BOUNDS, RESULTING FROM UNITARITY AND ANALYTICITY,
ON FORWARD SLOPES OF OVERLAP FUNCTIONS

Z. Ajduk ^{*)}
CERN - Geneva

A B S T R A C T

Lower and upper bounds on forward slopes of overlap functions are obtained with the help of unitarity conditions, the polynomial boundedness and analyticity of elastic amplitudes. The results are compared with values of slopes obtained in Regge pole models.

---

^{*)} On leave of absence from the Institute of Theoretical Physics, University of Warsaw, Warsaw.

1. INTRODUCTION

The overlap function $F(s, t)$ for a reaction with unpolarized initial particles is at small $|t|$ equal to

$$F(s, t) = \sigma(s) \left[1 + \frac{1}{4} \langle b^2 \rangle t + \dots \right], \quad (1)$$

where $\sigma(s) = F(s, t=0)$ and the slope $\frac{1}{4} \langle b^2 \rangle = d \ln F / dt(s, t=0)$ give, respectively, the cross-section $\sigma(s)$ and the expectation value $\langle b^2 \rangle$ of the squared impact parameter, both of them corresponding to the considered transition. The impact parameter squared is by definition

$$b^2 = \frac{\vec{J}_\perp^2}{p^2}, \quad (2)$$

where in the c.m.s. of the initial particles, $\vec{J}_\perp$ is the component of their total angular momentum perpendicular to the momentum $\vec{p}$ of one of them. The quantity $\langle b^2 \rangle$ is a measure of the reaction peripherality (or centrality) and its knowledge, besides the cross-section, is very interesting from the physical point of view.

In this paper we derive lower (Section 2) and upper (Section 3) bounds on $\langle b^2 \rangle$ in the high energy region. They follow from the general optical theorem (unitarity conditions for elastic reactions) as well as from the polynomial bounds of the asymptotic growth and the analyticity (in the Martin-Lehmann ellipse) of elastic amplitudes. Analogous results for cross-sections are commonly known⁵⁾. In the calculations we use the impact parameter approximation, which allows to replace the summation over discrete values of the total angular momentum by the integration over them as it is justified at high energies. For simplicity we consider in the following the reactions between spinless particles. It is easy to show that the obtained results remain true also for any reaction with unpolarized initial particles.

2. LOWER BOUNDS

First let us calculate the lower bounds on $\langle b^2 \rangle$ for total, elastic and inelastic transitions. Our assumptions are

1) the values of the total cross-section

$$\sigma_{\text{tot}}(s) = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l+1) \text{Im} f_l(s) \quad (3)$$

and of the elastic cross-section

$$\sigma_{\text{el}}(s) = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l+1) |f_l(s)|^2 \quad (4)$$

are fixed;

2) an inequality constraint, which follows from the general optical theorem,

$$\text{Im} f_l(s) - |f_l(s)|^2 \geq 0 \quad (5)$$

is satisfied.

Using the standard method of Lagrange multipliers, generalized ⁶⁾ to include inequality constraints, we look for such values of f_l , which minimize successively the following quantities:

$$\sigma_{\text{tot}}(s) p^2 \langle b^2 \rangle_{\text{tot}} = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l+1) l(l+1) \text{Im} f_l(s), \quad (6a)$$

$$\sigma_{\text{el}}(s) p^2 \langle b^2 \rangle_{\text{el}} = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l+1) l(l+1) |f_l(s)|^2, \quad (6b)$$

$$\sigma_{\text{inel}}(s) p^2 \langle b^2 \rangle_{\text{inel}} = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l+1) l(l+1) [\text{Im} f_l(s) - |f_l(s)|^2]. \quad (6c)$$

The results are the following.

a) The lower bound on $\langle b^2 \rangle_{\text{tot}}$ is given by

$$\langle b^2 \rangle_{\text{tot}}^{\text{lb}} = \begin{cases} \frac{1}{9\pi} \frac{\sigma_{\text{tot}}^2}{\sigma_{\text{el}}} & \text{for } \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \leq \frac{2}{3}, \\ \frac{\sigma_{\text{tot}}}{8\pi} [1 + 3(1 - \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}})^2] & \text{for } \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \geq \frac{2}{3}. \end{cases} \quad (7a)$$

