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AN EXPLICIT SOLUTION OF THE STATISTICAL BOOTSTRAP

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A B S T R A C T

An explicit solution of the statistical bootstrap proposed by Frautschi is exhibited and discussed. The associated density of states is shown to be linked to the solution of a one-dimensional partition problem, arising from the infinite compositeness of hadrons in the statistical world. A configuration space interpretation of the compositeness condition is given, and the possibility that a field-theoretic analogue of such a system exists is raised in the context of non-polynomial Lagrangians.

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1. - INTRODUCTION

In the following note we present an explicit solution of the statistical bootstrap system proposed some time ago by Frautschi ¹⁾. The sole dynamical quantity in that system is the density of (hadronic) states per unit (mass)², which will be referred to here as $\tau(M^2)$. Our solution for $\tau(M^2)$ has the form

$$\tau(M^2) = d_1 \delta(M^2 - M_0^2) + \sum_{n=2}^{\infty} d_n \Theta(M - n M_0) \rho^{(n)}(M; M_0, M_0, \dots, M_0) \quad (1.1)$$

where $\rho^{(n)}$ is the covariant (kinematic) density of states in energy-momentum for the decay of a mass M object into n mass M_0 objects

$$\rho^{(n)}(M; M_0, \dots, M_0) = \frac{1}{4\pi} \int \prod_{j=1}^n d^3 \vec{p}_j / E_j \delta^4(P - \sum_i p_i) \quad (1.2)$$

and the d_n are positive definite numbers, fixed, in terms of the "degeneracy" d_1 of the lowest state, by a compositeness condition which will be specified below. Their positivity will make it straightforward for us to give both an iterative rule for explicitly calculating the d 's, and also their asymptotic form for large n . The d 's can be thought of as the solutions to a certain counting or partition problem most naturally analyzed in terms of the properties of the associated generating function

$$r(t) = \sum_{n=1}^{\infty} d_n t^n \quad (1.3)$$

This counting problem has been previously discussed by Nahm ²⁾, who Laplace transformed Frautschi's bootstrap equation and investigated its analytic properties in the complex temperature plane. As we shall see, the series (1.3) has a finite radius of convergence which may be interpreted as arising from a non-locality associated with the lowest mass M_0 .

From the statistical point of view the state density (1.1) may be thought of as an average over all external and internal hadronic quantum numbers, with general requirements such as unitarity and crossing ignored. On the other hand, (1.1) has the same general structure as various familiar objects from scalar field theory. If one likes one may loosely identify τ , for example, with the imaginary part of the mass operator $(s - D^{-1}(s)) = Y(s)$. Exhibiting the two-particle contribution only we have the standard result ³⁾

$$\text{Im } Y = \rho^{(2)}(s) |\Gamma(s)|^2 + \dots \quad (1.4)$$

where Γ is the sum of all proper vertex diagrams, and $D(s)$ the renormalized propagator, all in, say, a $\lambda\phi^3$ field theory. If one takes the analogy between (1.1) and (1.4) seriously, then in some average sense the d_n are to be identified with the square moduli of the 1 to n particle transition amplitudes which appear in $\text{Im } Y$. In fact, as we shall see below, the compositeness condition forces $\tau(s)$ to grow faster than any power, for large s , and therefore any such identification must take place in the context of field theories of a highly singular type ⁴⁾. Whether the connection between (1.1) and (1.4) is more than mere bookkeeping is an open question whose significance will be further discussed in Section 4 below.

The organization of this paper is as follows. In Section 2 we introduce and solve the compositeness condition in the context of a fictitious world with one timelike dimension, at the same time giving a configuration space interpretation of the underlying physics. In Section 3 we extend the considerations of Section 2 to the ordinary four-dimensional case, using as a tool certain simple relations, between causal propagators, familiar from work on non-polynomial Lagrangian field theory. In Section 4 we conclude with some further observations regarding solutions of the form (1.1), contrasting the expressions obtained here with their opposite numbers in factorized narrow resonance models, and suggesting several avenues for further investigation.

