CM-P00058491

Ref.TH.1577-CERN

Archives

CRITICAL QUESTIONS CONCERNING A STATISTICAL APPROACH

TO THE HADRON MASS SPECTRUM

A. Bassetto *)

CERN - Geneva

and

L. Sertorio

INFN, Sezione di Torino

and

CERN - Geneva

A B S T R A C T

We study the problem of the hadron mass spectrum starting from an S matrix which is explicitly considered as a unitary operator. Making use of the additional property that the connected part T_c is a compact operator, we discuss the approximation of T_c by resonances. The connection with statistics is made by using the Bernstein-Dashen-Ma formalism together with a bootstrap assumption. Although our approach remains open, it allows a critical discussion of subtle issues concerning both the Hagedorn and the dual theories.

*) Address after 1 November 1972 : Istituto di Fisica Galilei, Padova.

1. - INTRODUCTION

The theoretical formulation of a new hadron thermodynamics developed by Hagedorn ¹⁾ has in our opinion two branches which are weakly dependent and their connection is not yet completed. They are first the construction of a self-consistent partition function (and following thermodynamical properties) and second the hadron hydrodynamical treatment of the collision processes.

The relationship between the two branches is well illustrated if we consider the recent developments due to Frautschi ²⁾ who has modified the hadronic partition function starting from a reanalysis of the self-consistency equation that relates phase-space and mass spectrum. The dynamical properties of the fireball modified by Frautschi, when introduced into the Hagedorn scheme for the collision processes, had to fit logically with the treatment of scattering processes. Actually they did and improved additionally the agreement with experiments ³⁾.

In the above approaches the statistical bootstrap is considered in a fixed volume at rest. The actual scattering processes are successively constructed with an additional model which is independent, but coherent with the thermodynamical properties of the system at rest previously found.

On the other hand, if the hadronic S matrix were known, the corresponding partition function could be evaluated by the general procedure proposed by Bernstein, Dashen and Ma (BDM) ⁴⁾. In our work we accept that the BDM formula for expressing the grand potential in terms of the S matrix has general validity, although it has been obtained in the framework of a special model. We accept in particular that it is valid for treating a hadron gas.

In principle the dual model is a candidate for the input of the BDM formula. Veneziano indeed inserted the dual model matrix elements into the BDM formula obtaining a partition function which exhibits a limiting temperature ⁵⁾. He showed that the counting resulting from the BDM formula is proportional to the free resonance spectrum of the dual model provided that spurious states have been suppressed. At the present time, however, the removal of the spurious states is only reached in models which possess unphysical features ⁶⁾. The unitarization program, although very fascinating, still seems to us a formidable problem.

The motivation of our work is to explore several deep questions which arise when use of the BDM theory is made starting from general properties of the operator S , in particular taking into account from the very beginning that it is unitary. In this way we have no model to start with. We proceed in two steps. First we define "eigenstates" of the operator S at a given energy and angular momentum ; this is discussed in Section 2. Clearly no volume appears there. It appears instead a rather deep substitute, namely the concept of minimum norm of the part of the operator S which cannot be expressed through resonances. Secondly we make use of the dynamical information contained in the couplings of these states. These couplings are the partial decay widths (Sections 3 and 4).

At this point the theoretical ingredients necessary to formulate a bootstrap are clearly defined : "the counting of the thermodynamical mass spectrum coïncides asymptotically with the counting of the S matrix resonances". This assumption finds its mathematical realization into two coupled integral equations [Eqs. (3.14) and (4.3)], having as unknowns the mass spectrum $\rho(m,J)$ and the total width $\Gamma(m,J)$. This is not a complete bootstrap since the partial widths enter as driving terms ; however, we believe that this is a good feature of our approach. In fact, if we are able to derive expressions for the partial widths from theoretical models or from phenomenology, we shall succeed in closing the bootstrap. Moreover the link between the S matrix phenomenology and hadron thermodynamics is established a priori.

2. - S MATRIX STATES AND S MATRIX RESONANT STATES

We wish to discuss how the S matrix is susceptible of being expressed in terms of states and how such states can possibly enter to contribute to a partition function.

We start from general assumptions on the S matrix :

- (i) connectedness structure,
- (ii) relativistic invariance,
- (iii) some general requirements of analyticity.

If we wish to consider states of definite energy w and angular momentum J intermediate between initial and final scattering states, we have to consider the connected part of the S matrix, T_c . Moreover we have in mind to link such abstract states to resonances which only are present in connected graphs.

The first question is if the operator $T_c(w, J)$ at fixed energy and angular momentum can be exactly written as a sum of a finite number of states. The answer is negative because the operator $T_c(w, J)$ is not a finite rank operator. The chances are :

- (i) T_c is an operator enjoying the properties of a generalized kernel of Fredholm type, namely having an infinite discrete spectrum of eigenvalues ;
- (ii) T_c is an operator with a continuum spectrum for which no discrete counting of eigenvalues is possible ;
- (iii) T_c has both discrete and continuum spectrum of eigenvalues.

