utility of our form of the projection operator: it is well known how to commute L_n 's with vertices and

propagators

$$(L_n - L_0 + 1) x^{L_0 - 1} V = x^{L_0 + n - 1} V (L_n - L_0 + 1) - x \frac{d}{dx} \left[(1 - x^n) x^{L_0 - 1} V \right] \quad (3.13)$$

and the algebra (3.3) tells us how to commute L_n with y^E and D_{-n} and bring it back to its original position. The result of the calculation is

$$\begin{aligned} \left[1 - \left(\frac{x_{1N}}{y} \right)^n \right] \text{Tr} (y^E D_{-n} L_n P(x)) &= -n \left(\frac{x_{1N}}{y} \right)^n \text{Tr} (y^E D_0 P(x)) \\ &+ \sum_{i=2}^N \mathcal{D}_i \left[x_{1i-1}^n (1 - x_i^n) \text{Tr} (y^E D_{-n} P(x)) \right] \end{aligned} \quad (3.14)$$

We have defined the differential operator

$$\mathcal{D}_i = x_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_i} \quad (3.15)$$

which has the following property that we shall use repeatedly: it commutes with any function of x_1 and x_i which depends on the variables only via the product $x_1 x_i$. This enables us to deduce

$$\begin{aligned} \text{Tr} (y^E D_{-n} L_n P(x)) &= \frac{-n \left(\frac{x_{1N}}{y} \right)^n}{1 - \left(\frac{x_{1N}}{y} \right)^n} \text{Tr} (y^E D_0 P(x)) + \\ &+ \sum_{i=2}^N \mathcal{D}_i \left[\frac{x_{1i-1}^n (1 - x_i^n)}{1 - \left(\frac{x_{1N}}{y} \right)^n} \text{Tr} (y^E D_{-n} P(x)) \right] \end{aligned} \quad (3.16)$$

The point of always keeping the last term as a total derivative is that eventually when the x integrations are applied it will integrate out. Similarly working the L_{-n} to the left we find

$$\begin{aligned} \text{Tr} (y^E L_{-n} D_n P(x)) &= \frac{-n \left(\frac{x_{1N}}{y} \right)^n}{1 - \left(\frac{x_{1N}}{y} \right)^n} \text{Tr} (y^E D_0 P(x)) \\ &+ \sum_{i=2}^N \mathcal{D}_i \left[\frac{x_{1i-1}^n (1 - x_i^n)}{y^n (1 - \left(\frac{x_{1N}}{y} \right)^n)} \text{Tr} (y^E D_n P(x)) \right] \end{aligned} \quad (3.17)$$

Notice that the right-hand sides of the last two terms differ only in the final terms which will integrate out. This is the origin of the two inverse powers of the partition function mentioned earlier: one power comes from the L_n 's and one from the L_{-n} 's.

If we denote the partition function ¹²⁾ by

$$f(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} \quad (3.18)$$

we can now assemble Eqs. (3.12), (3.16) and (3.17) to give the differential equation

$$\begin{aligned} y \frac{\partial}{\partial y} \text{Tr}(y^E P(x)) &= x_1 \frac{\partial}{\partial x_1} \text{Tr}(y^E (D_0 - 1) P(x)) + \\ &+ \text{Tr}(y^E D_0 P(x)) y \frac{\partial}{\partial y} \left[\ln f^2\left(\frac{x_{1N}}{y}\right) \right] + \sum_{i=2}^N \mathcal{D}_i A_i(x, y) \end{aligned} \quad (3.19)$$

where we have introduced a new shorthand notation for the terms which will integrate out. By using the fact that

$$x_1 \frac{\partial}{\partial x_1} f^2\left(\frac{x_{1N}}{y}\right) = -y \frac{\partial}{\partial y} f^2\left(\frac{x_{1N}}{y}\right) \quad (3.20)$$

one can rearrange the differential equation into a more convenient form

$$\begin{aligned} y \frac{\partial}{\partial y} \left[f^{-2}\left(\frac{x_{1N}}{y}\right) \text{Tr}(y^E P(x)) \right] &= x_1 \frac{\partial}{\partial x_1} \left[f^{-2}\left(\frac{x_{1N}}{y}\right) \cdot \right. \\ &\left. \text{Tr}(y^E (D_0 - 1) P(x)) \right] + \sum_{i=2}^N \mathcal{D}_i \left[A_i(x, y) f^{-2}\left(\frac{x_{1N}}{y}\right) \right] \end{aligned} \quad (3.21)$$

If the right-hand were zero, it is now fairly clear that we would reach the desired result. What we now wish to show is that both terms on the right-hand side will integrate to zero when the x_i and y integrations are carried out.

