Archives

CM-P00058503

Ref.TH.1589-CERN

BOUNDS ON THE $K_{\ell 3}$ SCALAR FORM FACTOR
FROM LOW ENERGY $K\pi$ PHASE SHIFT *)

C. Bourrely **)
CERN - Geneva

A B S T R A C T

Upper and lower bounds on the scalar form factor of $K_{\ell 3}$ decay are calculated using the analytic properties of the form factor, the experimental low energy S wave $K\pi$ phase shift, and an estimate of the propagator $\Delta(t)$ of the divergence of the strangeness changing current at zero momentum. We release the condition given by the soft pion theorem on $f(m_K^2)$ that we considered in a previous work. For this reason the new bounds are less restrictive. However they are sufficiently strong to be inconsistent with some experimental data irrespective of the fit used for $K\pi$ scattering.

*) Work supported in part by the Centre National de la Recherche Scientifique.

**) On leave from the Centre de Physique Théorique, CNRS, 13274 - Marseille.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The K_{l3} decay process has received considerable interest during the last few years both from an experimental and theoretical point of view. The experimental situation at present is still rather unclear, since new data ¹⁾ on the "scalar" form factor appear to be incompatible with older data ²⁾. On the theoretical side much progress has been made since the soft pion theorem of Callan and Treiman ³⁾ was obtained. From a model of $SU(3) \times SU(3)$ symmetry breaking one has obtained an estimate of $\Delta(0)$, where $\Delta(t)$ is the propagator built with the divergence of the strangeness violating vector current. Combining this estimate with the soft pion theorem and analyticity - unitarity considerations, Okubo ^{4),5)} was able to derive a very stringent bound on the derivative of the scalar form factor at $t=0$. Recently we were able to generalize this method to get upper and lower bounds on $f(t)$ in the physical region ⁶⁾. From this calculation it appeared that $f(t)$ could only deviate very little from a linear fit going through the Callan-Treiman point, a fact which was incompatible with the existing experimental data at the time. This is why we tried to see what happens if we ignore the soft pion theorem but keep the estimate of $\Delta(0)$. Then one still gets a bound, but a very loose one. There is, however, one piece of information which has not been taken into account in this calculation and which is that, if one accepts the $\Delta I = \frac{1}{2}$ rule, the phase of the scalar form factor $f(t)$ for $(M_K + m_\pi)^2 < t < (M_K + 3m_\pi)^2$ is by Watson's theorem equal to the $I = \frac{1}{2}$, $J = 0$, $K\pi$ scattering phase. If one were to neglect inelastic intermediate states one could apply this condition to arbitrarily large energies and get for $f(t)$ an Omnès factor times a polynomial in t . This has been done in the past ⁷⁾, but it is not what we want to do here. Here we want to impose the phase conditions only where it can be applied, i.e., in the elastic region for the $K\pi$ S wave or possibly up to a somewhat higher energy for which inelasticity is experimentally known to be small. The high energy part is controlled, as in the previous calculations ⁶⁾ by the knowledge of $\Delta(0)$.

If one uses existing "phase shift analyses" of $K\pi \rightarrow K\pi$ based on extrapolations of $Kn \rightarrow Kn\pi$ cross-sections, one finds that these bounds are still very restrictive. In particular in the physical region $f(t)/f(0) > 0.98$ and, on the other hand, the Callan-Treiman point (which was not a priori imposed) coincides with the upper bound of $f(t)$ for $t = M_K^2$. Comparison with experiment shows that the allowed region lies more or less between the "old" and "new" experimental data.

The only doubtful assumption, in this calculation and in all previous ones, is that the propagator $\Delta(t)$ satisfies an unsubtracted dispersion relation. There is no difficulty in adapting these calculations to the case with subtractions. However, the problem is to know the subtraction constants one starts from. We believe that if one takes "reasonable" subtractions the results will not be very different. More specifically we believe that the results are less sensitive to $\Delta(0)$ and subtraction constants since the phase constraint is imposed.

In Section 2, making use of a conformal transformation and properties of Hilbert space of analytic functions, we derive an upper and a lower bound for the normalized scalar form factor, which are now $K\pi$ phase shift dependent.

We calculate numerical values of the bounds in Section 3, where we use as information:

- i) a bound on $\Delta(0)$ calculated from the chiral SW(3) model⁸⁾;
- ii) the experimental value of the $K\pi$ S wave phase shift^{9),10)}.

The point ii) suffers from the "up" and "down" ambiguity, these two solutions are taken into account in the calculation in order to compare their effect. We conclude in Section 4 by a discussion of the theoretical analysis made so far when compared with the experimental situation.