In Ref. 7), it is suggested that the T_c matrix belongs to class (i). In fact an incomplete proof is given in favour of the statement that T_c is a compact operator, thus enjoying the properties under (i). The difficulty lies in the existence of cluster diagrams of the kind shown in Fig. 1. Such diagrams of T_c have a pole singularity in the physical region of the initial and final momenta. Consequently such amplitudes have the Fredholm norm which diverges. In spite of that such diagrams still correspond to compact kernels and consequently are susceptible of being expressed by a set of eigenstate contributions plus a rest of fixed norm arbitrarily small.

It is important to notice that in the proof of Ref. 7) the existence of a minimum finite mass affecting the asymptotic scattering states (the pion mass) is crucial. Consequently the compactness of T_c is deeply related to the possibility of defining a finite Fermi volume. If we assume that the partial results of Ref. 7) have general validity we can say that it is possible to express the connected part of the S matrix in an exact way as the sum of infinite numerable contributions at fixed total energy.

Reference 7) allows the definition of "states" according to the following formula :

$$[T_c(w, J)]_{\alpha\beta} = \sum_{i=1}^{\infty} V_{\alpha i}(w, J) \mu_i(w, J) \overline{U_{i\beta}(w, J)} \quad (2.1)$$

The quantities μ_i are generalized eigenphases ; from general theorems for compact operators follows :

- (i) the μ_i are positive ;
- (ii) they have only one accumulation point at the value $\mu = 0$;
- (iii) the vectors U_i and V_i belong to two orthonormal sets of vectors.

Moreover, the quantities μ_i have the following property :

$$\sum_{i=1}^{\infty} [\mu_i(w, J)]^2 = \|T_c(w, J)\|^2 = \infty \quad (2.2)$$

as a consequence of the appearance of poles in the physical region (Landau singularities).

Since T_c is a compact operator, it can be approximated by finite rank operators in norm ; therefore for a given energy w and a positive number μ , there is a non-negative number $N(w, \mu)$, which is the smallest rank of a finite rank operator A satisfying the following equation *) :

$$\|T_c - A\| \leq \mu \quad (2.3)$$

In Ref. 7), one shows that $N(w, \mu)$ is also the number of eigenphases obeying the following inequality

$$\mu_i > \mu \quad (2.4)$$

Finally we specify the concept of an approximating operator : if the compact operator $\hat{T}$ approximates T_c in the following sense

$$\|T_c - \hat{T}\| \leq \epsilon \quad (2.5)$$

*) Here and in the following sometimes we leave understood indices or variables when they are not explicitly needed.

then we have

$$|\mu_i - \hat{\mu}_i| \leq \epsilon \quad (2.6)$$

and

$$\hat{N}(\mu + \epsilon) \leq N(\mu) \leq \hat{N}(\mu - \epsilon) \quad (2.7)$$

It is worth remembering that, if $\hat{T}$ is compact, but not of finite rank, we have

$$\hat{N}(0) = \infty \quad (2.8)$$

We now discuss the assumptions we need for a statistical treatment of resonances ; we start by defining the concept of relevant eigenphases and relevant resonances. We first assume that

$$T_c = A + \hat{T} \quad (2.9)$$

with

$$\left\{ \begin{array}{l} \text{rank } \hat{T} = 1 \\ \|A\| \leq \epsilon \end{array} \right. \quad (2.10)$$

Then, from Eqs. (2.5)-(2.7), follows

$$\left\{ \begin{array}{l} \hat{N}(\mu + \epsilon) \leq N(\mu) \leq \hat{N}(\mu - \epsilon) , \\ N = 1 , \quad |\mu_1 - \hat{\mu}_1| \leq \epsilon , \\ \mu_i \leq \epsilon \quad i = 2, 3, \dots \end{array} \right. \quad (2.11)$$

Therefore in this case there is only one relevant eigenphase that approximates T_c with an error smaller than the norm of A .