2. - STATISTICAL BOOTSTRAP OF A ONE-DIMENSIONAL WORLD

We will write Frautschi's bootstrap constraint on the density of states ⁵⁾ in the form

$$\tau(M^2) = \tau_0(M^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \prod_{j=1}^n \int \tau(m_j^2) dm_j^2 \rho^{(n)}(M; m_1, \dots, m_n) \quad (2.1)$$

where $\tau_0(M^2)$ is the lowest input, $d_1 \delta(M^2-1)$, and the n fold multiple integral runs over the region

$$R: m_j \geq 1 \ ; \ m_1 + m_2 + \dots + m_m \leq M \quad (2.2)$$

(Above, and henceforward, we set the lowest mass M_0 , which fixes the energy scale of our system, equal to 1.)

The violently non-linear, Volterra integral equation (2.1) expresses the compositeness constraint. It states that the chance of discovering a state at mass M is the sum over the chance of finding any number of states at mass M .

Nothing in (2.1) restricts the metric of the space we work in, or the number of dimensions. To make the content of the compositeness condition transparent we are therefore free to choose the simplest case : a world with only one timelike dimension. In such a space, the appropriate (kinematic) state density is clearly

$$\rho^{(n)}(M; m_1, \dots, m_m) = \delta(M - m_1 - \dots - m_m) \quad (2.3)$$

Using (2.3), we immediately suspect (2.1) has a solution of the form

$$\sum_{n=1}^{\infty} c_n \delta(M-m) \quad (2.4)$$

To confirm our suspicions we transform the one-dimensional form of (2.1) to configuration space. Using (2.4) we then have

$$\sum_{n=1}^{\infty} c_n e^{imx_0} = d_1 e^{ix_0} + \sum_{n=2}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} c_k e^{ikx_0} \right]^n \quad (2.5)$$

which clearly has an iterative solution, with c_1 equal to d_1 , and c_n calculable in terms of c_{n-1} , c_{n-2} , $\dots$ c_1 .

If a physical interpretation of the compositeness condition is desired, we may note that, modulo an uninteresting factor of m , the probability of finding a positive mass state at time x_0 , is just e^{imx_0} , so

that (2.5) relates collections of such "elementary" states, and fixes the "strengths" of the elementary components. Explicitly, ignoring normalization factors, the probability of finding a positive energy state of definite mass propagating along space-time direction x , in N dimensions, is just the familiar expression

$$D^{(+)}(x^2, m^2) = \int d^N p \, e^{ipx} \delta(p^2 - m^2) \Theta(p_0) \quad (2.6)$$

which reduces, modulo a constant, to e^{imx_0} for our one-dimensional case. For the purpose of extracting the content of (2.5) it is convenient to introduce a new variable t to replace e^{ix_0} , so that (2.5) may be written as

$$H(r(t), t) = e^r - 2r - 1 + d_1 t = 0. \quad (2.7)$$

As is explicitly shown in the Appendix below, the function $r(t)$ defined by (2.7) has the following properties :

- (a) its power series about $t = 0$ has a finite radius of convergence

$$R = (\log 4 - 1)/d_1 \quad (2.8)$$

- (b) near R , $r(t)$ behaves like $(R-t)^{\frac{1}{2}}$;
 (c) from (a) and (b), the coefficients c_K in (2.5) have asymptotic behaviour

$$c_K \underset{K \rightarrow \infty}{\sim} \exp(-K \log R) K^{-3/2} (8d_1 R \pi)^{-1/2} \quad (2.9)$$

The exponential in (2.9) is symptomatic of the finite radius of convergence, while the $K^{-\frac{3}{2}}$ results from the square root branch.

We will see in the following section that this reasoning can be taken over essentially unchanged in treating the four-dimensional case.

3. - THE STATISTICAL BOOTSTRAP IN FOUR DIMENSIONS

We now wish to generalize the reasoning of Section 2 to include the existence of space-like dimensions. We will in fact reduce the (N dimensional) problem to the one-dimensional one by introducing an invariant auxiliary function which is related to the (N dimensional) kinematic phase space in the same way that e^{imx_0} is related to the delta function (2.3). To accomplish this we utilize a slight generalization of an integral representation for integer powers of the scalar, causal, free particle Green's function, well known to those who deal with non-renormalizable field theories ⁶⁾. We have

$$\prod_{j=1}^n (-iD^c(x^2; m_j^2)) = \int_{s_0=(m_1+\dots+m_n)^2}^{\infty} ds g^{(n)}(\sqrt{s}; m_1, \dots, m_n) (-iD^c(x^2; s)) \quad (3.1)$$