Before going into the derivation, we mention the basic formula of the K_{l3} decay problem. The K_{l3} form factors are defined in terms of the strangeness changing current $V_\mu^{(4-i5)}$

$$\langle \pi^0(p) | V_\mu^{(4-i5)}(0) | K^+(k) \rangle = -(\delta E_p E_k)^{-\frac{1}{2}} (2\pi)^3 [(k+p)_\mu f_+(t) + (k-p)_\mu f_-(t)]^{(1)}$$

the scalar form factor $f(t)$ proportional to the current divergence is related to the combination $f_\pm(t)$

$$f(t) = d(t)/(m_K^2 - m_\pi^2) = f_+(t) + t f_-(t)/(m_K^2 - m_\pi^2) \quad (2)$$

Here m_π and m_K are the pion and kaon masses, $d(t)$ is a real analytic function of t with a cut on the real axis $t_0 \leq t \leq \infty$, $t_0 = (m_\pi + m_K)^2$. The propagator of the strangeness changing vector divergence is defined as follows:

$$\Delta(t) = \frac{i}{2} \int d^4x e^{iqx} \langle 0 | (j_\mu V_\mu^{(4-i5)}(x), j_\lambda V_\lambda^{(4+i5)}(0))_+ | 0 \rangle \quad (3)$$

and we suppose that $\Delta(t)$ obeys an unsubtracted dispersion relation

$$\Delta(t) = \int_{t_0}^{\infty} \frac{\rho(t') dt'}{t' - t} \quad (4)$$

From the positivity of the spectral weight $\rho(t)$, Li and Pagels¹¹⁾ derived the inequality

$$\rho(t) \geq \frac{3}{64\pi^2} t^{-2} [(t-t_0)(t-t_1)]^{\frac{1}{2}} |d(t)|^2 \quad (5)$$

where $t_1 = (m_K - m_\pi)^2$.

Inserting (5) into (4) we obtain the basic inequality

$$\Delta(0) \geq \frac{3}{64\pi^2} \int_{t_0}^{\infty} dt t^{-2} [(t-t_0)(t-t_1)]^{\frac{1}{2}} |d(t)|^2 \quad (6)$$

2. DERIVATION OF THE BOUNDS

We suppose that the $K\pi$ scattering is elastic within a domain of energy $E = \{t_0, t_1\}$ and let $\delta(t)$ be the S wave phase shift, we define two functions $u(t)$ and $X(t)$ in the following way:

$$u(t) = \frac{(t-t_s)}{\pi} \int_{t_0}^{t_1} dt' \frac{\delta(t')}{(t'-t_s)(t'-t)} \quad (7)$$

$$X(t) = e^{u(t)/(t-t_1)} \quad (8)$$

$u(t)$ is once subtracted ($t_s < t_0$) in order to maintain the convergence in the hypothetical limit t_1 goes to infinity.

By construction $X(t)$ is holomorphic in the plane cut along E , with no zero in E , and it has the same phase (mod. π) as the form factor $f(t)$ in this domain. In the limit $t \rightarrow t_1$, $X(t)$ behaves like

$$A.(t-t_1)^{\frac{\delta(t_1)}{\pi}} - 1$$

we notice that in the domain of energy considered later $\delta(t_1) < \pi$; near the beginning of the cut t_0 , $\delta(t) \sim t-t_0$ and $X(t_0)$ is finite.

Now we consider the analytic function $g(t)$ related to the form factor $f(t)$ by

$$f(t) = X(t)g(t) \quad (9)$$

from the properties of $X(t)$ it follows that $g(t)$ is holomorphic in the cut plane $[t_1, \infty)$. In Eq. (8) we introduced a denominator $(t-t_1)$ to remove an unexpected zero of $f(t)$ at $t=t_1$; the zero comes from $\exp(u(t))$ and the fact that $\delta(t)$ is positive in the low energy region.

To the previous relations $d(t) = (m_K^2 - m_\pi^2)f(t) = (m_K^2 - m_\pi^2)X(t)g(t)$ we add a new definition

$$D(t) = (m_K^2 - m_\pi^2)g(t) \quad (10)$$

and decompose the basic inequality (6) into a sum of two integrals

$$\Delta(t) \geq \frac{3}{64\pi^2} \left\{ \int_{t_0}^{t_1} + \int_{t_1}^{\infty} \right\} (dt t^{-2} [(t-t_0)/(t-t_1')]^{\frac{1}{2}} |X(t)|^2 |D(t)|^2) \quad (11)$$