It is reached if

$$\operatorname{Re} f_l = 0 \quad (8a)$$

and

1) for $\sigma_{el}/\sigma_{tot} \leq 2/3$, if

$$\operatorname{Im} f_l = \begin{cases} \alpha [1 - \beta l(l+1)] & l(l+1) \leq \frac{1}{\beta} \\ 0 & l(l+1) \geq \frac{1}{\beta} \end{cases}, \quad (8a')$$

where

$$\alpha = \frac{3\sigma_{el}}{2\sigma_{tot}},$$

$$\beta = \frac{3\pi\sigma_{el}}{p^2\sigma_{tot}^2},$$

2) for $\sigma_{el}/\sigma_{tot} \geq 2/3$, if

$$\operatorname{Im} f_l = \begin{cases} 1 & l(l+1) \leq \frac{1-\alpha}{\beta} \\ \frac{1}{2} [1 - \beta l(l+1)] & \frac{1-\alpha}{\beta} \leq l(l+1) \leq \frac{1}{\beta} \\ 0 & l(l+1) \geq \frac{1}{\beta} \end{cases}, \quad (8a'')$$

where

$$\alpha = \frac{6(\sigma_{tot} - \sigma_{el})}{4\sigma_{tot} - 3\sigma_{el}},$$

$$\beta = \frac{4\pi}{p^2(4\sigma_{tot} - 3\sigma_{el})}.$$

The result is identical with that ⁷⁾ of the forward slope of $\operatorname{Im} f(s, t)$ where $f(s, t)$ is the elastic amplitude, because, according to the general optical theorem, the total overlap function is equal to $(4\pi/p)\operatorname{Im} f(s, t)$.

b) The lower bound on $\langle b^2 \rangle_{el}$ is given by

$$\langle b^2 \rangle_{el}^{lb} = \frac{\sigma_{el}}{8\pi} \quad (7b)$$

and it corresponds, in the limit $k \rightarrow \infty$, to the following values of f_l :

$$\text{Re } f_l = 0, \quad (8b)$$

$$\text{Im } f_l = \begin{cases} 1 & l(l+1) \leq \frac{1+\gamma}{\beta} \\ \gamma [l(l+1)\beta - 1]^{-1} & \frac{1+\gamma}{\beta} \leq l(l+1) \leq \frac{1+k\gamma}{\beta} \\ 0 & l(l+1) \geq \frac{1+k\gamma}{\beta} \end{cases} \quad (8b')$$

where

$$\gamma = \frac{\sigma_{tot} - \sigma_{el}}{(1 + \ln k)\sigma_{el} - (2 - \frac{1}{k})\sigma_{tot}},$$

$$\beta = \frac{4\pi (\ln k + \frac{1}{k} - 1)}{p^2 [(1 + \ln k)\sigma_{el} - (2 - \frac{1}{k})\sigma_{tot}]}.$$

c) The lower bound on $\langle b^2 \rangle_{inel}$ is given by the formula

$$\langle b^2 \rangle_{inel}^{lb} = \begin{cases} \frac{\sigma_{tot}}{4\pi} \frac{1 - 4\alpha + \alpha^2 (3 - 2 \ln \alpha)}{(1 - \alpha)^2 (1 - \alpha + \alpha \ln \alpha)} & \text{for } \frac{\sigma_{el}}{\sigma_{tot}} \leq \frac{1}{2} \\ \frac{\sigma_{tot} - \sigma_{el}}{2\pi} & \text{for } \frac{\sigma_{el}}{\sigma_{tot}} \geq \frac{1}{2} \end{cases} \quad (7c)$$