Using (3.1), (2.1) remains valid if we remove $\rho^{(n)}$, integrate over M^2 , while multiplying each $\tau(y^2)$ by $-iD^c(x^2; y^2)$. Defining

$$D(x^2) = \int_1^{\infty} dm^2 D^c(x^2; m^2) \tau(m^2) \quad (3.2)$$

(2.1) then becomes, suppressing the x dependence of D and absorbing $-i$'s,

$$D = D_0 + \exp(D) - D - 1 \quad (3.3)$$

where $D_0 = D^c(x^2; 1)d_1$.

If, for the sake of aiding our intuition, we identify $\tau(M^2)$ with the Lehmann spectral function in scalar field theory ⁷⁾, then $D(x^2)$ in (3.2) is the renormalized propagator, and (2.1), now transformed to (3.3), states that the probability of finding a (physical) particle propagating along space-time direction x_β is the sum of the probability that 2, 3, 4... such particles do so, plus the probability that a single free particle of mass M_0 does also. We will pursue the field theory analogy no further, except to remark that relations of the form (3.3) may possibly arise from self-consistency conditions on interaction Lagrangians of the form Ge^{ϕ} , applied for lowest order in G , and to all orders in g . If they exist, such conditions would be the analogue of $Z = 0$ compositeness conditions in renormalizable field theories ^{3), 8)}.

Let us now solve (3.3). If we rewrite it in the form used in Section 2, we have $H(D, D_0) = 0$, and D may be expressed as a formal power series in D_0 ,

$$D = \sum_{k=1}^{\infty} (D_0)^k C_k ; C_1 = 1 \quad (3.4)$$

Disregarding any mathematical difficulties involved in multiplying singular distributions such as D_0 ⁹⁾, we can then equate (3.4) and (3.2) to obtain

$$\int_1^{\infty} dm^2 D^c(x^2; m^2) \tau(m^2) = \sum_{k=2}^{\infty} d_k \int_{s_0}^{\infty} ds \rho^{(k)}(\sqrt{s}; 1, \dots, 1) D^c(x^2; s) + d_1 D^c(x^2; 1) \quad (3.5)$$

To derive (3.5) we have used $c_k = d_k/d_1^k$, and have also employed (3.1) to evaluate $(D_0)^k$. The solution of (3.5) is clearly (1.1).

To conclude the calculation we would like to evaluate $\tau(x)$ for large x . Using the results above, we rewrite (1.1) in integral form

$$\tau(M^2) \cong \int_1^M dm \exp(m \log(d_1/R)) \rho^{(n)}(M; 1, \dots, 1) (8R\pi m^3)^{-1/2} \quad (3.6)$$

and must now look for the appropriate approximation for

$$\rho^{(n)}(M) = \frac{1}{4\pi} \int \prod_{j=1}^n d^3 \vec{p}_j / E_j \delta^4(P - \sum_i p_i); P^2 = M^2, p_i^2 = 1.$$

Consider first $d_1/R \ll 1$. For large M we can clearly take the highly relativistic limit of $\rho^{(n)}$, since the important pieces of the integrand will have $M \gg n$. In that case

$$\rho^{(n)}(M) \underset{[p_i^2 \rightarrow 0]}{\sim} \frac{1}{2} (\pi M^2)^{n-2} / [\Gamma(n) \Gamma(n-1)] \quad (3.7)$$

Inserting (3.7) in (3.6), and noting that the resulting integrand peaks for $n \cong (d_1 \pi / R)^{\frac{1}{2}} M$, we get

$$\tau(M^2) \cong \frac{\sqrt{d_1/R}}{2M^3 (8\pi R)^{1/2}} \exp[2M\sqrt{\pi d_1 R}] ; d_1 \pi / R \ll 1 \quad (3.8)$$

