INTEGRAL EQUATIONS FOR THE CHAIN DECAY OF FIREBALLS

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A B S T R A C T

Integral equations for the chain decay of fireballs are derived in the thermodynamic model. The single particle distribution and multiplicity distribution of the decay products are considered. The asymptotic solutions are given explicitly and the deviation from them for finite fireball mass is determined numerically.

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1. INTRODUCTION

The mechanism of particle production in the thermodynamic model ¹⁾⁻⁴⁾ is composed of two consecutive processes. In the first one the fireballs are produced by a non-statistical mechanism. This process is out of the scope of the model and it is taken into account by some empirical formulae describing the velocity distribution of the produced fireballs. The second process is the statistical decay of fireballs determined essentially by the spectrum of hadronic states (fireballs) which are disposable for the decay. In the papers of Hagedorn and Ranft ^{1),2)} the asymptotic level density was taken to be $M^{-5/2} \exp(M/T_0)$ according to the "weak bootstrap condition" and a plausibility argument fixing the exponent of M to $-5/2$. The "strong bootstrap solution" of Frautschi and Hamer ^{5),6)} which takes into account energy-momentum conservation differs in the exponent of M , namely, it is $\rho(M) \sim M^{-3} \exp(M/T_0)$. This difference implies a change in the over-all picture of fireball decay which is like an explosion producing many new fireballs at the same time in the weak bootstrap solution and a more gentle sequential decay in the strong bootstrap solution. The purpose of the present paper is to discuss this sequential decay ("chain decay") of fireballs in which the large mass fireball emits light particles ("like pions") in a chain "cooling down" at the end into the world of individual resonances and then into a stable particle.

Many of the ingredients of our paper were in fact dealt with or touched upon already in earlier publications by Hagedorn and Ranft ⁴⁾ and especially by Frautschi and Hamer ^{7),8)}. Neither of these papers used, however, the technique of integral equations which allows a simple derivation of the asymptotic solutions for large fireball mass and an economical way of doing numerical calculation for the approaching of this asymptotic limit. In the present paper the problem is considered in a simplified model where the fireball with (say) proton quantum numbers emits neutral, spinless mesons (we shall call them "pions"). The three and more-body decays will be completely neglected [this is like in model 1 of Ref. 8)] and the quantum numbers as charge, baryonic number, isospin or strangeness will be disregarded. The neglect of other than two-body decays is almost justified as it is known that the two-body decay is dominant ⁵⁾, the average number of decay products per decay step being ~ 2.4 . The proper handling of quantum numbers seems to be only a technical complication having certainly no effect on the main features of the physical picture. Comparing our assumptions to those of Ref. 8) it is in order to note a difference. We assume that the decay probability is proportional (in our case) to the two-body invariant phase space

available for a given fireball. Frautschi and Hamer, on the other hand, considered the non-invariant phase space in the fireball rest frame. This makes a difference only in the form of the probability distribution $p(M,E)$ discussed in Section 3. We prefer this choice as for the decay the covariant phase space seems to be more natural. The integral equations derived in Sections 2 and 5 are, of course, valid for any $p(M,E)$.

The plan of the paper is the following: the integral equations for the single particle distributions and for certain moments of it are derived in Section 2. In Section 3 the emission probability, $p(M,E)$ is determined in the strong thermodynamic bootstrap solution and the asymptotic behaviour of the single particle distribution is determined for large fireball mass. Section 4 deals with the numerical calculations concerning the deviations at finite mass from the asymptotic solution. Finally, in Section 5, the integral equation for the generating function of the multiplicity distribution is derived and the asymptotic behaviour of the solution is investigated. Section 6 contains some final remarks and conclusions.

2. THE INTEGRAL EQUATION FOR THE SINGLE PARTICLE DISTRIBUTION

A simple kind of distributions reflecting many of the physical characteristics of the fireball is represented by the single particle distributions of its decay products. In our simple model there are two kinds of hadrons in the final state: a single proton and several pions. Due to the fact that proton-antiproton pair creation is not built into our simple model the proton distribution will not be considered here. In fact it could be treated along similar lines. Besides, we shall see that some essential properties of the proton distribution can be read out also from the pion distribution.

Let us consider a fireball of mass M which emits (neutral) pions of mass m sequentially until it is completely "cooled down" to a proton (mass M_p). The first act of pion emission is depicted on Fig. 1, where also the four momenta of the fireball and of its decay products are defined. We have, of course

$$k'^2 = m^2, \quad r^2 = M^2, \quad r'^2 = (r - k')^2 = M^2 + m^2 - 2kr. \quad (2.1)$$

The single pion distribution $d^3D(r,k)/dk^3$ resulting from the decay of the fireball is defined by the requirement that

$$\frac{d^3D(r,k)}{dk^3} dk^3$$

be the mean number of pions emitted in the cell d^3k of momentum space about k . (Therefore, also the name "differential mean-multiplicity" could be used for it.) The dependence of $d^3D(r,k)/dk^3$ on the four-momentum of the fireball r is trivial in the sense that once it is known in the rest frame of the fireball it can be determined in all other frames by a Lorentz-transformation. This fact is expressed by the covariance property

$$k_0 \frac{d^3D(r,k)}{dk^3} = D(r^2, r \cdot k) \equiv d(\sqrt{r^2}, \frac{r \cdot k}{\sqrt{r^2}}), \quad (2.2)$$

where k_0 is the zeroth component of the four-vector k and $D(r^2, r \cdot k)$ is an invariant function of the invariants r^2 and $r \cdot k$ (k^2 is of course fixed to m^2). The function d is identically equal to D , it has only different arguments. It is introduced in Eq. (2.2) for later convenience, as the variables of d have simple interpretation in the fireball rest frame: $\sqrt{r^2} = M$ is the mass of the fireball and $r \cdot k / \sqrt{r^2} \equiv E$ is the energy of the observed pion in this frame.