First we solve (3.21) by an integration of both sides from 1 to y . The lower limit is chosen in order to express the trace with y^E in terms of the trace without this factor, which is the known result ¹⁾. We then obtain

$$\begin{aligned}
 & f^{-2}\left(\frac{x_{1N}}{y}\right) \text{Tr}(y^E P(x)) - f^{-2}(x_{1N}) \text{Tr}(P(x)) = \\
 & = x_1 \frac{\partial}{\partial x_1} \int_1^y \frac{dy'}{y'} f^{-2}\left(\frac{x_{1N}}{y'}\right) \text{Tr}(y'^E (D_0 - 1) P(x)) + \\
 & + \sum_{i=2}^N \mathcal{D}_i \int_1^y \frac{dy'}{y'} A_i(x, y') f^{-2}\left(\frac{x_{1N}}{y'}\right)
 \end{aligned} \tag{3.22}$$

Now we multiply by $f^2(x_{1N}/y)$ and perform the x integrations. In the shorthand

$$\int \frac{dx}{x} \equiv \oint \frac{dx_1}{2\pi i x_1} \prod_{i=2}^N \int_0^1 \frac{dx_i}{x_i} \tag{3.23}$$

we have

$$\begin{aligned}
 \int \frac{dx}{x} \text{Tr}(y^E P(x)) &= \int \frac{dx}{x} \frac{f^2\left(\frac{x_{1N}}{y}\right)}{f^2(x_{1N})} \text{Tr}(P(x)) - \\
 &- \int \frac{dx}{x} x_1 \frac{\partial}{\partial x_1} \left[f^2\left(\frac{x_{1N}}{y}\right) \right] \int_1^y \frac{dy'}{y'} f^{-2}\left(\frac{x_{1N}}{y'}\right) \text{Tr}(y'^E (D_0 - 1) P(x))
 \end{aligned} \tag{3.24}$$

The term with the A_i 's has integrated out since A_i vanishes when $x_i = 0$ and 1 and since the x_1 integration is closed. We have also performed an integration by parts with respect to x_1 in the final remaining term. We can now perform the y integration. The last term is then seen to be zero because we can use Eq. (3.20) and perform a partial integration in y to arrive at

$$- \int \frac{dx}{x} \oint \frac{dy}{2\pi i y} \text{Tr}(y^E (D_0 - 1) P(x))$$

which is zero by Eq. (3.4). We thus arrive at our final result

$$\oint \frac{dy}{y} \int \frac{dx}{x} \text{Tr}(y^E P(x)) = \int \frac{dx}{x} f^{-2}(x_{1N}) \text{Tr}(P(x)) \tag{3.25}$$

where the y integration in the remaining term has been performed by distorting the contour to infinity. Equation (3.25) is in fact the announced result. The reader will notice that all the terms which appear to break Lorentz covariance have finally disappeared by some rather subtle integrations by parts. So far we have no better understanding of the mechanism.

4. COLLECTION OF TERMS IN THE FEYNMAN TREE THEOREM

The last section dealt with a single term in the Feynman tree expansion equation (2.8); the term with just line 1 cut and so on mass shell. This term is enough to make the pattern clear for as we shall now see, all the other terms can be deduced from it. Finally it will be possible to add them together and perform the sums over the mass shells of the cut lines.