To put in correspondence the rank one contribution with one resonance contribution requires some specification. We express the T_c matrix in the following way :

$$T_c = B + R \quad (2.12)$$

and assume that

$$R = Q |c\rangle \langle c| [w - w_R + i\Gamma]^{-1} 2i\Gamma, \quad (2.13)$$

where Q is a unitary operator slowly varying with energy in the interval $|w - w_R| \leq \Gamma$. Then the following restriction is obtained from Eq. (2.11):

$$\mu_1 - \epsilon \leq 2\Gamma [(w - w_R)^2 + \Gamma^2]^{-1/2} \leq \mu_1 + \epsilon. \quad (2.14)$$

As a second step, we assume to know there are n resonances in a small interval of energy, at the positions

$$(w_1, \Gamma_1), \dots, (w_n, \Gamma_n) \quad (2.15)$$

and residues specified by the vectors $|c_1\rangle, \dots, |c_n\rangle$. We put again

$$\begin{cases} T_c = B + R, \\ \|B\| \leq \epsilon, \\ R = Q \sum_{j=1}^n |c_j\rangle \langle c_j| [w - w_j + i\Gamma_j]^{-1} 2i\Gamma_j. \end{cases} \quad (2.16)$$

The rank $\hat{N}$ of R is less or equal to n depending on whether the vectors $|c_j\rangle$ are linear dependent or independent. Therefore, if we want to repeat in this case the conclusions reached before, namely that

$$\mu_j - \epsilon \leq 2\Gamma_j [(w - w_j)^2 + \Gamma_j^2]^{-1/2} \leq \mu_j + \epsilon, \quad (2.17)$$

we can require the orthogonality condition $*) \quad j = 1, \dots, n$,

$$\langle c_i | c_j \rangle = \delta_{ij} \quad i, j = 1, 2, \dots, n. \quad (2.18)$$

This condition guarantees that the relevant resonances are not mutually coupled.

*) This condition can be imposed since the resonances we consider lie in a small energy interval where their vector states can be considered energy independent.

Finally we consider the case in which the number of resonances is infinite without accumulation at a finite value of the energy. We want to extend the previous considerations to this case. We first define as relevant eigenphases at a given energy those which give a contribution that approximate T_c with an error in norm less than ϵ , namely :

$$\begin{cases} \|T_c - A\| \leq \epsilon \\ A = \sum_{k=1}^{N(\epsilon)} V_k \mu_k \overline{U_k} \end{cases} \quad (2.19)$$

Remark that the possibility of approximating T_c with a finite number of eigenphases is consequence of the compactness of T_c and that the definition itself of relevant eigenphases depends on the error ϵ we consider.

At this point we make the following assumption : at any given energy the operator T_c can be expressed as

$$T_c = R + Q \quad (2.20)$$

with

$$R = Q \sum_{\{D(w, J, \epsilon)\}} \frac{|c_j\rangle \langle c_j|}{w - w_j + i\Gamma_j} 2i\Gamma_j, \quad (2.21)$$

$$\|Q\| \leq \epsilon \quad (2.22)$$

The set $\{D(w, J, \epsilon)\}$ contains those resonances which obey the condition

$$2\Gamma_j \left[(w - w_j)^2 + \Gamma_j^2 \right]^{-1/2} \geq \epsilon \quad (2.23)$$

Remember that for any given w, J and for any given ϵ , these resonances are in a finite number since we can assume from the properties of the S matrix that they have only accumulation at infinity.

We remark that Eq. (2.23) defines for a given value of the energy w a number $n(w, J, \epsilon)$ of relevant resonances. We repeat the statistical assumption that these resonances are linearly independent, namely

$$\langle c_i | c_j \rangle = \delta_{ij} \quad (i, j) \in \{D\} \quad (2.24)$$

Therefore the number n coincides with the rank of the operator R . Notice that the definition of the set $\{D\}$ involves the definition of an interval δw which contains the relevant resonances in the sense specified by Eq. (2.23). In the following we shall look for a uniform approximation of the operator T_c namely we shall choose the small quantity ϵ independent of w . The expansion (2.21) can be explained if we imagine that to each energy phase larger than ϵ only one relevant resonance corresponds in the energy interval $\{D\}$. In this case the orthogonality of the vectors $|c_i\rangle$ follows from the orthogonality of the vectors U_i and their nearly energy independence follows from the slow energy dependence of the U_i once an energy dependent phase is extracted. It is also useful to consider a continuous distribution of resonances with density (w, J) .

3. - THE INTEGRAL EQUATION FOR THE MASS SPECTRUM

From the side of the S matrix we have seen how we can define a distribution of resonances with density $\rho(w, J)$. The properties of these resonances have been considered as "statistical" where the meaning of this word is defined within the S matrix language.

The connection between resonant states and thermodynamic functions is made in the Hagedorn procedure by using the Beth-Uhlenbeck prescription. In that way from the scattering amplitude to the partition function the hadron volume enters exactly "à la Fermi". The procedure of Frautschi does not even use the scattering amplitudes and instead treats directly a bootstrap bound-state problem in a given box.