To describe the chain decay of fireball one has to know the probability law governing each emission step in the chain. As we are disregarding quantum number complications we can assume that the only physical variable of the fireball along the chain is its total mass. The other physical quantities (like temperature, energy density, etc.) are supposed to be uniquely determined by the mass. Taking into account also recoil the appropriate variable, instead of the mass, is the four-momentum r , therefore, the probability of the pion emission will depend generally on r . Let us denote the probability that the fireball with four-momentum r emits a pion of four-momentum in the cell d^3k about k by

$$\frac{d^3P(r,k)}{dk^3} dk^3.$$

The probability density $d^3P(r,k)/dk^3$ has the same covariance property as $d^3D(r,k)/dk^3$, hence in analogy with Eq. (2.2) we can write

$$k_0 \frac{d^3P(r,k)}{dk^3} = P(r^2, r \cdot k) \equiv p(M, E). \quad (2.3)$$

The problem to be solved is the following: given the probability density $d^3P(r,k)/dk^3$ for every r and k , what is the single particle distribution $d^3D(r,k)/dk^3$? Considering the first emission in the chain the pions in the final state can be divided into two groups: the first contains the pion emitted in this first step, the second is the rest coming from the subsequent decay of the "daughter" fireball (which has four-momentum r' , see Fig. 1). In mathematical terms

$$\frac{d^3D(r,k)}{dk^3} = \frac{d^3P(r,k)}{dk^3} + \int d^3k' \frac{d^3P(r,k')}{dk'^3} \frac{d^3D(r-k',k)}{dk^3}. \quad (2.4)$$

This is the integral equation which determines the single particle distribution $d^3D(r,k)/dk^3$ once $d^3P(r,k)/dk^3$ is known.

It is possible and more convenient to go over from $d^3D(r,k)/dk^3$ to the invariant single particle distribution $D(r^2, r \cdot k)$. From Eqs. (2.2), (2.3) we have

$$D(r^2, r \cdot k) = P(r^2, r \cdot k) + \int \frac{d^3k'}{k'_0} P(r^2, r \cdot k') D[(r-k')^2, (r-k') \cdot k]. \quad (2.5)$$

The integrand depends on k' only through $r \cdot k'$ and $k' \cdot k$, therefore, one of the three integrations can be performed. An appropriate set of variables is defined in the fireball rest frame (see Fig. 2). $E' \equiv k'_0$ is the energy in that frame and F' is the projection of $\underline{k}'$ along the direction of $\underline{k}$. In that case $r \cdot k' = ME'$ and $k \cdot k' = EE' - F' \sqrt{E^2 - m^2}$, and one can easily perform the integration on the remaining angular variable corresponding to a rotation about $\underline{k}$. Hence, expressed in terms of the fireball rest frame variables the integral equation becomes

$$d(M, E) = p(M, E) + 2\pi \int_{T(M, p)} dE' dF' p(M, E') d \left\{ \sqrt{M^2 + m^2 - 2ME'}, \frac{(M-E')E + F' \sqrt{E^2 - m^2}}{\sqrt{M^2 + m^2 - 2ME'}} \right\}. \quad (2.6)$$

The complicated expressions in the arguments of d take into account exactly the recoil of the fireball during the emission. Neglecting the recoil (for large enough M) would be equivalent to putting $d(M-E', E)$ in the integral. For details of the asymptotic considerations (valid for large M) see the next Section.

The integration in Eq. (2.6) is done over the kinematically allowed domain $T(M_p)$ for E', F' . $T(M_p)$ obviously depends on M, E and it is easy to show that it is defined by the inequalities

$$E'^2 - F'^2 \geq m^2; \quad E' \leq \frac{1}{2M} (M^2 + m^2 - M_p^2); \quad (2.7a)$$

$$\frac{1}{2} [M^2 + 2m^2 - 2ME' - M_p^2] \geq (M-E')E + F' \sqrt{E^2 - m^2} \geq m \sqrt{M^2 + m^2 - 2ME'}. \quad (2.7b)$$

The typical shape of $T(M_p)$ is illustrated on Fig. 3, by the shaded area.

The solution of the two-variable integral equation, Eq. (2.6) can be obtained by numerical methods (see Section 4). Numerically, the single variable integral equations are, however, much easier to handle. This motivates the search for physical quantities depending only on a single variable but being also of interest for the characterization of the fireball decay. An obvious candidate for that is the mean multiplicity of the emitted pions. (Mean number of step in which the fireball decays into the proton.) The mean multiplicity $n(M)$ can be expressed as the integral of the single particle distributions

$$n(M) = \int d^3k \frac{d^3D(r, k)}{dk^3} = \int \frac{d^3k}{k_0} D(r, r, k) = 4\pi \int_m^{E(M, M_p)} dE \sqrt{E^2 - m^2} d(M, E) \quad (2.8)$$

where

$$E(M, M_p) = \frac{1}{2M} (M^2 + m^2 - M_p^2). \quad (2.9)$$

The integral equation for $n(M)$ can be obtained from Eq. (2.6) by integration over E . The integral of the inhomogeneous term is unity as $d^3P(r, k)/dk^3$ is a normalized probability density, therefore

$$\int d^3k \frac{d^3P(r,k)}{dk^3} = 4\pi \int_m^{E(M,M_p)} dE \sqrt{E^2 - m^2} p(M,E) = 1. \quad (2.10)$$

The equation for $n(M)$ is, consequently

$$n(M) = 1 + 4\pi \int_m^{E(M,M_p)} dE' \sqrt{E'^2 - m^2} p(M,E') n(\sqrt{M^2 + m^2 - 2ME'}). \quad (2.11)$$

The simplicity of the equation is due to the fact that the kernel of Eq. (2.6) is independent of F' which made possible to perform the integration over F' . This suggests that a general way of transforming Eq. (2.6) into a set of single-variable integral equations is to multiply it by certain functions of the energy E [in the case of $n(M)$ this function is identically 1] and to integrate over E . An economic way of doing this would be, of course, if this "expansion" into functions of E would at the same time diagonalize the equation. A way of achieving this goal is the following. Let us go back to Eq. (2.5) and multiply it by the ℓ th rank totally symmetric, traceless tensor $T_{\{\mu_1 \dots \mu_\ell\}}(k)$ which can be built from the four-momentum k . The first few such tensors are given by

$$\begin{aligned} T(k) &\equiv 1; & T_{\{\mu\}}(k) &\equiv k_\mu; \\ T_{\{\mu\nu\}}(k) &= k_\mu k_\nu - g_{\mu\nu} \frac{m^2}{4}; \\ T_{\{\lambda\mu\nu\}}(k) &= k_\lambda k_\mu k_\nu - \frac{m^2}{6} (g_{\lambda\mu} k_\nu + g_{\mu\nu} k_\lambda + g_{\nu\lambda} k_\mu). \end{aligned} \quad (2.12)$$

It follows from covariance that

$$\int \frac{d^3k}{k_0} D(r^2, r, k) T_{\{\mu_1 \dots \mu_\ell\}}(k) = T_{\{\mu_1 \dots \mu_\ell\}}(r) I_\ell(r^2), \quad (2.13)$$

where I_ℓ is some invariant function which can depend only on r^2 . If the function analogous to I_ℓ but defined with $P(r^2, r, k)$ instead of $D(r^2, r, k)$ is denoted by I_ℓ^P , then from Eq. (2.5) it follows

$$\begin{aligned} T_{\{\mu_1 \dots \mu_\ell\}}(r) I_\ell(r^2) &= T_{\{\mu_1 \dots \mu_\ell\}}(r) I_\ell^P(r^2) + \int \frac{d^3k'}{k'_0} P(r^2, r, k') \cdot \\ &\cdot T_{\{\mu_1 \dots \mu_\ell\}}(r-k') I_\ell((r-k')^2). \end{aligned} \quad (2.14)$$