To derive the bounds we will apply a mathematical method developed in I, to this end we consider a mapping of the cut plane $[t_1, \infty)$ into the interior of the unit circle ($|z| < 1$)

$$(t-t_1)^{\frac{1}{2}} = i t_1^{\frac{1}{2}} (1-z)/(1+z)^{-1} \quad (12)$$

In terms of this conformal transformation, we set

$$G(z) \equiv D(t) \quad (13)$$

$G(z)$ is a real analytic function of z inside the unit circle. Next, we apply the conformal transformation to the left-hand side of inequality (11) and we get

$$\frac{3}{64\pi^2} \int_{t_0}^{t_1} dt t^{-2} [(t-t_0)/(t-t'_1)]^{\frac{1}{2}} |X(t) D(t)|^2 = \int_{x_0}^1 w(x) [G(x)]^2 dx \quad (14)$$

where

$$w(x) = \frac{3(1-b_0^2)^{\frac{1}{2}}(1-b_1^2)^{\frac{1}{2}}}{256\pi^2} \cdot \frac{(1-x)}{x^2(1+x)} \cdot (1-x_0x)^{\frac{1}{2}} \left(\frac{x}{x_0} - 1\right)^{\frac{1}{2}} \cdot (1-x_1x)^{\frac{1}{2}} \left(\frac{x}{x_1} - 1\right)^{\frac{1}{2}} |X(x)|^2 \quad (15)$$

$$b_i = \left(\frac{t_1 - t_i}{t_1} \right)^{\frac{1}{2}} \quad x_i = \frac{1 - b_i}{1 + b_i} \quad (16)$$

and

$$\frac{3}{64\pi^2} \int_{t_1}^{\infty} dt t^{-2} [(t-t_0)/(t-t'_1)]^{\frac{1}{2}} |X(t) D(t)|^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta p(e^{i\theta}) |G(e^{i\theta})|^2 \quad (17)$$

with $p(z) = \pi zw(z)$. The inequality (11) takes the form

$$\Delta(0) \geq \int_{x_0}^1 w(x) [G(x)]^2 + \frac{1}{2\pi} \int_0^{2\pi} d\theta p(e^{i\theta}) |G(e^{i\theta})|^2 \quad (18)$$

We need a further transformation on the right-hand side of (18) in order to write the integrals (14) and (17) as an inner product of analytic functions according to the definition

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta f^*(e^{i\theta}) g(e^{i\theta})$$

The integral (17) can be expressed as a norm of an analytic function in H^2 ; to that end we define a function $\psi_1(z)$ belonging to H^2 (see Appendix I, Refs. 12) and 13).

$$\psi_1(z) = \exp \left[\frac{1}{4\pi} \int_0^{2\pi} d\theta (e^{i\theta} + z)(e^{i\theta} - z)^{-1} \log p(\theta) \right] \quad (19)$$

with the property $|\psi_1(e^{i\theta})|^2 = p(\theta)$.

From the analyticity of $G(z)$ inside the unit circle and the existence of the integral (17) we conclude that the product function

$$h_1(z) = \psi_1(z) G(z) \quad (20)$$

belongs to H^2 ; then we have

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta p(e^{i\theta}) |G(e^{i\theta})|^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta |h_1(e^{i\theta})|^2 = \langle h_1, h_1 \rangle \quad (21)$$

The integral (14) has not a suitable form for writing an inner product; therefore, we transform the integral by the Schwarz inequality

$$\int_{x_0}^1 w(x) [G(x)]^2 dx \geq \left| \int_{x_0}^1 G(x) v(x) dx \right|^2 \left[\int_{x_0}^1 \frac{|v(x)|^2}{w(x)} dx \right]^{-1} \quad (22)$$

$v(x)$ is an arbitrary function of x to be determined later. Let us set:

$$w_1(x) = v(x) / \psi_1(x) \quad (23)$$

$$\varphi_1(z) = \int_{x_0}^1 \frac{w_1(x) x z^*}{1 - x z^*} dx \quad (24)$$

$$K = \int_{x_0}^1 \frac{|v(x)|^2}{w(x)} dx, \quad M = \int_{x_0}^1 w_1(x) dx \quad (25)$$

with these definitions, Ineq. (22) reads

$$\int_{x_0}^1 w(x) [G(x)]^2 dx \geq K^{-1} / \langle \varphi_1, h_1 \rangle + h_1(0) M / 2$$

Then the basic inequality (6) takes the form

$$S^2 \geq \Delta(0) \geq \kappa^{-1} \langle \varphi_2, h_2 \rangle + h_2(0) M / 2 + \langle h_2, h_2 \rangle \quad (26)$$

where S^2 is an estimated bound to $\Delta(0)$.