It is reached, if

$$\text{Re } f_l = 0 \quad (8c)$$

and

1) for $\sigma_{el}/\sigma_{tot} \leq \frac{1}{2}$, if

$$\text{Im } f_l = \begin{cases} \frac{1}{2} \left[1 + \frac{\alpha}{\beta(l+1)l-1} \right] & l(l+1) \leq \frac{1-\alpha}{\beta} \\ 0 & l(l+1) \geq \frac{1-\alpha}{\beta} \end{cases} \quad (8c')$$

where the quantity α is the solution of the algebraic equation

$$\frac{\sigma_{el}}{\sigma_{tot}} = \frac{1}{2} \frac{1 + 2\alpha \ln \alpha - \alpha^2}{1 - \alpha + \alpha \ln \alpha}$$

and

$$\beta = \frac{2\pi}{p^2 \sigma_{tot}} (1 - \alpha + \alpha \ln \alpha),$$

2) for $\sigma_{el}/\sigma_{tot} \geq \frac{1}{2}$, if

$$\text{Im } f_l = \begin{cases} \frac{1}{2} & l(l+1) \leq \frac{1}{\beta} \\ 1 & \frac{1}{\beta} \leq l(l+1) \leq \frac{1+\gamma}{\beta} \\ 0 & l(l+1) \geq \frac{1+\gamma}{\beta} \end{cases}, \quad (8c'')$$

where

$$\gamma = \frac{1}{4} \frac{2\sigma_{el} - \sigma_{tot}}{\sigma_{tot} - \sigma_{el}},$$

$$\beta = \frac{\pi}{p^2 (\sigma_{tot} - \sigma_{el})}.$$

The real parameters α , β , γ , k in Eqs. (7), (8) are dimensionless and satisfy the inequalities: $0 \leq \alpha \leq 1$, $\beta \geq 0$, $\gamma \geq 0$, $k \geq 1$. Each bound from those given above can be written in the form

$$\langle b^2 \rangle_{(\dots)}^{lb} = \frac{\sigma_{tot}}{4\pi} g_{(\dots)}\left(\frac{\sigma_{el}}{\sigma_{tot}}\right), \quad (9)$$

where (...) $\equiv$ tot, el or inel. The graphs of functions $g(\dots)(\sigma_{el}/\sigma_{tot})$ for total, elastic and inelastic transitions are drawn in the Figure.

Knowing σ_{el} and σ_{tot} , one can determine directly from these graphs the lower bounds $\langle b^2 \rangle_{lb}$ for the considered reactions.

Next, let us find a lower bound on $\langle b^2 \rangle_r$ for an inelastic reaction r at:

- 1) fixed values of the total cross-section $\sigma_{tot}(s)$ and of the cross-section $\sigma_r(s)$ for the considered inelastic reaction

$$\sigma_r(s) = \frac{4\pi}{p^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{i \in r} |f_{\ell}^{(i)}(s)|^2, \quad (10)$$

- 2) an inequality constraint, following from the general optical theorem,

$$\text{Im} f_{\ell} - |f_{\ell}|^2 - \sum_{i \in r} |f_{\ell}^{(i)}|^2 \geq 0. \quad (11)$$

It appears that the result is identical to the case c) considered above after two formal substitutions: $\sigma_{tot} - \sigma_r \rightarrow \sigma_{el}$ and $\sum_{i \in r} |f_{\ell}^{(i)}|^2 \rightarrow \text{Im} f_{\ell} - |f_{\ell}|^2$.

To determine exact values of $\langle b^2 \rangle$ one needs to know the reaction amplitudes, including their spin structure and phases ⁴⁾. Using Regge pole exchange models ⁸⁾⁻¹²⁾, we have calculated ¹³⁾ $\langle b^2 \rangle$ for total, elastic and inelastic transitions in π^+p , K^+p , $\bar{K}^-p$, pp , $\bar{p}p$ collisions and $\langle b^2 \rangle_r$ for some two-body inelastic processes in π^+p interactions. Tables 1 and 2 give the comparison between the exact values $\langle b^2 \rangle$ and $\langle b^2 \rangle_{lb}$, at $s=10, 30, 50$ (GeV/c)². They show the lower bounds $\langle b^2 \rangle_{lb}$ to have the magnitude order of $\langle b^2 \rangle$ for total and inelastic interactions but to be one or more magnitude orders smaller for the elastic or the considered inelastic reactions. In the latter cases it is possible to obtain another lower bound on $\langle b^2 \rangle$ by taking into account only the c.m. differential cross-sections $d\sigma/d\Omega(p, \theta)$ for the considered reactions. Denoting the lower bound by $\langle b^2 \rangle'$, we get then ⁴⁾

$$\langle b^2 \rangle' = \frac{\int_{-1}^1 d \cos \theta \frac{d\sigma}{d\Omega} \left[\frac{\partial}{\partial \theta} \left(\ln \frac{d\sigma}{d\Omega} \right) \right]^2}{4p^2 \int_{-1}^1 d \cos \theta \frac{d\sigma}{d\Omega}}. \quad (12)$$