Let us go into the fireball rest frame, put every Lorentz index equal to 0 and denote

$$T_{\{0\dots 0\}}(k) \equiv t_l(E, m) \quad (l=0,1,2,\dots). \quad (2.15)$$

The first few of the polynomials t_l is from Eq. (2.12) given by

$$\begin{aligned} t_0(E, m) &= 1 & t_1(E, m) &= E \\ t_2(E, m) &= E^2 - \frac{1}{4}m^2 & t_3(E, m) &= E^3 - \frac{1}{2}m^2 E. \end{aligned} \quad (2.16)$$

Then it follows that

$$T_{\{0\dots 0\}}(\tau) I_l(\tau^2) = t_l(M, M) I_l(M^2). \quad (2.17)$$

Let us denote this function by $e_l(M)$. Comparing Eq. (2.17) with Eq. (2.13) and performing the angular integration we have

$$e_l(M) = 4\pi \int_m^{E(M, M_p)} dE \sqrt{E^2 - m^2} d(M, E) t_l(E, m) \quad (l=0,1,\dots) \quad (2.18)$$

It follows from Eq. (2.14) that $e_l(M)$ satisfies the equation

$$\begin{aligned} e_l(M) &= \int_m^{E(M, M_p)} dE w(M, E) t_l(E, m) + \\ &+ \int_m^{E(M, M_p)} dE w(M, E) \frac{t_l(M-E, m)}{t_l(\sqrt{M^2 + m^2 - 2ME}, m)} e_l(\sqrt{M^2 + m^2 - 2ME}). \end{aligned} \quad (2.19)$$

Here the function $w(M, E)$ is defined as

$$w(M, E) = 4\pi \sqrt{E^2 - m^2} p(M, E) \quad (2.20)$$

which is the probability density of the emission in the energy variable E satisfying the normalization condition

$$\int_m^{E(M, M_p)} dE w(M, E) = 1. \quad (2.21)$$

Two particular cases of the Eq. (2.19) are worth mentioning. $l=0$ gives back obviously the Eq. (2.11) for the mean-multiplicity

$$n(M) = e_0(M). \quad (2.22)$$

The case $l=1$ is interesting because of energy conservation. It follows from Eqs. (2.16) and (2.18) that $e_1(M)$ is the mean energy of the emitted pions in the rest frame of the fireball. As the total energy is M

$$e_p(M) = M - e_1(M) \quad (2.23)$$

gives the mean energy of the recoil proton in the fireball rest frame. This is the way how the pion distribution determines also some features of the proton distribution.

The solution of the equations (2.6) and (2.19) will be dealt with in the following sections. Before entering into the details of the solution it is, however, necessary to precise a little more the physical situation in which the equations are applied. According to the thermodynamic model the level density of fireballs grows exponentially with M , therefore, above some mass M_0 the levels overlap completely and a continuous spectrum is a good approximation of the real situation. We shall call this region ($M \geq M_0$) the "fireball region". At the lower edge of the hadron spectrum, however, there exist prominent resonances well distinguishable from each other. This will be called the "resonance region" which extends from M_p to M_0 . (In reality there is, of course, no sharp frontier between the two regions. The assumption of a definite M_0 is made in order to simplify the discussion.) For a fireball of mass $M > M_0 + m$ there is a possibility to decay into a pion plus either another fireball [probability $p_F(M, E)$] or a resonance [probability $p_R(M, E)$]. The total probability is

$$p(M, E) = p_F(M, E) + p_R(M, E). \quad (2.24)$$

In the case of sharp resonances p_R contains δ functions corresponding to the well defined mass M_r of the resonance. Using the notation in Eq. (2.9):

$$p_R(M, E) = \sum_r \frac{\delta(E - E(M, M_r))}{4\pi \sqrt{E(M, M_r)^2 - m^2}} w_r(M), \quad (2.25)$$

where $w_r(M)$ is the probability of the decay into the r th resonance. Introducing the splitting of Eq. (2.24) into Eq. (2.6), one obtains

$$d(M, E) = d_R(M, E) + p_F(M, E) + 2\pi \int_{T(M_0)} dE' dF' p_F(M, E') d \left\{ \sqrt{M^2 + m^2 - 2ME'}, \frac{(M-E')E + F'\sqrt{E'^2 - m^2}}{\sqrt{M^2 + m^2 - 2ME'}} \right\} \quad (2.26)$$

where $T(M_0)$ is the domain defined by Eqs. (2.7a) and (2.7b) with M_0 instead of M_p and

$$d_R(M, E) = \sum_r \frac{\delta(E - E(M, M_r))}{4\pi \sqrt{E(M, M_r)^2 - m^2}} w_r(M) + \\ + \sum_r \Theta(M, M_r, E) w_r(M) \frac{M_r}{8\pi \sqrt{(E(M, M_r)^2 - m^2)(E(M_r, M_p)^2 - m^2)(E^2 - m^2)}}. \quad (2.27)$$

The second term in d_R comes from the integral of $p_R(M, E')$ which is multiplied by the single particle distribution of the resonance. In deriving Eq. (2.27) it was assumed for simplicity that all the resonances are purely elastic, that is they all decay only into a pion plus a proton. In that case the single-particle distribution of the r th resonance is given by

$$d(M_r, E) = \frac{\delta(E - E(M_r, M_p))}{4\pi \sqrt{E(M_r, M_p)^2 - m^2}}. \quad (2.28)$$

The function $\Theta(M, M_r, E)$ in Eq. (2.27) takes into account whether or not the emission of the pion of energy E from the fireball of mass M is kinematically allowed through the intermediary of the resonance with mass M_r . In the former case Θ is equal to 1 in the latter one to 0. The other kinematical constraint, namely that $p_F(M, E)$ vanishes if $E > E(M, M_0)$ was taken into account in Eq. (2.26) when $T(M_p)$ was replaced by $T(M_0)$.