In order to calculate the bounds, additional inequalities must be added to (26). Consider the projectors 1 , $\varphi_1(z)$, $\phi_\lambda(z) = -z/(1 - \lambda^* z)$, $[\lambda \equiv \lambda(t)$ is a point inside the unit circle], we have, for instance $\langle 1, h_1 \rangle = h_1(0)$, $\langle \phi_\lambda, h_1 \rangle = (h_1(\lambda) - h_1(0))/\lambda$. If one takes the set of linearly independent functions $h_1(z)$, 1 , $\varphi_1(z)$, $\phi_\lambda(z)$, the norm of a vector made from a linear combination of the functions is positive definite.

This property is equivalent to the statement that a symmetric matrix $\mathcal{M}$ whose elements are all the inner products of such functions is positive definite

$$\mathcal{M} = \begin{vmatrix} \langle h_2, h_2 \rangle & h_2^*(0) & \langle h_2, \varphi_2 \rangle & \langle h_2, \phi_\lambda \rangle \\ h_2(0) & 1 & 0 & 0 \\ \langle \varphi_2, h_2 \rangle & 0 & \langle \varphi_2, \varphi_2 \rangle & \langle \varphi_2, \phi_\lambda \rangle \\ \langle \phi_\lambda, h_2 \rangle & 0 & \langle \phi_\lambda, \varphi_2 \rangle & \langle \phi_\lambda, \phi_\lambda \rangle \end{vmatrix} \quad (27)$$

The condition of positivity is fulfilled when all the determinants of the submatrices of $\mathcal{M}$ are positive (see I), the equality being possible when the functions are not linearly independent.

The calculation of the successive determinants of $\mathcal{M}$ combined with Ineq. (26) gives inequalities involving $h_1(0)$ and $\langle \varphi_1, h_1 \rangle$, solving the inequalities one gets bounds on such quantities, but, at present, we are mainly interested in the determinant of $\mathcal{M}$ which yields the following inequalities for the normalized form factor

$$B^+(t) \leq \frac{f(t)}{f(0)} \leq B^-(t) \quad (28)$$

$$B^\pm(t) = \frac{X(\lambda(t))}{X(0)} \cdot \frac{\psi_2(0)}{\psi_2(\lambda(t))} \left[1 - \frac{\lambda(t) Y_\lambda^\pm}{h_2(0)} \right] \quad (29)$$

$$Y_\lambda^\pm = [\langle \varphi_2, h_2 \rangle \langle \varphi_2, \phi_\lambda \rangle \pm \sqrt{\delta_2}] / \langle \varphi_2, \varphi_2 \rangle \quad (30)$$

$$\delta_1 = N_\lambda \det \mathcal{M}_2 \quad (31)$$

$$\begin{aligned} \det \mathcal{M}_2 = & S^2 \langle \varphi_2, \varphi_2 \rangle - (\langle \varphi_2, h_2 \rangle + h_2(0)M)^2 (\langle \varphi_2, \varphi_2 \rangle + K) / K \\ & + 2(\langle \varphi_2, h_2 \rangle + h_2(0)M) M h_2(0) - (\langle \varphi_2, \varphi_2 \rangle + M^2) (h_2(0))^2 \end{aligned} \quad (32)$$

$$N_\lambda = \langle \varphi_2, \varphi_2 \rangle \langle \phi_\lambda, \phi_\lambda \rangle - \langle \varphi_2, \phi_\lambda \rangle^2 \quad (33)$$

δ_1 is positive as a result of the positivity of the previous determinant ($\det \mathcal{M}_2$) and the positivity of N_λ (Schwarz inequality). It is easy to see from the expressions (29)-(33) that the inequalities (28) cannot be computed effectively because they involve unknown quantities like $\langle \varphi_1, h_1 \rangle$, and also $v(z)$ which is implicit in most of the integrals, the other quantities such as $f_+(0)$, S^2 , t_1 , $\delta(t)$ are generally treated as known parameters.

The expression $\langle \varphi_1, h_1 \rangle$ is unknown and only restricted by $\det \mathcal{H}_2 \geq 0$. Nevertheless, the dependence of $B^\pm(t)$ on the inner product $\langle \varphi_1, h_1 \rangle$ appears only through $Y_\lambda^\pm$, and in consequence we can maximize $B^+(t)$ and minimize $B^-(t)$ with respect to $\langle \varphi_1, h_1 \rangle$.