The general formal procedure for constructing the partition function of a system of which one does not know the Hamiltonian but the S matrix is proposed by BDM. In such a formalism the dualism interaction volume-thermodynamic volume is overcome. The authors are in fact able to derive the following formula for the grand potential (we are neglecting internal quantum numbers)

$$\Omega = \Omega_0 + \frac{V}{2\pi^2\beta^2} \int_{m_\pi}^{\infty} \omega^2 d\omega K_2(\beta\omega) \sum_J \rho(\omega, J), \quad (3.1)$$

where

$$\rho(\omega, J) = -\frac{1}{4\pi i} \left[\text{Tr}_J \left(A S^{-1} \frac{\partial}{\partial \omega} S \right) \right]_c \quad (3.2)$$

and A is the appropriate permutation operator according to the statistics of the contributing particles. Ω_0 is the grand potential for the ideal gas ; in our approach no elementary particles are present and consequently Ω_0 will be put equal to zero. The subscript c denotes that only connected diagrams are kept ; consequently Eq. (3.1) is linear in the thermodynamic volume V as it is explicitly shown ⁴⁾. The grand partition function follows by calculation once the S matrix is known.

We want now to use the formal apparatus developed in the previous paragraph as the S matrix input. Such input is still largely unspecified. A more detailed input could be given by dual models. However at the present stage of their development, they do not yet agree with unitarity. In our approach instead we have made a particular effort to treat since the very beginning the S matrix as unitary and consequently we accept in our scheme only resonances of finite width.

Apart from theoretical models, we have a certain wealth of phenomenological information on S matrix elements. Unfortunately both cosmic ray and accelerator phenomenology only gives data concerning a small class of amplitudes, namely the ones with two particles in the initial state. The most general feature of these data is the universal damping on the transverse momenta of the produced particles. This information, for the full consistency of our approach, has to be introduced through the widths of the resonances ; in the present work we do not exploit it in detail. As, however, we shall find a limiting temperature, the limitation on the transverse momenta will not be contradicted a posteriori.

In the previous section we have discussed how we are able to specify the concept of an S matrix approximated at a given energy w by a definite number of states plus a residual background of which we can fix

the operator norm. The important problem one faces now is that in the BDM formalism the expression of the partition function requires the use of the entire S matrix and not only of its connected part T_c . More precisely the knowledge of the spectrum $\rho(m, J)$ requires the extra knowledge of the interferences among the various connected and disconnected parts in addition to what is known simply through $D(m, J)$.

In order to discuss more closely this point, let us write the contributions to the S matrix at a given energy, as follows :

$$S = R + Q + S_d = T_c + S_d, \quad (3.3)$$

where S_d is the disconnected part and R and Q are defined in the previous section. This decomposition gives rise to two contributions to $\rho(m, J)$:

$$\rho(m, J) = \rho_R(m, J) + \tilde{\rho}(m, J), \quad (3.4)$$

where

$$\rho_R(m, J) = - \frac{1}{4\pi i} \left[T_{z, J} \left[A(Q-R)^{-1} \frac{\partial}{\partial m} (Q-R) \right] \right]_c \quad (3.5)$$

is the contribution due to the resonant part alone. The quantity $\tilde{\rho}(m, J)$ has contributions from disconnected parts, background and interference between them and with the resonant part. We like to notice that in this formalism the contribution $\tilde{\rho}$ cannot be trivially neglected. In fact we cannot make the rest Q vanish as discussed in Section 2. This is a priori a rather deep problem which could in principle question the entire thermodynamical consistency philosophy.

At this point we introduce the expression (2.21) for R into Eq. (3.5) ; we get, after some calculations

$$\rho_R(m, J) = \sum_{\{D(m, J, \epsilon)\}} \frac{1}{\pi} \frac{\Gamma_j}{(m - m_j)^2 + \Gamma_j^2}, \quad (3.6)$$

where essential use has been made of the properties of the vectors $|c_j\rangle$.

It is useful now to imagine a continuous distribution of resonances with a density $\mathcal{D}(w, J)$ and a width $\Gamma(w, J)$. Then we convert Eq. (3.6) into an integral equation :

$$\rho_R(m, J) = \frac{1}{\pi} \int_{\{D(m, J, \epsilon)\}} \mathcal{D}(w, J) dw \frac{\Gamma(w, J)}{(w - m)^2 + \Gamma^2(w, J)} \quad (3.7)$$

We recall that the domain $\{D\}$ of relevant resonances which is defined by Eq. (2.23) depends on the quantity ϵ and is a constraint between mass and width of the resonances. Although for low masses w_R we have experimental information on the relation $\Gamma(w_R)$ (namely the Rosenfeld tables), what happens for all values of w_R is a matter of speculation.

In our approach the widths will be related to the decay matrix elements and cannot be a priori determined, even knowing the mass spectrum. They are also depending on the spin of the resonances involved ; again we have not at this stage any definite statement about this dependence. Under this respect our approach is incomplete and should be implemented with judicious hypotheses concerning the decays. This problem, which we do not face in the present work, is also crucial in developing the second part of a statistical approach, namely the one concerned with scattering processes. We are convinced that a proper treatment of that phenomenology may suggest and need some extra assumptions ; in our scheme these assumptions should find their place in the study of the decay matrix elements, and, through the widths, they could affect in turn the entire scheme, in particular the mass spectrum.