We shall now rewrite (2.8) in a slightly different way:

$$-\sum_{\substack{\text{ways of} \\ \text{cutting}}} \int \frac{i d^4 k}{(2\pi)^4} \prod_{l=1}^N E_l(q_l) \int_{C_l} \frac{dx_l}{x_l} \text{Tr} (x_1^{L_0-1} \mathcal{I}_1 V_1 \\ x_2^{L_0-1} \mathcal{I}_2 V_2 \dots x_N^{L_0-1} \mathcal{I}_N V_N) \quad (4.1)$$

where

$$\left. \begin{aligned} E_l(q_l) &= \sum_{n_l=0}^{\infty} \theta(q_l^0) \delta(q_l^2 - 2(n_l-1)) \\ \mathcal{I}_l &= \oint \frac{dy_l}{2\pi i y_l} y_l^E \\ C_l &\text{ is a contour encircling the origin} \end{aligned} \right\} \begin{array}{l} \text{if line } l \\ \text{is cut} \end{array} \quad (4.2)$$

and

$$\left. \begin{aligned} E_l(q_l) &= -1/2 \\ \mathcal{I}_l &= 1 \\ C_l &\text{ runs from } 0 \text{ to } 1 \end{aligned} \right\} \begin{array}{l} \text{if line } l \\ \text{is not cut} \end{array}$$

Now it follows from the work of our previous paper ⁴⁾ that in any term with several lines cut, all but one of the $\mathcal{T}$'s can be replaced by 1. In the last section we saw that if there was just one $\mathcal{T}$, say on line 1

$$\begin{aligned} & \prod_i \int_{C_i} \frac{dx_i}{x_i} \text{Tr} (x_1^{L_0-1} \mathcal{T} V_1 x_2^{L_0-1} V_2 \dots x_N^{L_0-1} V_N) = \\ & = \prod_i \left[\int_{C_i} \frac{dx_i}{x_i} x_i^{-d_i-1} \right] \prod_{n=1}^{\infty} (1-x_{iN}^n)^2 \text{Tr} (x_1^H V_1 x_2^H V_2 \dots x_N^H V_N) \end{aligned} \quad (4.3)$$

where $\alpha_i = 1 + q_i^2/2$ is the trajectory function associated with the i 'th internal line. Notice that the argument of the previous section is unaffected if some of the contours C_i enclose the origin instead of running from 0 to 1. Also notice that the only way the integral on the right-hand side of (4.3) distinguishes which particular lines are cut is via the assignment of contours C_i . In particular the integrand is quite independent of the choice of which lines are cut and it is this which will enable us to collect together different terms of the sum in (4.1).

Since the operator H has a positive spectrum the expression

$$\prod_{n=1}^{\infty} (1-x_{iN}^n)^2 \text{Tr} (x_1^H V_1 x_2^H V_2 \dots x_N^H V_N)$$

is analytic in the x_i at the origin and so possesses a multiple Taylor expansion in these variables

$$\sum_{m_i=0}^{\infty} \prod_{i=1}^N (x_i^{m_i}) T_{m_1, m_2 \dots m_N}$$

Now

$$-\frac{1}{2} \int_0^1 \frac{dx_i}{x_i} x_i^{m_i - d_i - 1} = \frac{1}{2(d_i - m_i)} = \frac{1}{q_i^2 - 2(m_i - 1)} \quad (4.4)$$

and

$$\oint \frac{dx_i}{2\pi i x_i} x_i^{m_i - d_i - 1} = \delta_{\alpha_i m_i}$$

Therefore, putting together all these comments (4.1) can be rewritten as

$$-\sum_{\text{ways of cutting}} \int \frac{i d^4 k}{(2\pi)^4} \sum_{n_i=0}^{\infty} \prod_i \Delta_i^0(q_i) T_{n_1, n_2 \dots n_N} \quad (4.5)$$

where $\Delta_i^0(q_i)$ are given by Eqs. (2.6) with

$$m_i^2 = 2(n_i - 1)$$

If we can interchange the summation over n_i with the remaining sums and integrations we can use the Feynman tree theorem for ϕ^3 theory Eq. (2.5) to rewrite (4.5) as

$$\int \frac{i d^4 k}{(2\pi)^4} \sum_{n_i=0}^{\infty} \prod_i \frac{1}{q_i^2 - 2(n_i - 1) + i\epsilon} T_{n_1, n_2 \dots n_N}$$