As a function of $\langle \varphi_1, h_1 \rangle$ the $Y_\lambda^\pm$ describe a part of an ellipse whose axes are rotated from the original axes. It is not difficult to show that Y_λ^+ has a maximum value and Y_λ^- a minimum value independent of $\langle \varphi_1, h_1 \rangle$. The expression of the maximum and the minimum are:

$$Y_\lambda^{0\pm} = h_2(0) \left[\frac{-\langle \varphi_2, \phi_2 \rangle M}{\langle \varphi_2, \varphi_2 \rangle + K} \pm \sqrt{\delta_2} \right] \quad (34)$$

$$\delta_2 = \left[\langle \phi_2, \phi_2 \rangle - \frac{\langle \varphi_2, \phi_2 \rangle^2}{\langle \varphi_2, \varphi_2 \rangle + K} \right] \left[\left(\frac{S^2}{h_2^2(0)} - 1 \right) - \frac{M^2}{\langle \varphi_2, \varphi_2 \rangle + K} \right] \quad (35)$$

The corresponding bounds $B^{0\pm}(t)$ follow immediately from the replacement of $Y_\lambda^\pm$ by $Y_\lambda^{0\pm}$ into $B^\pm(t)$, then we have:

$$B^{0+}(t) \leq \frac{f(t)}{f(0)} \leq B^{0-}(t) \quad (36)$$

The function $v(x)$ that we introduced in the Schwarz inequality (22) appears in the results through φ_1 and K ; in principle this function could be freely chosen to optimize inequality (22) or to find an optimal value of $Y_\lambda^\pm$ in order to get the most constraining bounds. We have not tried to do so because the best one could hope is to reduce the problem to some integral equation which remains to be solved. Besides, numerical calculations have, as we shall see, shown that the results are not very sensitive to $v(x)$. In the opposite direction one could try to find a bound valid for any $v(x)$. This can be done, but then one finds back the result obtained when the phase information is ignored.

3. NUMERICAL RESULTS

The numerical computation of the bounds (36) involves different assumptions that we outlined in Section 1 and Section 2. Now let us give their precise content used in the calculation.

The main feature of the derivation is the introduction of the $K\pi$ phase shift, and in order to get results which are less dependent on the theory we take the values of the phase from experiment or some fitted parametrization. We see by inspection of the formula (A.1.6) that the phase appears in ψ_1 through a non-singular integral taken over a finite range of energy, the upper limit of integration being fixed by the inelasticity effect. We use a simple parametrization as follows ^{9c)}

$$\cot(\delta_0^1)_{down} = q^{-1} \left(\frac{1}{a} + \frac{1}{2} r q^2 \right) \quad (37)$$

$$(\delta_0^1)_{up} = (\delta_0^1)_{down} + (\delta_0^1)_{res} \quad (38)$$

where q is the meson momentum in the $K\pi$ rest frame, and $(\delta_0^1)_{res}$ is given by an S wave Breit-Wigner distribution

$$\tan(\delta_0^1)_{res} = \frac{m_s \Gamma_s [(t-t_0)(t-t_1')]^{\frac{1}{2}}}{2 q_s \sqrt{t} (m_s^2 - t)} \quad (39)$$

The "up" and "down" correspond to the ambiguity of the two possible solutions for the determination of the phase shift from experiment; the value of the parameters with a comment on their validity will be found in Ref. 9c).

The other input is S^2 , an estimated bound to $\Delta(0)$, (26). We know that different evaluations exist ^{4),8),11)}, namely $S = 1.01 m_{\pi} f_{\pi}$, also

$$S = (m_K f_K - m_{\pi} f_{\pi}) / \sqrt{2} \quad (40)$$

We retain the last expression although it probably gives an over-estimate for $\Delta(0)$, but this choice corresponds to weakening the theoretical assumption.

Finally, concerning the arbitrary function $v(x)$ we have chosen

$$v(x) = \frac{\psi_1(x) P_n(x)}{x} \quad (41)$$

where $\psi_1(x)$ is defined through (19) and (A.1.3), $P_n(x)$ is a polynomial of degree n . In practice we use a linear form $P_1(x) = 1 + a_1 x$, (a_1 is an arbitrary real number). With this form the inner products involved in the bounds are expressible in terms of elementary functions (see Appendix II).

The numerical value of the bounds is presented in the Figure with the following set of parameters: $t_1 = (m_K + 3m_\pi)^2$, S^2 given by Eq. (40), $a_1 = 1$, $f_+(0) = 1$ ^{*)}. The difference between the up and down solutions cannot be seen on the plot, this results from the very small difference of the two solutions up to the first inelastic threshold.