For the time being we shall only require some smoothness for the mass dependence of the total widths. This is reasonable, since possible statistical fluctuations affecting partial widths should average in the total widths.

Let us come back to Eq. (3.7). Remark that the domain $\{D\}$ is compact : in fact, from Eq. (2.23) we get

$$|w - m| \leq \alpha \Gamma(w, J) \quad (3.8)$$

where

$$\alpha = \left(4 \epsilon^{-2} - 1 \right)^{1/2} . \quad (3.9)$$

Let us now define an "equivalent width" $\bar{\Gamma}(m, J, \alpha)$ through the following relation

$$\int_{\{D(m, J, \epsilon)\}} D(w, J) \frac{\Gamma(w, J) dw}{(m-w)^2 + \Gamma^2(w, J)} = \int_{m-\alpha \bar{\Gamma}(m, J, \alpha)}^{m+\alpha \bar{\Gamma}(m, J, \alpha)} D(w, J) \frac{\Gamma(w, J) dw}{(m-w)^2 + \Gamma^2(w, J)} . \quad (3.10)$$

It is not obvious that Eq. (3.10) makes sense in general. It certainly does, however, when the integrand is a continuous function. In this case we can write finally :

$$\rho_R(m, J, \alpha) = \frac{1}{\pi} \int_{m-\alpha \bar{\Gamma}(m, J, \alpha)}^{m+\alpha \bar{\Gamma}(m, J, \alpha)} D(w, J) \frac{\Gamma(w, J) dw}{(w-m)^2 + \Gamma^2(w, J)} . \quad (3.11)$$

Up to now we have specified some general properties of resonances to be counted by D ; we know neither the function $D(w, J)$ nor its relevance to $\rho = \rho_R + \tilde{\rho}$.

At this point we make the hypothesis :

$$\lim_{w \rightarrow \infty} \frac{\tilde{\rho}(w)}{\rho(w)} = 0 \quad (3.12)$$

and impose the bootstrap condition

$$\rho(m, J, \bar{\alpha}) = D(m, J) , \quad (3.13)$$

to be satisfied for a particular value $\alpha = \bar{\alpha}(J)$ and for large m .

We get in this way the following integral equation

$$\rho(m, J) = \frac{1}{\pi} \int_{m - \bar{\alpha} \bar{\Gamma}(m, J, \bar{\alpha})}^{m + \bar{\alpha} \bar{\Gamma}(m, J, \bar{\alpha})} \rho(w, J) \frac{\Gamma(w, J) dw}{(w - m)^2 + \Gamma^2(w, J)} \quad (3.14)$$

which involves the mass spectrum ρ , and holds for large m .

Equations (3.12), (3.13), (3.14) need several comments which we make orderly.

A - From the point of view of thermodynamics the assumption (3.12) implies that asymptotically the entropy of hadronic matter is dominated by the resonance contribution. In terms of S matrix it implies that the main part of the mass spectrum comes from independent resonances as specified in Section 2. In this approach, Eq. (3.12) is translated into the requirement that resonances outside the domain $\{D\}$, giving rise to tail effects and the background due to disconnected parts undergo interference effects and are slowly varying with energy. In other words we have explicitly assumed that the resonances which give small contribution in norm to T_c , give also small contribution in trace in the BDM formula, when considered together with the background and the disconnected parts. There is no general theorem in favour of this statement and moreover one can construct counter-examples. A further study of this problem by means for instance of S matrix models would be interesting, in our opinion. It is nevertheless encouraging in this regard that the solution of Eq. (3.14), we shall find in the next section with a certain choice of $\Gamma(w, J)$, turns out to be a fast increasing function of the energy.

B - The condition (3.13) is satisfied for a fixed value of α . This parameter in our approach bears a certain analogy with the interaction volume V_H appearing in the Hagedorn theory. This can be seen in two ways. First of all in Hagedorn the bootstrap is imposed in a particular hadronic volume V_H ; here the bootstrap is imposed for a given α . Secondly the very possibility of introducing a parameter α is due to the existence of a minimum mass (the pion mass) for hadron interactions. The same holds for the existence of a finite Fermi volume.

C - If in Eq. (3.14) we perform the limit $\Gamma \rightarrow 0$, $\bar{\alpha} \rightarrow \infty$, the equation is identically satisfied. In our formalism indeed the narrow width approximation, although it can be considered in a limiting sense, is void of information.