In solving the integral equations we shall always assume that the functions $p_F(M, E)$ and $d_R(M, E)$ are known. The probability density p_F is determined by the thermodynamical model whereas $d_R(M, E)$ containing the resonance parameters has to be taken from experiment. It is important to note that the consistency of the equation (2.6) requires the presence of this resonance region. In other words, it is inconsistent to assume that the continuum fireball region is extended down to the stable particle (proton) with mass M_p . Namely, if this were the case then due to the normalization condition Eq. (2.10), the equation in the form (2.6) would allow an arbitrary constant added to $d(M, E)$ which is physically unacceptable. Furthermore, this would only be possible if the pions had zero mass. Equation (2.26) is, however, already free from this ambiguity as p_F is not normalized to unity

any more but we have instead

$$4\pi \int_{m_0}^{E(M, M_0)} dE \sqrt{E^2 - m^2} p_F(M, E) = 1 - \sum_r w_r(M) \equiv 1 - w_R(M). \quad (2.29)$$

Here $w_R(M)$ denotes the total probability that the fireball of mass M emits the pion together with some of the resonances.

3. THE ASYMPTOTIC SOLUTION

The numerical solution of the equations (2.6) and (2.19) will be dealt with in the next section. For asymptotically large mass of the fireball it is expected on intuitive grounds that everything simplifies considerably due to the fact that at high enough M the recoil of the fireball can be neglected. This is a consequence of the small average energy of the emitted pions. Namely, in the thermodynamic model, for fixed M , $p_F(M, E)$ is a rapidly (exponentially) decreasing function of the pion energy E .

In the strong thermodynamic bootstrap solution ^{5), 6)} the density of states at high mass M' is

$$\rho(M') = \text{const. } M'^{-3} e^{\frac{M'}{T_0}} \quad (3.1)$$

T_0 denoting Hagedorn's limiting temperature. Taking into account four-momentum conservation in the two-body decay the invariant probability for the pion emission in d^3k about k is ^{*)}

$$p_F(M, E) \frac{d^3k}{k_0} = \frac{d^3k}{k_0} \int \frac{d^3r'}{r'_0} dM' \rho(M') \delta^4(r - r' - k) \quad (3.2)$$

Performing the r' and M' integrations with the help of the δ function and introducing $C^{-1}(M)$ as normalization factor it follows from (3.2) that

$$p_F(M, E) = C^{-1}(M) \rho(\sqrt{M^2 + m^2 - 2ME}) \frac{1}{\sqrt{M^2 + m^2 - 2ME}}. \quad (3.3)$$

*) It was mentioned in the Introduction that we assume the emission probability to be proportional to the invariant phase space. To take, instead, the non-covariant phase space in the fireball rest frame would be equivalent to multiply $p_F(M, E)$ by $E(M-E)$.

The final form of $p_F(M, E)$ can be obtained substituting here Eq. (3.1) for ρ . In order to have a finite limit at $M = \infty$ for the normalization constant let us introduce $Z(M) \sim C(M) M^4 e^{-M/T_0}$ as a new normalization factor. We have then

$$p_F(M, E) = Z(M)^{-1} \left(1 - \frac{2E}{M} + \frac{m^2}{M^2}\right)^{-2} \exp\left\{\frac{1}{T_0}(\sqrt{M^2 + m^2 - 2ME} - M)\right\} \quad (3.4)$$

where, in order to satisfy Eq. (2.29)

$$Z(M) = (1 - w_R(M))^{-1} 4\pi \int_m^{E(M, M_0)} dE \sqrt{E^2 - m^2} \left(1 - \frac{2E}{M} + \frac{m^2}{M^2}\right)^{-2} \exp\left\{\frac{1}{T_0}(\sqrt{M^2 + m^2 - 2ME} - M)\right\}. \quad (3.5)$$

For M very large compared to the other mass parameters in the problem (that is for $M \gg m, T_0$) the upper limit of the integration in Eq. (3.5) can be extended safely to infinity due to the exponential decrease of the integrand in E . The probability of emitting resonances $w_R(M)$ is also exponentially suppressed for high M , therefore

$$\lim_{M \rightarrow \infty} Z(M) \equiv Z_\infty = 4\pi \int_m^\infty dE \sqrt{E^2 - m^2} e^{-\frac{E}{T_0}} = 4\pi m T_0 K_1\left(\frac{m}{T_0}\right). \quad (3.6)$$

Here we used that for $M \rightarrow \infty$

$$\sqrt{M^2 + m^2 - 2ME} - M = -E - \frac{E^2 - m^2}{2M} + \dots \quad (3.7)$$

From Eq. (3.4) the limit of $p_F(M, E)$ can also be easily obtained. Determining also the term of order $1/M$ we have

$$\lim_{M \rightarrow \infty} p_F(M, E) \equiv p_\infty(E) = Z_\infty^{-1} e^{-\frac{E}{T_0}}; \quad (3.8)$$

$$p_F(M, E) = p_\infty(E) \left[1 + \frac{1}{M} \left(4E - \frac{E^2 - m^2}{2T_0} - \frac{5}{2} E_\infty\right) + \dots\right].$$

Here $\bar{E}_\infty$ is the average energy of the first emitted pion at infinite fireball mass, that is

$$\bar{E}_\infty = \frac{4\pi \int_m^\infty dE \sqrt{E^2 - m^2} E e^{-\frac{E}{T_0}}}{4\pi \int_m^\infty dE \sqrt{E^2 - m^2} e^{-\frac{E}{T_0}}} = m \frac{K_2\left(\frac{m}{T_0}\right)}{K_1\left(\frac{m}{T_0}\right)} = 2T_0 + m \frac{K_0\left(\frac{m}{T_0}\right)}{K_1\left(\frac{m}{T_0}\right)}. \quad (3.9)$$

Let us now consider the high M asymptotic limit of the equation for the single particle distribution, Eq. (2.26). For $M \rightarrow \infty$ the contribution of the resonance region $d_R(M, E)$ decreases exponentially so it can be neglected. Analogously, the substitution of the integration domain $T(M_0)$ by

$$\{E' \in [m, \infty]; F' \in [-\sqrt{E'^2 - m^2}, +\sqrt{E'^2 - m^2}]\}$$

makes only an exponentially decreasing difference as $p_F(M, E)$ decreases exponentially at large E . The arguments of the d function under the integral sign can also be expanded in $1/M$ as in the region giving the essential contribution to the integral we have $E', F' \ll M$. After these simplifications the equation for the asymptotic solution $d_\infty(M, E)$ is

$$\begin{aligned} d_\infty(M, E) &= p_\infty(E) + 2\pi \int_m^\infty dE' \int_{-\sqrt{E'^2 - m^2}}^{+\sqrt{E'^2 - m^2}} dF' p_\infty(E') d_\infty(M - E', E) = \\ &= p_\infty(E) + 4\pi \int_m^\infty dE' \sqrt{E'^2 - m^2} p_\infty(E') d_\infty(M - E', E). \end{aligned} \quad (3.10)$$

It could seem at the first sight inconsistent that in the first argument of d_∞ we kept the first two terms in the $1/M$ expansion whereas in the second argument only the first one. The neglected terms are, however, in both cases of order $1/M$. In any case, it can be shown that the second term in front of the second argument E is proportional to F' hence it is cancelled by the F' integration. So it makes no difference if in the second argument too, we keep the first two terms of the expansion.