Now we examine the influence of the different inputs on the bounds. The effect of the phase shift can be roughly understood by the fact that the more information on the phase shift we put into the calculation, the more refined bounds we get. For instance, if we believe that inelasticity is negligible up to the second inelastic threshold, then for a fixed value of t the distance between the lower and the upper bound decrease, and also, the bounds corresponding to each type of solution separate; we want to point out that all these effects are small in the narrow energy range considered for the phase. Although Eq. (40) is an overestimate of the parameter S , we can take it as being a free parameter. Then we observe the same effect that we reported in I, where any increase of its value produces an enlargement of the domain between the bounds, we note that such an effect is less sensitive here.

The linear dimensionless coefficient a_1 gives a negligible effect within the domain $|a_1| \leq 1000$, moreover, we see that the homogeneity of the numerator and denominator of $Y_\lambda^{0\pm}$ (Eq. 34) with respect to a_1 yields anyway a finite limit when $|a_1| \rightarrow \infty$, and in the limit the effect is small.

The last parameter $f_+(0)$ is the value of the vector form factor at the origin. The dependence on this parameter is small in the interval $0.5 \leq f_+(0) \leq 1.5$, i.e., including $f_+(0) = 1$ which corresponds to a SU(3) prediction valid to second order in the symmetry breaking.

*) For a general angular momentum the inelastic threshold is at $t = (m_K + 2m_\pi)^2$, but for $J = 0$ it is at $t = (m_K + 3m_\pi)^2$.

The dependence of the normalized bounds on $f_+(0)$ appears only through $S^2/h_1^2(0)$, Eq. (35), this quantity increases to a finite limit when $f_+(0) \rightarrow \infty$, and in this limit we have checked that the absolute variation of each bound compared to the case $f_+(0) = 0.5$ is less than 0.15.

In the present analysis, we would like to mention that phenomenological bounds for $\xi(t) = f_-(t)/f_+(t)$ are obtained by a combination of the bounds (36) and some experimental value of $f_+(t)/f_+(0)$. It follows from a simple calculation that

$$A^+(t) \leq \xi(t) \leq A^-(t)$$

$$A^\pm(t) = \frac{m_K^2 - m_\pi^2}{t} \left[\frac{f_+(0)}{f_+(t)} B^{0\pm}(t) - 1 \right] \quad (42)$$

Taking for $f_+(t)/f_+(0)$ a quadratic fit to experiment ¹⁴⁾, and the bounds plotted in the Figure, we get bounds close to zero, more precisely we can summarize the result for any t in the decay region by the inequalities

$$-8.6 \cdot 10^{-3} \leq \xi(t) \leq 1.10^{-3}$$

4. DISCUSSION AND CONCLUSION

The new bounds derived in this work put less constraint on the scalar form factor than the ones obtained previously and in which the soft pion theorem was taken into account. However, they are very restrictive since at $t = m_K^2$ the upper bound obtained is just above the Callan-Treiman value while the lower bound of $f(t)/f(0)$ is extremely close to unity.

For a given $\Delta(0)$ the bounds become tighter as we increase t_1 the energy limit at which inelasticity in $K\pi$ scattering can be neglected. We know that strictly speaking $t_1 = (m_K + 3m_\pi)^2$, but it is an experimental fact that inelasticity becomes important at appreciably higher energy. Clearly, as one increases t_1 , for a given model of $K\pi$ phase shift, the gap between the upper and lower bounds will get narrower.

It is very difficult, at the present time, to make a comparison of these bounds with experiment since the most recent (preliminary) experiment ¹⁾ (open circles on the Figure) is statistically incompatible with the previous ones ²⁾ (black circles). The domain allowed by our bounds falls more or less between the two sets of experiments. However, neither set is compatible with our bounds.

It is therefore very important to follow the evolution of the experimental situation very closely. If it turns out that experiment is in definite disagreement with the bounds we may try to see where to put the blame. First, one might increase $\Delta(0)$. However, we know by a numerical investigation that the bounds are not very sensitive to $\Delta(0)$. Modifications in the parametrization of the phase shift do not produce big changes. The weakest point in the obtention of the bounds is undoubtedly the assumption that $\Delta(t)$ satisfies an unsubtracted dispersion relation ¹⁵⁾.

One should re-examine this problem, assuming a finite number of subtractions, for $\Delta(t)$. We know from light cone arguments ^{16),17)} that two subtractions could be necessary. Then there is no mathematical difficulty to extend the treatment that has been presented here. The physical difficulty is to estimate the subtraction constants. A reasonable postulate would be to choose subtraction constants which do not spoil the "pole dominance" in the low energy region.