4. - THE INTEGRAL EQUATION FOR THE TOTAL WIDTH

In order to specify Eq. (3.14) we need the knowledge of the function $\Gamma(w, J)$. However, as we shall discuss in detail in this section, the function Γ depends on the mass spectrum itself in addition to the decay matrix elements between resonances belonging to different domains $\{D\}$ and $\{D'\}$. Clearly the functions ρ and Γ have to be determined together in a consistent way.

In accord with the notations of Section 2 we define the partial widths by

$$|\gamma_i^x| = \sqrt{\Gamma_i} | \langle c_i | X \rangle |, \quad (4.1)$$

where $|X\rangle$ is a particular final state. Of course they satisfy the condition

$$\Gamma_i = \sum_x |\gamma_i^x|^2, \quad (4.2)$$

where the sum is extended to all possible final states.

The dynamical information which was not specified in the previous section is contained in the partial widths γ_i . If we make explicit the summation over the final states in Eq. (4.2), we realize that it involves the knowledge of the spectral function $\mathcal{O}(m, J)$. In fact we have

$$\Gamma(w, J) = \sum_n \sum_{J_1 \dots J_n; \alpha_n} \int d\omega_1 \dots d\omega_n \vartheta[w - \omega_1 - \dots - \omega_n] \mathcal{O}(w_1, J_1) \dots \mathcal{O}(w_n, J_n) |\gamma_n(w, J; \omega_1, \dots, \omega_n, J_1, \dots, J_n, \alpha_n)|^2, \quad (4.3)$$

where α_n is a suitable set of angular momentum variables.

The only model which in principle provides us with expressions for the partial widths is the dual model ⁸⁾. We have also phenomenological information concerning partial widths at low energy. They are rather clearly in favour of two-body decays. For the sake of simplicity we write in detail Eq. (4.3) by retaining only the two-body contribution :

$$\Gamma(w, J) \approx \sum_{J_1 J_2 J_{1z} J_{2z}} \int dw_1 dw_2 \theta[w - w_1 - w_2] d\Omega \cdot \rho(w_1, J_1) \rho(w_2, J_2) \left| \langle w, J J_z | \gamma | \Omega, w_1, w_2, J_{1z}, J_{2z} \rangle \right|^2 \quad (4.4)$$

where J_1 and J_2 are the spins of the decay resonances, J_{1z} and J_{2z} their z components and Ω is the solid angle of the relative momentum in the centre-of-mass frame.

The approximation (4.4) is rather interesting because it allows a detailed treatment of the problem of introducing in the counting the angular momentum with a reasonable simplicity. It is convenient to consider a complete set of states for the two final resonances which are eigenstates of J, J_z, l and σ , where

$$\begin{cases} \vec{J} = \vec{l} + \vec{\sigma} \\ \vec{\sigma} = \vec{J}_1 + \vec{J}_2 \end{cases} \quad (4.5)$$

In this way the angular integration in Eq. (4.4) can be performed giving the final result

$$\Gamma(w, J) \approx \sum_{l \sigma J_1 J_2} \int dw_1 dw_2 \theta[w - w_1 - w_2] \cdot \rho(w_1, J_1) \cdot \rho(w_2, J_2) \delta[l, \sigma, J] \delta[J_1, J_2, \sigma] \left| \langle w, J | \gamma | w_1, w_2, J_1 J_2 \rangle \right|^2, \quad (4.6)$$

where δ is one if $\vec{J} = \vec{l} + \vec{\sigma}$ and zero otherwise. Actually w_1 and w_2 must obey a further condition ; in fact, if we remember that resonances having a mass difference less than $\delta w = w_{\max} - m$ as specified in Eq. (3.8) are orthogonal in our approach, we realize that the partial width must vanish if $w - w_1 < \delta w$ or $w - w_2 < \delta w$.

It is also instructive to write again Eq. (4.6) in the particular case in which one of the decay particles is a pion. In this case

$$\mathcal{O}(w_1, J_1) = \delta(w_1 - m_\pi) \delta_{J_1, 0} \quad (4.7)$$

and we get

$$\Gamma(w, J) \simeq \sum_{\ell J_2} \int dw_2 \, \vartheta[w - m_\pi - w_2] \cdot \mathcal{O}(w_2, J_2) \cdot \delta[\ell, J_2, J] \left| \langle w, J | \gamma | w_2, J \ell J_2 \rangle \right|^2 \quad (4.8)$$

In conclusion we have written a system of two coupled integral equations : (3.14) and (4.3), which constitute a bootstrap containing driving terms, namely the partial widths.