The values of $\langle b^2 \rangle'$, as we can see from Tables 1 and 2, are much larger than $\langle b^2 \rangle^{lb}$ and, therefore, they give a stronger restriction on the reaction centrality.

3. UPPER BOUNDS

The analyticity in the Martin-Lehmann ellipse and the polynomial bounds on the asymptotic growth of the elastic amplitude imply that at high energy the partial elastic amplitudes $f_\ell(s)$ decrease with ℓ at least exponentially. This fact allows to take into consideration only the first $(L+1)$ partial amplitudes with $L = C \sqrt{s} \ln(s/s_0)$ (s_0 determines an energy scale and the constant C satisfies the condition $C \leq 1/4m_\pi$, m_π - pion mass). On the other hand, the general optical theorem imposes essential inequality constraints on elastic partial amplitudes $f_\ell(s)$ and on the sum over the squared moduli of inelastic partial amplitudes $\sum_{i \in r} |f_\ell^{(i)}(s)|^2$:

$$0 \leq |f_\ell|^2 \leq \text{Im} f_\ell \leq 1,$$

$$0 \leq \sum_{i \in r} |f_\ell^{(i)}|^2 \leq \text{Im} f_\ell - |f_\ell|^2 \leq \frac{1}{4}. \quad (13)$$

The above facts allow to determine upper bounds on $\langle b^2 \rangle$ for a total, elastic and any inelastic transition.

a) Upper bounds on $\langle b^2 \rangle_{\text{tot}}$

Starting from the definition (6a) we obtain successively ($p^2 \approx s/4$):

$$\begin{aligned} \sigma_{\text{tot}} \langle b^2 \rangle_{\text{tot}} &\approx \frac{64\pi}{s^2} \sum_{\ell=0}^L (2\ell+1) \ell(\ell+1) \text{Im} f_\ell \\ &\leq \frac{64\pi}{s^2} \sum_{\ell=0}^L (2\ell+1) \ell(\ell+1) \\ &\approx \frac{32\pi}{s^2} L^4 = 32\pi C^4 \ln^4\left(\frac{s}{s_0}\right) \end{aligned}$$

and finally

$$\langle b^2 \rangle_{\text{tot}}^{ub} = 32\pi \frac{C^4}{\sigma_{\text{tot}}} \ln^4\left(\frac{s}{s_0}\right). \quad (14a)$$

A different upper bound

$$\langle b^2 \rangle_{\text{tot}}^{\text{ub}} = 16 C^3 \sqrt{\frac{\pi \sigma_{\text{el}}}{3}} \frac{\ln^3 \left(\frac{s}{s_0} \right)}{\sigma_{\text{tot}}}, \quad (14a')$$

we can get, using the Schwarz inequality:

$$\begin{aligned} (\sigma_{\text{tot}} \langle b^2 \rangle_{\text{tot}})^2 &\approx \left[\frac{64\pi}{s^2} \sum_{l=0}^L (2l+1) l(l+1) \text{Im} f_l \right]^2 \\ &\leq \left[\frac{64\pi}{s^2} \sum_{l=0}^L (2l+1) \text{Im}^2 f_l \right] \left[\frac{64\pi}{s^2} \sum_{l=0}^L (2l+1) l^2 (l+1)^2 \right] \\ &\leq \frac{256\pi}{s^3} \sigma_{\text{el}} \frac{1}{3} L^6 = \frac{256}{3} \pi C^6 \sigma_{\text{el}} \ln^6 \left(\frac{s}{s_0} \right). \end{aligned}$$