To end up this section we would like to mention an approach to the knowledge of $\Delta(t)$ when subtractions are required. Let us suppose that two subtractions are necessary, in that case we can try to fit the data with the two subtraction points and a quantity which involves the subtraction constants [see I, Eq. (A.11)]. Another possibility is to find inequalities on $\Delta(t)$ by solving the original bounds on the form factor $f(t)$ with respect to $\Delta(t)$ ¹⁸⁾ [once some approximate expression for $f(t)$ is available]. If we define $D(t) = \Delta(t) - \Delta(0)$, we can prove from the positivity of the spectral weight that $D(t)$ is a non-decreasing function of t below the cut t_0 . More precisely let t_{i+1} and t_i be two subtraction points $t_0 > t_{i+1} \geq t_i \geq 0$, we derive from the bounds (36) the inequality

$$D(t_{i+1})/t_{i+1} \geq P(t_{i+1}, t_i) + D(t_i)/t_i \quad (43)$$

$P(t_{i+1}, t_i)$ being a known non-negative function [dependent on $f(t)$].

In conclusion, the bounds that we derived without the assumption of the Callan-Treiman point show that they are compatible with this point (and the soft pion theorem). Now we have to deal with the last theoretical assumption due to the propagator of the divergence of the strangeness changing current, a solution to this problem would give bounds strongly constraining.

ACKNOWLEDGMENTS

I am grateful to Professor A. Martin for suggesting this study and stimulating discussions throughout the course of this work. It is a pleasure to thank Dr. M.K. Gaillard for valuable comments when reading the manuscript.

APPENDIX I

We want to derive an explicit expression of $\psi_1(z)$ defined by Eq. (19). Let $\alpha(\theta)$ be a non-negative summable function of θ on the unit circle, we suppose also that $\log \alpha(\theta)$ is summable, and we define a function $\psi_1(z)$ by the relation

$$\psi_1(z) = \exp \left[\frac{1}{4\pi} \int_0^{2\pi} d\theta \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \alpha(\theta) \right] \quad (\text{A.1.1})$$

this function belongs to H^2 and is outer^{12),13)}; it satisfies almost everywhere on the circle the boundary value condition

$$|\psi_1(e^{i\theta})|^2 = \alpha(\theta)$$

Now, we consider the function defined through (15) and (17)

$$p(z) = \frac{3}{256\pi} \frac{(1-b_0^2)^{\frac{1}{2}}(1-b_1^2)^{\frac{1}{2}}(1-z)}{z(1+z)} \left[\frac{(1-x_0z)(1-x_1z)(\frac{z}{x_0}-1)(\frac{z}{x_1}-1)}{(1+z)} \right]^{\frac{1}{2}} |X(z)|^2 \quad (\text{A.1.2})$$

[although $p(\theta)$ may have singularities at $\theta=0$ and π we can cope with this difficulty⁴⁾].

The expression (A.1.1) can be calculated from (A.1.2) and the formulas (7), (8)

$$\psi_1(z) = \exp \left[\frac{1}{4\pi} \int_0^{2\pi} d\theta \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |p(\theta)| \right] = \exp [H_0(z) + H_1(z)] \quad (\text{A.1.3})$$

where $H_0(z)$ corresponds to the case $\delta=0$ [i.e., $X(z)=1$].

$$\exp(H_0(z)) = \sqrt{\frac{3}{\pi}} \frac{[(1+b_0)(1+b_1)]^{\frac{1}{2}}}{16} \left[\frac{(1-z)(1-2x_0)(1-2x_1)}{(1+z)} \right]^{\frac{1}{2}} \quad (\text{A.1.4})$$

The term $H_1(z)$ can be calculated after a careful interchange of the order of integration, and it yields

$$\exp(H_1(z)) = \exp [u(z) - H(z)] / (t_2 - t) \quad (\text{A.1.5})$$

$$H(z) = \frac{1-z^2}{\pi} \int_{t_0}^{t_2} \frac{dt' f(t')}{t'(1+z)^2 - 4zt_2} \sqrt{\frac{t_2}{t_2-t'}} \quad (\text{A.1.6})$$

For example it follows that the quantity [see (29)]

$$\frac{X(\lambda(t))}{X(0)} \cdot \frac{\psi_2(0)}{\psi_2(\lambda(t))} = \exp \left[H_0(0) - H_0(\lambda(t)) - H(0) + H(\lambda(t)) \right] \quad (\text{A.1.7})$$

APPENDIX II

We give the explicit expression of all the inner products involved in the numerical calculation of the bounds (34)-(36), the functions $v(x)$, $w_1(x)$ and $\psi_1(x)$ are given by the formulas (41), (23) and (A.1.3).