So, working back using (4.4) this becomes

$$\int \frac{i d^4 k}{(2\pi)^4} \prod_{i=1}^N \left[-\frac{1}{2} \int_0^1 \frac{dx_i}{x_i} \right] \prod_{n=1}^{\infty} (1 - x_{1N}^n)^2 \text{Tr} (x_1^{L_0-1} v_1 \dots x_N^{L_0-1} v_N) \quad (4.6)$$

We should remark that in the last few steps the mathematics is not very rigorous. Indeed the final integral, as is well known, diverges at $x_{1N} = 1$. This divergence in our calculation was probably first introduced by the sums over mass shells. This sum over mass shells also means that the poles extend to infinity along the real axis and this makes the dropping of the $i\omega$ contour in the lower k^0 plane (corresponding to the retardedness of Feynman) somewhat dubious. We suspect that a more careful treatment of these points might lead to a construction of the correct renormalization counterterm which renders the amplitude finite. We should mention that the problem of renormalizing these single planar amplitudes in the critical dimension is as yet unsolved ¹³⁾.

5. THE CONSTRUCTION OF DUAL SINGLE PLANAR LOOPS IN THE NEVEU-SCHWARZ MODEL

In our previous paper ⁴⁾ we also constructed the corresponding projection operator at any mass shell level onto the transverse subspace of physical states for the Neveu-Schwarz model ⁹⁾. This time it is

given by

$$\oint \frac{dy}{4\pi i y} y^{E_{NS}}$$

where the integration contour now encircles the origin twice ¹⁴⁾ (this is the meaning of the two circles on the integration sign), since E_{NS} could have half-integer eigen values, and E_{NS} is defined by

$$E_{NS} = (D_0 - 1)(L_0 - 1/2) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n) + \sum_{r=1/2}^{\infty} (E_{-r} G_r - G_{-r} E_r) \quad (5.1)$$

L_n and G_r are the conventional gauge operators for the Neveu-Schwarz model ^{9), 15)}, and D_n and E_r are new operators defined in Ref. 4). The following algebraic properties of these operators are needed for the calculation of planar single loops.

$$\left. \begin{aligned} [L_n, D_m] &= - (2n+m) D_{n+m} \\ \{G_r, E_s\} &= - \frac{3r+s}{2} D_{r+s} \\ L_n y^E &= y^{E-n} L_n \\ G_r y^E &= y^{E-r} G_r \end{aligned} \right\} \quad (5.2)$$

We also need as in Section 3 that

$$(D_0 - 1) \oint \frac{dy}{4\pi i y} y^{E_{NS}} = 0 = \oint \frac{dy}{4\pi i y} y^{E_{NS}} (D_0 - 1) \quad (5.3)$$

We shall follow the methods of Section 3 closely. We study the loop amplitude with lines as in Fig. 2.5, with external pions, and consider also this time the term in which just internal line 1 is on the mass shell. If we define

$$x_1^{L_0 - 1/2} V_1 x_2^{L_0 - 1/2} V_2 \dots x_N^{L_0 - 1/2} V_N = P_{NS}(x_1, \dots, x_N) = P_{NS}(x) \quad (5.4)$$

(with V_i being the Neveu-Schwarz vertex).

We need to evaluate

$$\oint \frac{dy}{4\pi i y} \oint \frac{dx_1}{4\pi i x_1} \prod_{i=2}^N \int_0^1 \frac{dx_i}{x_i} \text{Tr} (y^{E_{NS}} P_{NS}(x)) \quad (5.5)$$