The solution of the asymptotic equation (3.10) can be written down explicitly. Namely, it is easy to verify that

$$d_\infty(M, E) = \frac{M}{E_\infty} p_\infty(E) = \frac{M}{E_\infty z_\infty} e^{-\frac{E}{T_0}} \quad (3.11)$$

is indeed a solution. This is due to the normalization condition

$$4\pi \int_m^\infty dE \sqrt{E^2 - m^2} p_\infty(E) = 1 \quad (3.12)$$

and to the definition of $\bar{E}_\infty$ in Eq. (3.9). In particular, the moments of $d_\infty(M, E)$ defining the asymptotic solution of the equation (2.19) are given by

$$e_{l\infty}(M) = \frac{M}{\bar{E}_\infty} 4\pi \int_m^\infty dE \sqrt{E^2 - m^2} p_\infty(E) t_l(E, m). \quad (3.13)$$

The special case $l=0$ gives the asymptotic mean multiplicity:

$$n_\infty(M) = \frac{M}{\bar{E}_\infty}. \quad (3.14)$$

The asymptotic solution in Eq. (3.11) is very easy to interpret. The very heavy fireball does not practically recoil as the energy of the emitted pions is small. In consequence, the consecutive steps of emission do not differ from each other, every emission is producing the same pion distribution $p_\infty(E)$. The number of steps is determined by the average amount by which the mass of the fireball decreases in one step. Hence the number of steps for the total mass M is $M/\bar{E}_\infty$. This latter fact is expressed also by Eq. (3.14). These considerations hold, of course, only asymptotically, as for finite M the recoil is there and d is determined by Eq. (2.26).

After having determined the leading behaviour of the solution in M one is tempted to try to look for an expansion in $1/M$. The probability distribution of the first emitted pion $p_F(M, E)$ has, indeed, such an expansion given in Eq. (3.8). This equation shows that the integral over E of the correction term is zero, as we have

$$\overline{(E^2 - m^2)}_\infty = \bar{E}_\infty^{-1} 4\pi \int_m^\infty dE \sqrt{E^2 - m^2} (E^2 - m^2) e^{-\frac{E}{T_0}} = 3mT_0 \frac{K_2(\frac{m}{T_0})}{K_1(\frac{m}{T_0})} = 3T_0 \bar{E}_\infty \quad (3.15)$$

This must be so as in the normalization equation (3.5) it is possible to put $w_R = 0$ and $E(M, M_0) = \infty$ in the upper limit of the integral up to the accuracy of a decreasing exponential and as $p_\infty(E)$ is already normalized to unity there is nothing left from the norm to the correction term. [It has to be noted that this juggling with the omitted exponentially decreasing terms shows that the expansion in Eq. (3.8) cannot be an analytic expansion but only an asymptotic one.] Looking more closely at the correction term, it turns out that for pions ($m = 140$ MeV) and for $T_0 = 150$ MeV ($\bar{E}_\infty = 400$ MeV) it is positive between $E \sim 300$ MeV and $E \sim 900$ MeV and negative elsewhere. This refers, however, only to the first emission step. The same correction for the whole decay chain could be determined from the asymptotic expansion of $d(M, E)$. But this does not seem to exist as trying an expansion of the

solution of Eq. (2.26) leads to contradictions and inconsistencies. This can be seen easily also on the simpler equation for the mean multiplicity Eq. (2.11).

Our last remark in this Section concerns the result Eq. (3.15) which shows that

$$\frac{\overline{(E^2 - m^2)}_\infty}{2\bar{E}_\infty} = \frac{3T_0}{2} \quad (3.16)$$

This has an amusing resemblance to the non-relativistic expression one would write down for the kinetic energy of the pions.

4. NUMERICAL SOLUTION

The solution of the integral equations (2.26) or (2.19) is possible in general only by numerical methods. Nevertheless the knowledge of the asymptotic solutions determined in the preceding section can be exploited by calculating only the correction terms which are small, at least asymptotically, compared to the leading contribution. Let us define the correction term μ_d to the single-particle distribution d by the equations

$$d(M, E) = p_\infty(E) \frac{M + \mu_d(M, E)}{\bar{E}_\infty} ; \quad (4.1)$$

$$\mu_d(M, E) = \frac{\bar{E}_\infty}{p_\infty(E)} d(M, E) - M .$$

It follows from (2.26) that μ_d satisfies the integral equation

$$\begin{aligned} \mu_d(M, E) = & \frac{\bar{E}_\infty}{p_\infty(E)} \left[p_F(M, E) + d_F(M, E) \right] - M + \\ & + 2\pi \int_{T(M_0)} dE' dF' \frac{p_F(M, E')}{p_\infty(E)} p_\infty \left(\frac{(M-E')E + F'\sqrt{E'^2 - m^2}}{\sqrt{M^2 + m^2 - 2ME'}} \right) \sqrt{M^2 + m^2 - 2ME'} + \\ & + 2\pi \int_{T(M_0)} dE' dF' \frac{p_F(M, E')}{p_\infty(E)} p_\infty \left(\frac{(M-E')E + F'\sqrt{E'^2 - m^2}}{\sqrt{M^2 + m^2 - 2ME'}} \right) \mu_d \left(\sqrt{M^2 + m^2 - 2ME'}, \frac{(M-E')E + F'\sqrt{E'^2 - m^2}}{\sqrt{M^2 + m^2 - 2ME'}} \right) \end{aligned} \quad (4.2)$$

The analogous equations for the "moments" $e_l(M)$ are

$$e_l(M) = \frac{M + \mu_e^l(M)}{\bar{E}_{l\infty}} ; \quad \mu_e^l(M) = \bar{E}_{l\infty} e_l(M) - M ; \quad (4.3)$$

$$\bar{E}_{l\infty} = \frac{\bar{E}_\infty}{4\pi \int_m^\infty dE \sqrt{E^2 - m^2} p_0(E) t_l(E, m)} \quad (l=0, 1, 2, \dots)$$

and μ_e^l satisfies, with $w_F(M, E) \equiv 4\pi \sqrt{E^2 - m^2} p_F(M, E)$:

$$\begin{aligned} \mu_e^l(M) = \bar{E}_{l\infty} & \left[\int_m^{E(M, M_0)} dE w_F(M, E) t_l(E, m) + e_{lR}(M) \right] - M + \\ & + \int_m^{E(M, M_0)} dE w_F(M, E) \frac{t_l(M-E, m)}{t_l(\sqrt{M^2 + m^2 - 2ME}, m)} \sqrt{M^2 + m^2 - 2ME} + \\ & + \int_m^{E(M, M_0)} dE w_F(M, E) \frac{t_l(M-E, m)}{t_l(\sqrt{M^2 + m^2 - 2ME}, m)} \mu_e^l(\sqrt{M^2 + m^2 - 2ME}). \end{aligned} \quad (4.4)$$