Our intention is to convert the large amount of information coming from data and from models into a method for deriving reasonable expressions for the partial widths. Our approach at this stage could seem still largely unspecified. Actually, it is, but we like to remark that the missing input is clearly defined and no more subject to further ad hoc hypotheses. We have not at present any proof of a theorem of existence and possibly uniqueness for the solution of the system. Nevertheless we shall prove in the next section that, starting from a given reasonable ansatz on $\Gamma(w, J)$, it is possible to find a solution $\vartheta(w, J)$ of Eq. (3.14) for each J . We do not try to solve also the second equation (4.3). Consequently at this stage the dependence of $\vartheta(w, J)$ on J remains unspecified. It is possible, however, to guess expressions for $|\gamma|^2$ which, inserted into Eq. (4.3), may suggest that, after all, the iteration procedure makes sense.

5. - A FIRST ORDER APPROXIMATION

As a first order treatment we shall consider Eq. (3.14) by choosing as starting point

$$\Gamma(w, J) = \Gamma_0(J) = \overline{\Gamma_0}(J) \quad (5.1)$$

Equation (3.14) becomes

$$\rho(m, J) = \frac{1}{\pi} \int_{m - \bar{\alpha}\Gamma_0(J)}^{m + \bar{\alpha}\Gamma_0(J)} d\omega \rho(\omega, J) \frac{\Gamma_0(J)}{(\omega - m)^2 + \Gamma_0^2(J)} \quad (5.2)$$

This equation is of convolution type and can be solved following the procedure given in Ref. 9). By putting

$$k[m - \omega, J] = \frac{\Gamma_0(J)}{\pi} \frac{\Theta[\bar{\alpha}\Gamma_0(J) - |\omega - m|]}{(\omega - m)^2 + \Gamma_0^2(J)} \quad (5.3)$$

the equation takes the form

$$\rho(m, J) = \int_{-\infty}^{+\infty} k[m - \omega, J] \rho(\omega, J) d\omega \quad (5.4)$$

and can be solved by a Fourier transformation.

The relevant theorem is the following.

Let $0 < c < c'$ and let $\exp\{c'|x|\} \cdot k(x)$ belong to L and $\exp\{-c|x|\} \cdot \rho(x)$ to $L^2(-\infty, +\infty)$. Then if $\rho(x)$ satisfies Eq. (4.4), it is of the form

$$\rho(x) = \sum_{\nu} \sum_{p=1}^q C_{\nu,p} x^{p-1} \exp\{-i\eta_{\nu}x\} \quad (5.5)$$

where η_{ν} runs through all the zeros of $1 - \sqrt{2\pi} K(\eta)$, where $K(\eta)$ is the Fourier transform of k , such that $|\text{Im}(\eta_{\nu})| \leq c$, the $C_{\nu,p}$ are constants, and q is the order of multiplicity of the zero η_{ν} .

In our case we have

$$1 - \sqrt{2\pi} K(\eta) = 1 - \frac{1}{\pi} \int_{-\bar{\alpha}}^{\bar{\alpha}} d\xi \frac{e^{i\eta \Gamma_0 \xi}}{1 + \xi^2} = 0 \quad (5.6)$$

Equation (5.6) can be also written in the form

$$\int_0^{\bar{\alpha}} \frac{d\xi}{1 + \xi^2} \cos(\eta \Gamma_0 \xi) = \frac{\pi}{2} \quad (5.7)$$

Each root of this equation gives rise to a solution of Eq. (5.2).

Equation (5.7) has no real solutions ; it has in general an infinite number of roots with the infinity as the only accumulation point, as Eq. (5.7) defines an analytic function in the entire η plane. One can show moreover that the real and the imaginary part of the roots do both accumulate at infinity. Consequently one gets for Eq. (5.5) solutions that are more and more increasing and oscillating.

Equation (5.7) has only two zeros at imaginary values of η ; in fact, by putting $\eta = i\beta$ and taking into account the fact that the cosine is an even function, it is easy to realize that the equation

$$\int_0^{\bar{\alpha}} \frac{d\xi}{1 + \xi^2} \cosh(\beta \Gamma_0 \xi) = \frac{\pi}{2} \quad (5.8)$$

has only two real solutions $\pm \beta_0$. Equation (5.8) establishes a relation among the quantities β_0 , $\bar{\alpha}$ and $\Gamma_0(J)$. Obviously we accept only those solutions of Eq. (5.4) which are real and positive. Therefore we get

$$\rho(m, J) = C_1(J) e^{\beta_0(J)m} + C_2(J) e^{-\beta_0(J)m} \quad (5.9)$$

A numerical investigation of Eq. (5.8) is interesting since it can give information on average widths, on β_0 (which later we shall connect to the limiting temperature) and on the "error" or background size ϵ . This in turn is related to the energy interval where relevant resonances are present.