This we do as in Section 3 by differentiating the trace with respect to y

$$y \frac{\partial}{\partial y} \text{Tr} (y^{E_{NS}} P_{NS}(x)) = x_1 \frac{\partial}{\partial x_1} \text{Tr} (y^{E_{NS}} (D_0 - 1) P_{NS}(x)) + \\ + \sum_{n=1}^{\infty} \text{Tr} (y^E (D_{-n} L_n + L_{-n} D_n) P_{NS}(x)) + \sum_{r=1/2}^{\infty} \text{Tr} (y^E (E_{-r} G_r - G_{-r} E_r) P_{NS}(x)) \quad (5.6)$$

and evaluating the terms in the sums by moving L_n and G_r to the right, and L_{-n} and G_{-r} to the left around the trace back to their initial positions. This can be done as we know how the L_n and G_r operators commute with propagators and vertices

$$\left. \begin{aligned} L_n x^{L_0} &= x^{L_0+n} L_n \\ (L_n - L_0 - n) V &= V (L_n - L_0) \\ G_r x^{L_0} &= x^{L_0+r} G_r \\ G_r V &= -V G_r - (L_0 + r - 1/2) V_0 + V_0 (L_0 - 1/2) \end{aligned} \right\} \quad (5.7)$$

where V_0 is the Veneziano model vertex. If we collect all terms together we end up with

$$y \frac{\partial}{\partial y} \text{Tr} (y^{E_{NS}} P_{NS}(x)) = x_1 \frac{\partial}{\partial x_1} \text{Tr} (y^{E_{NS}} (D_0 - 1) P_{NS}(x)) \\ + \text{Tr} (y^{E_{NS}} D_0 P_{NS}(x)) y \frac{\partial}{\partial y} \left\{ \ln f^2 \left(\frac{x_{1N}}{y} \right) + \ln g^2 \left(\frac{x_{1N}}{y} \right) \right\} \\ + \sum_{i=2}^N \mathcal{D}_i \left\{ \frac{x_{1i-1}^n (1-x_i^n)}{1 - (x_{1N}/y)^n} \text{Tr} (y^{E_{NS}} D_{-n} P_{NS}(x)) \right. \\ \left. + \frac{x_{i+1N}^n (1-x_i^n)}{y^n (1 - (x_{1N}/y)^n)} \text{Tr} (y^{E_{NS}} D_n P_{NS}(x)) \right\} \\ + \sum_{i=1}^N \tilde{\mathcal{D}}_i (-1)^i \left\{ \frac{x_{1i}^r}{1 + (x_{1N}/y)^r} \text{Tr} (y^{E_{NS}} E_{-r} \tilde{P}_{NS}^i(x)) \right. \\ \left. + \frac{x_{i+1N}^r}{y^r (1 - (x_{1N}/y)^r)} \text{Tr} (y^{E_{NS}} E_r \tilde{P}_{NS}^i(x)) \right\} \quad (5.8)$$

where

$$g(x) = \prod_{r=1/2}^{\infty} (1+x^r) \quad (5.9)$$

$\mathcal{D}_i$ is defined in (3.15) and

$$\tilde{\mathcal{D}}_i = x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} \quad (5.10)$$

$$\tilde{\mathcal{D}}_N = x_N \frac{\partial}{\partial x_N} - x_1 \frac{\partial}{\partial x_1} \quad (5.11)$$

$$\tilde{P}_{Ns}^i(x) = x_1^{L_0-1/2} V_1 x_2^{L_0-1/2} V_2 \dots x_i^{L_0-1/2} V_i \dots x_N^{L_0-1/2} V_N \quad (5.12)$$

We notice that also in this case we get differential operators $\mathcal{D}_i$ and $\tilde{\mathcal{D}}_i$ which commute with x_{1N} . We can then understand that the last two sums finally will integrate to zero.

By using (3.20) and the corresponding equation for the function g [Eq. (5.9)], Eq. (5.8) can be written as

$$\begin{aligned} & y \frac{\partial}{\partial y} \left[f^{-2} \left(\frac{x_{1N}}{y} \right) g^{-2} \left(\frac{x_{1N}}{y} \right) \text{Tr} (y^{E_{Ns}} P_{Ns}(x)) \right] = \\ & = x_1 \frac{\partial}{\partial x_1} \left[f^{-2} \left(\frac{x_{1N}}{y} \right) g^{-2} \left(\frac{x_{1N}}{y} \right) \text{Tr} (y^{E_{Ns}} (D_0 - 1) P_{Ns}(x)) \right] \\ & + \sum_{i=2}^N \mathcal{D}_i A_i^{(1)}(x, y) + \sum_{i=1}^N \tilde{\mathcal{D}}_i A_i^{(2)}(x, y) \end{aligned} \quad (5.13)$$