Here $e_{lR}(M)$ denotes the contribution of the resonance region to e_l . It can be determined from Eq. (2.27), too, but a simpler method is to start directly from the equation for e_l (2.19). One then easily obtains

$$e_{lR}(M) = \sum_r w_r(M) \left\{ t_l[E(M, M_r), m] + t_l[E(M_r, M_r), m] \frac{t_l[M-E(M, M_r), m]}{t_l(M_r, m)} \right\}. \quad (4.5)$$

From the point of view of numerical solutions the equations (4.2) and (4.4) are quite simple. This is due to the finiteness of the pion mass which forbids the fireball of mass M to decay into fireballs with mass larger than $M-m$. Therefore, it is possible to calculate numerically the exact solution by steps of length m starting from the lowest fireball mass M_0 . To illustrate the situation in more detail it is enough to consider the model equation

$$f(M) = \varphi(M) + \int_{M_0}^{M-m} dM' K(M', M) f(M') \Theta(M - M_0 - m). \quad (4.6)$$

Here φ and K are given functions, $m > 0$ (the pion mass in our case), and $\Theta(\xi)$ is the usual step function (0 for $\xi < 0$, 1 for $\xi > 0$) taking into account the threshold of fireball emission. It is obvious that Eq. (4.6) represents the general class of integral equations our equations belong to. It follows from Eq. (4.6) that the solution for $f(M)$ is:

$$\begin{aligned} f(M) &= \varphi(M) \quad \text{for } M \leq M_0 + m; \\ f(M) &= \varphi(M) + \Theta(M - M_0 - m) \int_{M_0}^{M-m} dM' K(M, M') \varphi(M') \quad \text{for } M \leq M_0 + 2m; \\ f(M) &= \varphi(M) + \Theta(M - M_0 - m) \int_{M_0}^{M-m} dM' K(M, M') \varphi(M') + \Theta(M - M_0 - 2m) \int_{M_0+m}^{M-2m} dM' K(M, M') \cdot \\ &\quad \cdot \int_{M_0}^{M'-m} dM'' K(M', M'') \varphi(M'') \quad \text{for } M \leq M_0 + 3m; \text{ etc.} \end{aligned} \quad (4.7)$$

In other words, for given M , the Neumann-series of Eq. (4.6) has only a finite number of non-zero terms and gives the exact solution. The different terms in (4.7) have in our case a simple meaning, namely the term with k fold integral gives the contribution of the decay chain with k intermediate fireballs.

A way of doing the numerical calculation is to calculate the integrals in (4.7) one after the others. That is compute first $\int K \varphi$ on a set of points, then from the result compute by a second single variable integration $\int K \int K \varphi$ using some interpolation formula for $\int K \varphi$ etc. The same method could, of course, be used also directly for Eqs. (2.26) and (2.19) instead of (4.2) and (4.4). The advantage of the latter equations is that they give only the correction terms which are small and the errors of the numerical calculation are thus less important.

The results of the numerical calculation are depicted on Figs. 4-7. The parameters were chosen as follows: $m = 140$ MeV for the pion mass, $T_0 = 150$ MeV for the Hagedorn temperature, $M_p = 940$ MeV for the proton mass. From Eq. (3.9) one has then $\bar{E}_\omega = 400$ MeV. In d_R and e_{LR} two resonances were taken together with the proton ground state on a linear trajectory with slope 1 GeV^{-2} , that is

$$M_r = \sqrt{M_p^2 + r} \quad (\text{GeV}) \quad (r=1,2). \quad (4.8)$$

The frontier between the fireball region and resonance region was taken at $M_0 = 2$ GeV. The probabilities of the direct resonance emission $w_r(M)$ were chosen to be proportional to $\exp(-M/T_0)$. All of these values are only crude estimates of the real ones but even so the solutions obtained reflect the general way how the asymptotic limit given in the preceding Section is approached by the real solution. A more careful choice of the parameters will be needed only if the more fundamental simplifications (neglect of the three and more-body decays and of the quantum numbers) will be properly cured.

In general it can be concluded from Figs. 4-7 that the corrections to the asymptotic value M are not large and negative and they are slowly varying functions of M, E except for a narrow region close to M_0 where the abrupt switch on of the fireball region and switch off of the resonance region cause some oscillations. The "mass losses" μ_n, μ_e, μ_d must in fact be counted not from zero but from $-M_p$ as the total mass available for the decay is $M - M_p$.

The role of the resonance parameters in our calculations is illustrated by Fig. 4 and the curve $\ell=0$ of Fig. 5, both giving the mass correction term for the mean multiplicity. The difference is that on Fig. 4 the mean multiplicity of the resonances was taken as $(M_r - M_p)2.5$ whereas on Fig. 5 (as well as on Figs. 6, 7) the resonances were assumed purely elastic, i.e., with mean pion multiplicity 1. This loss in multiplicity explains why the $\ell=0$ curve on Fig. 5 is shifted down compared to the curve on Fig. 4. A remarkable feature is the behaviour of the curve $\ell=1$ on Fig. 5 showing [See Eq. (2.23)] that the average recoil energy of the proton is essentially constant over the range $2 \text{ GeV} < M < 25 \text{ GeV}$. It can be seen from Fig. 6 that the mean energy per pion is in general above its limit for $M \rightarrow \infty$. This corresponds to an "effective temperature" T_{eff} somewhat higher than T_0 in accordance with the results of Ref. 8) [see also Ref. 9)]. The T_{eff} given in the Figure is defined by

$$\frac{e_1(M)}{e_0(M)} = \frac{4\pi \int_m^\infty dE \sqrt{E^2 - m^2} \left(e^{\frac{E}{T_{\text{eff}}(M)}} - 1 \right)^{-1} E}{4\pi \int_m^\infty dE \sqrt{E^2 - m^2} \left(e^{\frac{E}{T_{\text{eff}}(M)}} - 1 \right)^{-1}} \quad (4.9)$$

that is, like in most of the phenomenological applications of the thermodynamic model $(\exp(E/T_{\text{eff}})-1)^{-1}$ is used instead of $\exp(-E/T_{\text{eff}})$ corresponding to Eq. (3.8). This makes $T_{\text{eff}}(\omega)$ 10% higher than T_0 . In the fireball mass range $3 \text{ GeV} < M < 25 \text{ GeV}$ the recoil effects make T_{eff} a further 5-10% higher. The effect of the fireball threshold (a little artificial in our calculations) is well illustrated in Fig. 7. For M just above the threshold $M_0 + m$ the emission of low energy pions is suppressed due to the fact that the resonances all decay directly to the proton. This explains the dip at small E in the curves. The dip, however, disappears rapidly as M grows, that is the decay mechanism we are considering washes out the rapid variations due to the threshold. It favours the mass correction term $\mu_d(M, E)$ to be essentially constant.