Figure 2 shows the relationship between ϵ and $\beta_0 \Gamma_0$. It is also interesting to consider the function $\epsilon = \epsilon(\Gamma_0)$ for a fixed value of β_0 , which we take to be m_π^{-1} . This is exhibited in Fig. 3. One immediately sees how the norm of the background decreases when we move towards a narrow width limit. We have discussed in detail, however, that ϵ cannot vanish. Finally we can consider the quantity $\delta w = w_{\max} - m$ defined by Eq. (3.8). By making use of the numerical relation $\epsilon = \epsilon(\beta_0 \Gamma_0)$ we can obtain the plot δw versus Γ_0 (Fig. 4). The quantity δw is rather insensitive to Γ_0 and also to β_0 .

Remember that δw represents the interval of independent resonances. Consequently δw represents a lower limit for the difference of energy between the initial resonance and the decay products. For instance if the decay mechanism follows the cascade scheme, the light particle must have a minimum average energy of about 200 MeV. This result once more casts some light on the concept of minimum hadronic mass. In particular it could have been embarrassing to find a δw smaller than the pion mass. It is also interesting to notice that if the δw was instead very large (for instance an increasing function of w) the cascade decay scheme would be evidently forbidden.

We have now to discuss the dependence of our solution $\rho(m, J)$ on J . This dependence is indeed essential in constructing the grand potential as shown in Eq. (3.1). Since we are interested in large values of m we shall keep in Eq. (3.9) only the first addendum. The dependence on J can affect in principle both the coefficient C_1 and the parameter β_0 . However, for a given $\Gamma_0(J)$, which is bounded with respect to J , it is always possible to make a choice of $\epsilon(J)$, such that $\beta_0(J)$ coming from the eigenvalue condition is bounded :

$$\beta_0(J) \leq \beta_0^* \quad (5.10)$$

At this stage, although we can get a limiting temperature, we are unable to calculate the grand potential. We can put :

$$\hat{\rho}(m) = e^{\beta_0^* m} \sum_J C_1(J) e^{[\beta_0(J) - \beta_0^*] m} = \phi(m) e^{\beta_0^* m}. \quad (5.11)$$

As $\beta_o(J) - \beta_o^* \leq 0$ the function $\phi(m)$ is a non-increasing function of m . The possibility that $\phi(m)$ behaves at infinity like an inverse power of m is not excluded ; in this case the weak condition ¹⁾

$$\frac{\log \hat{\rho}(m)}{\log \sigma(m, V_o)} \rightarrow 1 \quad (5.12)$$

is satisfied, as can be seen by direct calculation of $\sigma(m, V_o)$.

We like to remark that the thermodynamic properties of hadron matter strongly depend on the power term affecting the exponential factor of the mass spectrum $\hat{\rho}$. In our opinion, without a complete treatment of the angular momentum dependence, the thermodynamical consequences derived from the mass spectrum may be doubtful.

ACKNOWLEDGEMENTS

It is a pleasure to thank R. Hagedorn for enlightening stimulation and M. Toller for critical discussions. We thank also A. Kobine for help in the numerical calculations.

REFERENCES

- 1) R. Hagedorn - Nuovo Cimento Suppl. 3, 147 (1965) ;
R. Hagedorn and J. Ranft - Nuovo Cimento Suppl. 6, 169 (1968) ;
R. Hagedorn - "Thermodynamics of Strong Interactions", Lectures given
in the Academic Training Programme of CERN, Geneva (1971).
- 2) S.C. Frautschi - Phys.Rev. D3, 2821 (1971).
- 3) R. Hagedorn and J. Ranft - CERN Preprint TH. 1440 (1972) ;
G. Ranft and J. Ranft - CERN Preprints TH. 1532 and 1544 (1972).
- 4) R. Dashen, S. Ma and H.J. Bernstein - Phys.Rev. 187, 345 (1969).
- 5) G. Veneziano - Talk presented at the International Conference on
Duality and Symmetry in Hadron Physics, Tel-Aviv (1971).
- 6) P. Goddard and C.B. Thorn - CERN Preprint TH. 1493 (1972) ;
R.C. Brower - MIT Preprint CTP 277 (1972).
- 7) L. Sertorio and M. Toller - CERN Preprint TH. 1421 (1971).
- 8) H.M. Chan and S.T. Tsou - Phys.Rev. D4, 156 (1971) ;
F. von Hippel and C. Quigg - Phys.Rev. D5, 624 (1972).
- 9) E.C. Titchmarsh - "Introduction to the Theory of Fourier Integrals",
Oxford (1937).

FIGURE CAPTIONS

- Figure_1 A double scattering process giving rise to a pole in the physical region.
- Figure_2 Norm of the background ϵ versus $\beta_0 \Gamma_0$ from Eq. (5.8).
- Figure_3 Norm of the background ϵ versus Γ_0 for $\beta_0 = m_T^{-1}$.
- Figure_4 Energy interval of independent resonances versus their average total width.

Fig. 1

Fig. 2

Fig. 3

Fig. 4