In the case of $\sigma_{\text{el}} < \frac{3}{4} \sigma_{\text{tot}}$ the upper bound (14a') is smaller than (14a) because of the Froissart bound $\sigma_{\text{tot}} \leq 16\pi C^2 \ln^2(s/s_0)$, derived ⁵⁾ at assumptions identical to those used here.

b) Upper bounds on $\langle b^2 \rangle_{\text{el}}$

From the definition

$$\langle b^2 \rangle_{\text{tot}} = \frac{\sigma_{\text{el}} \langle b^2 \rangle_{\text{el}} + \sigma_{\text{inel}} \langle b^2 \rangle_{\text{inel}}}{\sigma_{\text{tot}}} \quad (15)$$

it follows

$$\langle b^2 \rangle_{\text{el}} \leq \frac{\sigma_{\text{tot}}}{\sigma_{\text{el}}} \langle b^2 \rangle_{\text{tot}} \quad (15')$$

and, therefore, using (14a), (14a'), we obtain

$$\langle b^2 \rangle_{\text{el}}^{\text{ub}} = 32 \pi \frac{C^4}{\sigma_{\text{el}}} \ln^4 \left(\frac{s}{s_0} \right), \quad (14b)$$

$$\langle b^2 \rangle_{\text{el}}^{\text{ub}} = 16 C^3 \sqrt{\frac{\pi}{3 \sigma_{\text{el}}}} \ln^3 \left(\frac{s}{s_0} \right). \quad (14b')$$

c) Upper bounds on $\langle b^2 \rangle_r$, r any inelastic reaction

Starting from the definition (6c) one can get analogously as in

a) the following upper bounds $\langle b^2 \rangle_r^{\text{ub}}$:

$$\langle b^2 \rangle_\pi^{ub} = 8\pi \frac{C^4}{\sigma_\pi} \ln^4 \left(\frac{s}{s_0} \right), \quad (14c)$$

$$\langle b^2 \rangle_\pi^{ub} = 8C^3 \sqrt{\frac{\pi}{3\sigma_\pi}} \ln^3 \left(\frac{s}{s_0} \right). \quad (14c')$$

The formula

$$\langle b^2 \rangle_\pi \leq \frac{\sigma_{tot}}{\sigma_\pi} \langle b^2 \rangle_{tot}$$

analogous to Eq. (15'), is also true and it gives with the help of (14a')

$$\langle b^2 \rangle_\pi^{ub} = 16 \frac{C^3}{\sigma_\pi} \sqrt{\frac{\pi \sigma_{tot}}{3}} \ln^3 \left(\frac{s}{s_0} \right). \quad (14c'')$$

Using stronger assumptions about the reaction amplitudes we can get smaller upper bounds. For example, in a Regge pole exchange model for two-body reactions at very high energy we get ⁴⁾

$$\langle b^2 \rangle = 2\alpha' \ln \left(\frac{s}{s_0} \right), \quad (16)$$

where α' is the slope of the dominant Regge trajectory.

The lower and upper bounds obtained in this paper have the advantage that they follow directly from some results derived in the axiomatic quantum fields theory.

I thank Professors L. Van Hove, A. Martin and S. Pokorski for a critical reading of the manuscript and the CERN Theoretical Study Division for its hospitality.