5. THE EQUATION FOR THE MULTIPLICITY DISTRIBUTION

Beside the single particle distribution another characteristic feature of the fireball decay is the multiplicity distribution of the decay products (pions). Some information about the multiplicity is contained also in the single particle distribution, namely, the mean multiplicity which is the integral of the single particle distribution. But in order to have a complete information it is necessary to start once more in a way somewhat different from that of Section 2.

Let us consider again the fireball of mass M and denote by $p_n(M)$ the probability that during the chain decay exactly n pions are emitted. If in the first step $w(M, E)$ is the probability for the emission of a pion of energy E [see Eq. (2.20)] then we have for $n \geq 2$:

$$p_n(M) = \int_m^{E(M, M_p)} dE w(M, E) p_{n-1}(\sqrt{M^2 - m^2 - 2ME}). \quad (5.1)$$

If the probability $p_1(M)$ for the decay in one step is known then using this recursion formula it is possible to calculate all the $p_n(M)$ one after the other. Equation (5.1) is a simple case of the Chapman-Kolmogorov equation¹⁰⁾ for the simple Markov chain we are considering. Its solution gives the multiplicity distribution $p_n(M)$ we are looking for.

Experimentally the quantities often measured are the different moments of the multiplicity distribution. In order to have equations directly for the moments let us introduce the generating function of the multiplicity distribution (see for instance ¹¹⁾) :

$$\psi(M, h) = \sum_{n=1}^{\infty} p_n(M) (1+h)^n, \quad (5.2)$$

$$p_n(M) = \frac{1}{n!} \left. \frac{\partial^n \psi(M, h)}{\partial h^n} \right|_{h=-1}.$$

The binomial moments of the multiplicity distribution are expressed then as

$$\varphi_N(M) \equiv \left. \frac{\partial^N \psi(M, h)}{\partial h^N} \right|_{h=0} = \sum_{n=N}^{\infty} n(n-1) \dots (n-N+1) p_n(M). \quad (5.3)$$

In particular

$$\varphi_0(M) = \sum_{n=1}^{\infty} p_n(M) = 1$$

$$\varphi_1(M) = \sum_{n=1}^{\infty} n p_n(M) = n(M) \quad (5.4)$$

where $n(M)$ is the mean multiplicity defined before, in Eq. (2.8). The Chapman-Kolmogorov equations (5.1) are equivalent to the following integral equation for $\psi(M, h)$:

$$\psi(M, h) = (1+h) p_1(M) + (1+h) \int_m^{E(M, M_p)} dE w(M, E) \psi(\sqrt{M^2 + m^2 - 2ME}, h). \quad (5.5)$$

From the definition of the binomial moments Eq. (5.3) we have for $n \geq 2$

$$\varphi_n(M) = n \int_m^{E(M, M_p)} dE w(M, E) \varphi_{n-1}(\sqrt{M^2 + m^2 - 2ME}) + \int_m^{E(M, M_p)} dE w(M, E) \varphi_n(\sqrt{M^2 + m^2 - 2ME}). \quad (5.6)$$

The special case $n=1$ gives back Eq. (2.11) for the mean multiplicity. It can be seen from Eq. (5.6) that knowing the first moment $\varphi_1(M)$ one is able to calculate the second one $\varphi_2(M)$, from $\varphi_2(M)$ the third one $\varphi_3(M)$, etc.

Before discussing Eqs. (5.5), (5.6) let us note that the mean energy per particle in the final state (that is including also the proton at the end of the chain) can be simply expressed by the generating function ψ . The total energy in the final state (measured in the fireball rest frame) is M , therefore, the mean energy per particle $e(M)$ is given by

$$e(M) = \sum_{n=1}^{\infty} p_n(M) \frac{M}{n+1} = M \int_{-1}^0 dh \psi(M, h). \quad (5.7)$$

This quantity is similar but not exactly the same as $e_1(M)/e_0(M)$ defined in Section 2. [The latter would be difficult to define from $\psi(M, h)$.]

Let us now consider Eq. (5.5) after the splitting up due to Eq. (2.24):

$$\psi(M, h) = (1+h) \left(p_1(M) + \psi_R(M, h) \right) + (1+h) \int_m^{\infty} dE w_F(M, E) \psi(\sqrt{M^2 + m^2 - 2ME}) \quad (5.8)$$

where $\psi_R(M, h)$ is the contribution of the resonance region, that is, from Eqs. (2.25), (2.20):

$$\psi_R(M, h) = \sum_r w_r(M) \psi(M_r, h). \quad (5.9)$$

At asymptotically large M the probability for the direct emission of the resonances or of the ground state proton is exponentially decreasing, hence, it can be neglected. $w_F(M, E)$ tends at the same time to

$$w_{\infty}(M, E) = 4\pi \sqrt{E^2 - m^2} p_0(E) = Z_{\infty}^{-1} 4\pi \sqrt{E^2 - m^2} e^{-\frac{E}{T_0}}. \quad (5.10)$$

The asymptotic equation for ψ can be written down by similar arguments like those leading to Eq. (3.10):

$$\psi_{\infty}(M, h) = (1+h) \int_m^{\infty} dE w_{\infty}(E) \psi_{\infty}(M-E, h). \quad (5.11)$$

The solution of this equation goes as follows: dividing by $(1+h) \psi_{\infty}(M, h)$ one has

$$\frac{1}{1+h} = \int_m^{\infty} dE w_{\infty}(E) \frac{\psi_{\infty}(M-E, h)}{\psi_{\infty}(M, h)}. \quad (5.12)$$

Therefore,

$$\frac{\psi_{\infty}(M-E, h)}{\psi_{\infty}(M, h)}$$

is independent of M , that is

$$\psi_{\infty}(M, h) = e^{\frac{M}{E_{\infty}} C(h)} \quad (5.13)$$

with some function $C(h)$ depending only on h . Substituting back into Eq. (5.12) one obtains

$$\frac{1}{1+h} = \int_m^{\infty} dE w_{\infty}(E) e^{-\frac{E}{E_{\infty}} C(h)} \quad (5.14)$$

This is an equation determining the function $C(h)$. More explicitly, using Eq. (5.10)

$$\frac{1}{1+h} = \frac{1}{1 + \frac{T_0}{E_{\infty}} C(h)} \frac{K_1 \left[\frac{m}{T_0} \left(1 + \frac{T_0}{E_{\infty}} C(h) \right) \right]}{K_1 \left(\frac{m}{T_0} \right)} \quad (5.15)$$

This transcendent equation for $C(h)$ is still implicit enough but it is possible to conclude from it that $C(h)$ is negative everywhere in $-1 < h < 0$. Namely, if $C(h) = 0$ then, obviously $h = 0$. Inversely, if $h = 0$ then $C(h) = 0$ and $C(h)$ behaves as h , therefore $C(h)$ is negative (for $h < 0$) in the vicinity of $h = 0$.