$$\langle \varphi_1, \phi_1 \rangle = \int_{x_0}^1 dx \phi_1(x) w_1(x) = \left[\left(1 + \frac{a_1}{\lambda}\right) \log \left(\frac{1-\lambda}{1-\lambda x_0} \right) + a_1(1-x_0) \right] / \lambda$$

$$\langle \phi_1, \phi_1 \rangle = (1 - |\lambda|^2)^{-1}$$

where $\lambda \equiv \lambda(t)$ is a point inside the unit circle.

$$M = \int_{x_0}^1 w_1(x) dx = -\log x_0 + a_1(1-x_0)$$

$$\begin{aligned} \langle \varphi_1, \varphi_1 \rangle &= \int_{x_0}^1 dx \int_{x_0}^1 dx' \frac{w_1(x) x w_1(x') x'}{1 - x x'} \\ &= V(x_0) - a_1 \left[a_1(1-x_0^2) + (1-x_0) \log x_0 - a_1 V(x_0) \right. \\ &\quad \left. + \left(3 + \frac{2}{x_0}\right) (1-x_0) \log(1-x_0) - \frac{2(1-x_0)^2}{x_0} \log(1-x_0^2) \right] \end{aligned}$$

$$V(x_0) = \frac{\pi^2}{6} + \text{Li}_2(x_0^2) - 2 \text{Li}_2(x_0)$$

$\text{Li}_2(x)$ is the dilogarithm function.

The constant K cannot be expressed in terms of elementary functions

$$K = \pi \sqrt{x_0 x_1} \int_{x_0}^1 dx \left[\frac{\left(\frac{1}{x_0} - x\right) \left(\frac{1}{x_1} - x\right)}{(x-x_0)(x-x_1)} \right]^{\frac{1}{2}} (1+a_1 x)^2 \exp[-2H(x)]$$

where $H(x)$ is defined by (A.1.6).

R E F E R E N C E S

- 1) G. Donaldson et al., Preliminary report submitted to the XVI International Conference on High Energy Physics, Chicago (1972).
- 2) See a compilation of data contained in Refs. 14) and 19).
Once this work was achieved we became aware of the paper
E. Dally et al., Phys.Letters 41B (1972), 647.
- 3) C. Callan and S. Treiman, Phys.Rev.Letters 16 (1966), 153.
- 4) S. Okubo and I-Fu Shih, Phys.Rev. D4 (1971), 2020;
S. Okubo, Phys.Rev. D4 (1971), 725;
S. Okubo and I-Fu Shih, Phys.Rev. D4 (1971), 3519.
- 5) See also:
G.J. Aubrecht, D.M. Scott, K. Tanaka and R. Torgerson, Phys.Rev. D4
(1971), 1423;
M. Micu and E.E. Radescu, Phys.Rev. 6D (1972), 1943;
M. Micu, Nuclear Phys. B44 (1972), 531.
- 6) C. Bourrely, Nuclear Phys. B43 (1972), 434 (referred to as I).
- 7) For a review see Ref. 14) and Ref. 15).
- 8) V.S. Mathur and S. Okubo, Phys.Rev. D1 (1970), 3468.
- 9) a) R. Mercer et al., Nuclear Phys. B32 (1971), 381;
b) H. Yuta et al., Phys.Rev.Letters 26 (1971), 1502;
c) H.H. Bingham et al., Nuclear Phys. B41 (1972), 1.
- 10) For a review see:
T.G. Trippe, LBL-763 (1971), preprint.
- 11) L.F. Li and H. Pagels, Phys.Rev. D3 (1971), 2192; D4 (1971), 255.
- 12) K. Hoffman, Banach spaces of analytic functions (Prentice Hall,
Englewood Cliffs, N.J., 1962).
- 13) H. Helson, Lectures on invariant subspaces (Academic Press, New York,
1964).
- 14) L.M. Chounet, J.M. Gaillard and M.K. Gaillard, Leptonic decays of
hadrons, Physics Reports 4C (1972), 199.
- 15) M.K. Gaillard, talk presented at the winter meeting of the American
Physical Society, Cambridge, Mass., 27-29 Dec. 1971.
- 16) D.T. Cornwell and D. Eimerl, DESY preprint 1972, Nuclear Phys. to be
published.

- 17) B. Renner, private communication.
- 18) With the bounds derived in Ref. 6), the problem has been investigated in case of no subtraction, Ref. 16), and with two subtractions by M.K. Gaillard (private communication).

FIGURE CAPTION

Bounds on the normalized scalar form factor, solid lines. Previous bounds of Ref. 6), dashed lines. Experimental values, full circles, Ref. 1), open circles, Ref. 2). Callan-Treiman point, cross. Bounds without $K\pi$ phase shift nor Callan-Treiman point, dash-dotted lines.

FIG.1