This equation can be solved in the same way as the corresponding equation in Section 3 (3.21) was solved, by integration from 1 to y . This expression for the trace is then integrated over

$$\oint \frac{dx_1}{4\pi i x_1} \prod_{i=2}^N \int_0^1 \frac{dx_i}{x_i} \equiv \int \frac{dx}{x} \quad (5.14)$$

The last two sums will now integrate to zero. The first sum is trivially integrated to zero because $A_i^{(1)}$ is directly seen to be zero at $x_i = 0$ and 1 and the x_i integration is closed. To see that $A_i^{(2)}$ is zero at $x_i = 1$ we have to use an argument common for many calculations in the Neveu-Schwarz model. When $x_i = 1$ we have an expression with two vertices together and this can be found to be zero.

We then finally perform the y integration and integrate by parts using (3.20) and the corresponding equation for g to obtain that:

$$\oint \frac{dy}{4\pi i y} \int \frac{dx}{x} \text{Tr} (y^{E_{NS}} P_{NS}(x)) = \int \frac{dx}{x} f^{-2}(x_{1N}) g^{-2}(x_{1N}) \text{Tr} (P_{NS}(x)) \quad (5.15)$$

Thus, also in this case the resultant effect is to multiply the trace without a projector with two inverse powers of the corresponding partition function for the system.

In order to construct the single planar loop in this model we use the Feynman tree theorem exactly as for the ordinary dual model. We use the formula corresponding to (2.8) but this time V_i means the Neveu-Schwarz vertex and

$$\Delta_i(q_i) = \begin{cases} -\frac{1}{2} \int_0^1 \frac{dx_i}{x_i} x_i^{L_0-1/2} & \text{if line } i \text{ is uncut} \\ \sum_{n_i=0}^{\infty} [2\pi i \theta(q_i^0) \delta(q_i^2 - 2n_i - 1)] \left[\oint \frac{dx_i}{4\pi i x_i} x_i^{H-n_i} \right] \tilde{J}_{NS} & (5.16) \end{cases}$$

Now we can redo all the computations of Section 4 with (5.15) and its generalization to traces with arbitrary number of cut lines (which also in this case is only reflected in the integration contours) instead of (4.3). We will then end up with the result for the single planar loop:

$$\int \frac{i d^4 k}{(2\pi)^4} \prod_{i=1}^N \left[-\frac{1}{2} \int_0^1 \frac{dx_i}{x_i} \right] \prod_{n=1}^{\infty} (1-x_{1N}^n)^2 (1+x_{1N}^{n-\frac{1}{2}})^{-2} ,$$

(5.17)

$$\text{Tr} \left[x_1^{L_0-1/2} V_1 x_2^{L_0-1/2} V_2 \dots x_N^{L_0-1/2} V_N \right]$$

This result is, as the result (4.6), divergent so we have to investigate this divergence more carefully in order to find a renormalization counterterm. We see that the divergence enters in the same way as for the result (4.6) so finding the way to renormalize one model could probably be used to renormalize the other too.

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- 12) We call these functions partition functions since for the dual resonance
model this function is just the mathematical partition function. When
taken to the power $d-2$ it can also be related to the physical parti-
tion function for the system, if the product of Chan variables $x_1 \cdot x_2 \dots x_N$
(N the number of internal lines) is changed into e^{-1/k_T} . For the
Neveu-Schwarz model the past point is also true. This is the reason
for calling the corresponding function, somewhat carelessly, a parti-
tion function.
- 13) J. Scherk, private communication. (Though we understand that there is
some recent progress).

- 14) In our previous paper we defined the projection operator by

$$\int \frac{dy}{2\pi i y} y^{2E_N s}$$

which is equivalent to the definition of this paper.

- 15) We work in the so-called $\mathcal{H}_2$ space [see the second paper of Ref. 9].