The function $C(h)$ defines the usual correlation parameters C_{ℓ} by the equation

$$C(h) = \sum_{\ell=1}^{\infty} \frac{h^{\ell}}{\ell!} C_{\ell} \quad (5.16)$$

For a Poisson distribution (i.e., uncorrelated emission) $C(h) = h$ and, therefore, all of the correlation parameters C_{ℓ} ($\ell \geq 2$) vanish. To obtain a recursion formula for C_{ℓ} let us expand both sides of Eq. (5.14) in Taylor series at $h = 0$ and equate the coefficients of h^{ℓ} :

$$(-1)^{\ell} = \sum_{i_1+2i_2+\dots+j_i=j, j=l} \frac{1}{i_1! \dots i_j!} \left(\frac{C_1}{1!} \right)^{i_1} \dots \left(\frac{C_j}{j!} \right)^{i_j} \int_m^{\infty} dE w_{\infty}(E) \left(-\frac{E}{E_{\infty}} \right)^{i_1+\dots+i_j} \quad (5.17)$$

Putting for instance $l=1$ and 2 the parameters C_1 and C_2 can be determined:

$$C_1 = 1, \quad C_2 = \frac{m^2 + 6T_0^2 + 3mT_0 \frac{K_0}{K_1}}{(2T_0 + m \frac{K_0}{K_1})^2} - 2; \quad (5.18)$$

where $K_1 \equiv K_1(m/T_0)$. With our parameters the value of C_2 turns out to be $C_2 \sim -3/4$. In general, also the higher correlation parameters are different from zero showing that even for very large fireball masses the multiplicity distribution is different from the Poisson distribution. This can also be seen from Fig. 8 where the function $C(h)$ is given as determined numerically from Eq. (5.15). It should be noted, however, that perhaps the neglected three and more-body decays play a more important role in the multiplicity distributions than in single particle distributions and can change the values of the correlation parameters calculated here. But the discussion in Section 7.4 of Ref. 3) shows that the Poisson distribution can at most be a good approximation,

Finally, let us determine the asymptotic behaviour of the mean energy per particle $e(M)$. As for $h \neq 0$ $C(h)$ is negative in the interval $[-1, 0]$ the asymptotic behaviour of the integral (5.7) is determined by the vicinity of the point $h=0$, that is

$$e_\infty(M) = M \int_{-1}^0 dh \psi_0(M, h) = M \int_{-1}^0 dh e^{\frac{M}{E_\infty} C(h)} \sim M \int_{-1}^0 dh e^{\frac{M}{E_\infty} h} \sim E_\infty. \quad (5.19)$$

Therefore, $e(M)$ tends to $\bar{E}_\infty$ in the same way as $e_1(M)/e_0(M)$ does. This is because for $M \rightarrow \infty$ the mean number of emitted pions goes to infinity like M and, therefore, the energy carried away by the proton can be neglected.

6. CONCLUSIONS

The beautiful idea of thermodynamic bootstrap determines the hadronic mass spectrum for large masses. The decay of the fireballs depends on the spectrum of the hadronic states at disposal, therefore, it is also determined by the bootstrap. All these considerations are, however, asymptotic and the question where the asymptotic regime begins is depending on the way how the lower part of the hadron spectrum is joined to the "fireball region".

The experience in the thermodynamic model shows that the asymptotic regime begins at relatively low masses (say, about 2 GeV). This "early asymptotics" is something like the "early scaling" observed in deep inelastic electron proton scattering. (It is very possible that the two phenomena are intimately connected!) Here the "early asymptotics" is partly due to the fact that the prominent resonances are not intrinsically different from the "continuum fireballs", but, in fact, possess similar properties. On the other hand, the early setting in the asymptotic relations derived in Section 3, is due also to the dynamics of the decay which is mainly sensitive to the part of the spectrum immediately below the decaying state. The influence of the lower resonance states is, therefore, rapidly dying out as the mass increases. This phenomenon is showed by the numerical calculations in Section 4. Even the sudden switch on of the fireball region causes rapid variations in the single particle distribution function only in a narrow mass region above the threshold (in the order of several hundreds of MeV's). Above that region the corrections to the fireball mass in the asymptotic relations are slowly varying and close in magnitude to the "natural" value $-M_p$ expressing that the mass available for the decay is not the total M but $M - M_p$.

The limitations of the simple model considered in the paper are mainly technical. The inclusion of quantum numbers and three and more-body decays seems straightforward. The same methods of integral equations can be incorporated also in other models of particle production in which the excited hadron states produced in the collision decay statistically. Examples of such models are the diffraction excitation models of Adair ¹²⁾, Hwa ¹³⁾ and Jacob and Slansky ¹⁴⁾.

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FIGURE CAPTIONS

- Figure 1 :
The first step in the fireball decay.
- Figure 2 :
The pion momenta in the rest frame of fireball.
- Figure 3 :
The integration domain in Eq. (2.6).
- Figure 4 :
The mass correction term μ_n for the mean multiplicity.
- Figure 5 :
The mass correction term μ_e^ℓ for the moments defined in Eq. (2.18) with $\ell = 0, 1, 2, 3$.
- Figure 6 :
The average energy per pion $e_1(M)/e_0(M)$ as a function of the fireball mass M . The "effective temperature" T_{eff} is defined by $(\exp(E/T_{\text{eff}})-1)^{-1}$ instead of $\exp(-E/T_{\text{eff}})$ [see Eq. (4.9)].
- Figure 7 :
The mass correction term $\mu_d(M, E)$ for the single particle distribution.
- Figure 8 :
The function $C(h)$ as determined from Eq. (5.15).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig 5

Fig. 6

Fig. 7

Fig. 8