TABLE 1

s [GeV/c] ²	$\langle b^2 \rangle_{\text{tot}}^{\ell b}$ [fm ²]	$\langle b^2 \rangle_{\text{tot}}$ [fm ²]	$\langle b^2 \rangle_{\text{el}}^{\ell b}$ [fm ²]	$\langle b^2 \rangle_{\text{el}}'$ [fm ²]	$\langle b^2 \rangle_{\text{el}}$ [fm ²]	$\langle b^2 \rangle_{\text{inel}}^{\ell b}$ [fm ²]	$\langle b^2 \rangle_{\text{inel}}$ [fm ²]
$\pi^+ p$ collisions ⁸⁾							
10	0.44	0.61	0.022	0.24	0.27	0.47	0.71
30	0.54	0.70	0.015	0.29	0.30	0.56	0.78
50	0.56	0.74	0.014	0.30	0.31	0.59	0.81
$\pi^- p$ collisions ⁸⁾							
10	0.53	0.71	0.023	0.28	0.30	0.55	0.81
30	0.58	0.76	0.016	0.30	0.31	0.61	0.84
50	0.61	0.78	0.014	0.32	0.32	0.63	0.85
$K^+ p$ collisions ⁹⁾							
10	0.26	0.25	0.015	0.16	0.18	0.27	0.27
30	0.37	0.42	0.011	0.22	0.23	0.38	0.45
50	0.41	0.47	0.010	0.24	0.24	0.43	0.50
$\bar{K}^- p$ collisions ⁹⁾							
10	0.57	0.70	0.015	0.30	0.32	0.60	0.77
30	0.54	0.69	0.011	0.28	0.29	0.56	0.75
50	0.54	0.67	0.010	0.28	0.29	0.56	0.73
pp collisions ¹⁰⁾							
10	0.42	0.84	0.058	0.28	0.52	0.46	1.01
30	0.56	0.97	0.039	0.35	0.60	0.60	1.24
50	0.59	1.09	0.035	0.37	0.63	0.62	1.22
$\bar{p} p$ collisions ¹⁰⁾							
10	0.77	1.40	0.070	0.49	0.98	0.83	1.56
30	0.78	1.38	0.044	0.50	0.89	0.83	1.52
50	0.75	1.28	0.039	0.49	0.84	0.80	1.40

TABLE 2

s [GeV/c] ²	$\langle b^2 \rangle_r^{\ell b}$ [fm ²]	$\langle b^2 \rangle_r^{\ell}$ [fm ²]	$\langle b^2 \rangle_r$ [fm ²]	$\langle b^2 \rangle_r^{\ell b}$ [fm ²]	$\langle b^2 \rangle_r^{\ell}$ [fm ²]	$\langle b^2 \rangle_r$ [fm ²]
	$\pi^- p \rightarrow \pi^0 n$ reaction 8)			$\pi^- p \rightarrow \eta^0 n$ reaction 11)		
10	0.0020	0.29	0.98	0.0108	0.38	1.90
30	0.0005	0.38	1.16	0.0010	0.58	2.12
50	0.0003	0.41	1.22	0.0003	0.67	1.94
	$\pi^+ p \rightarrow \eta^+ p$ reaction 11)			$\pi^+ p \rightarrow \pi^0 \Delta^{++}$ reaction 12)		
10	0.0082	0.29	1.40	0.0035	0.36	1.47
30	0.0007	0.48	1.76	0.0009	0.43	1.78
50	0.0003	0.51	1.64	0.0005	0.50	1.73

R E F E R E N C E S

- 1) L. Van Hove, Nuovo Cimento 28, 798 (1963).
- 2) L. Van Hove, Revs.Modern Phys. 36, 655 (1964).
- 3) Z. Koba and M. Namiki, Nuclear Phys. B8, 413 (1968).
- 4) Z. Ajduk, Warsaw University preprint IFT/72/4 (1972).
- 5) R.J. Eden, Revs.Modern Phys. 43, 15 (1971).
- 6) M.B. Einhorn and R. Blankenbecler, Ann.Phys. 67, 480 (1971).
- 7) S.W. MacDowell and A. Martin, Phys.Rev. 135B, 960 (1964).
- 8) V. Barger and R.J.N. Phillips, Phys.Rev. 187, 2210 (1969).
- 9) G.V. Dass, C. Michael and R.J.N. Phillips, Nuclear Phys. B9, 549 (1969),
erratum B14, 474 (1969).
- 10) W. Rarita, R.J. Riddell, C.B. Chiu and R.J.N. Phillips, Phys.Rev. 165,
1615 (1968).
- 11) G.V. Dass and C.D. Froggatt, Nuclear Phys. B8, 661 (1968).
- 12) M. Krammer and U. Maor, Nuclear Phys. B13, 477 (1969).
- 13) Z. Ajduk, Warsaw University preprint IFT/72/17 (1972).

FIGURE CAPTION

Lower bounds on $\langle b^2 \rangle$ for total, elastic
and inelastic transitions as functions of
 σ_{tot} and $\sigma_{\text{el}}/\sigma_{\text{tot}}$.