Next take $d_1/R \gg 1$, where high n terms in (3.6) must now be considered, with n of order M , where the edge of phase space plays a significant role. Since $n \gg 1$, we can use an approximate expression for $p^{(n)}$ given by Kolkunov ¹⁰⁾

$$\pi p^{(n)}(M; 1, \dots, 1) \cong (4\pi M^2 \alpha / \beta)^{n-2} \alpha^{-1/2} \exp(\beta/\alpha) M^{-n} K_1^n(\beta/M\alpha) \quad (3.9)$$

where $\alpha \equiv 1-n/M$ and $\beta \equiv 3(n-1)/2$. The approximation (3.9) holds for the (non-relativistic) regime $\alpha \ll 1$, where the mass of any particular particle is much greater than the total available kinetic energy. We would therefore expect that (3.9) makes sense when $d_1/R \gg 1$. For small α (and n/M of order 1) we can replace the K functions in (3.9) by their asymptotic values, so that ¹¹⁾

$$p^{(n)}(M) \cong e^\beta (8\pi\beta M^3)^{-1/2} (2\pi M\alpha/\beta)^{\beta-1} \quad (3.10)$$

Inserting (3.10) in (3.6) and defining $L \equiv \beta/M\alpha$, $K \equiv (2\pi)^{\frac{3}{2}}$. d_1/R , we find the integrand peaks near $K = e^L L^{\frac{3}{2}}$, or $L = \log K + O(\log \log K)$. Using the standard saddle-point method, the integral becomes

$$\tau(M^2) \cong \frac{e^{M \log K}}{32\pi^3 M^3 R^{1/2}} ; K \gg 1 \quad (3.11)$$

where we have dropped $M \log \log K$ corrections to the exponential and also lower powers of M . The appearance of the leading M^{-3} power in (3.11) and (3.8) has interesting consequences which have recently been discussed in detail by Carlitz ¹²⁾. Presumably the general $e^{aM} M^{-3}$ behaviour of $\tau(M^2)$ occurs for all d_1/R , including $d_1/R = O(1)$. We have not been able to find a clean proof that this is correct, and leave this problem as an exercise for the reader.

4. - CONCLUSION

We conclude with several observations regarding the bootstrap solutions obtained above, followed by suggestions for further work along the present lines.

A. -

The solution arising from (2.1) and (3.3) has a highly unsatisfactory asymmetry, in that there is a lowest state, represented by the inhomogeneous term τ_0 , which must be put in by hand. In the interpretation of Frautschi and Hamer ⁵⁾, this is a manifestation of the absence of a true interaction with a substantial binding energy. From our point of view it should be possible to write the compositeness condition so that it explicitly relates "composite" states only.

B. -

The power series (3.4) in the Green's function contains an implicit mapping from the complex plane of the space-time variable $\lambda = (x_0^2 - \underline{x}^2)^{\frac{1}{2}}$ to that of the Green's function itself. In a world with one timelike dimension, this mapping is particularly simple, taking e^{ix_0} into the power series variable t of (1.3), with the upper half x_0 plane mapped into the interior of the unit circle. If one adds K spatial dimensions (for boundary value considerations on the forward light cone, say) it is necessary to deal with Hankel functions of order $(K-1)/2$ times an equal number of inverse powers of λ . [These latter factors are responsible for the $M^{-\frac{3}{2}}$ factor which combined above with the $M^{-\frac{3}{2}}$ from the asymptotic behaviour (2.9) of the c_K to give the M^{-3} in the density $\tau(M^2)$.]

As pointed out above and explicitly demonstrated in the Appendix, the compositeness condition (2.1) leads to a non-locality which manifests itself as a divergence of the power series (1.3) at a point within the t plane unit circle. The square root branch point that results may be thought of as the co-operative effect of the strengths of the higher mass threshold cuts. In the λ plane the non-locality appears as an infinite sequence of branch points becoming equally spaced, for large $|\lambda|$, along the negative imaginary axis ¹³⁾.

C. -

In the factorized dual, narrow resonance model, the count of states has the form $e^{ma}m^{-b}$, just as does the one considered here ¹⁴⁾. Let us convince ourselves that this is an unsurprising fact, and that beyond similarities in general form, no further detailed agreement should be expected.

We remind the reader that an N dimensional dual world behaves as a set of N oscillator systems sharing a common energy which may be conveniently labelled by an integer K . The count of states in such a world is appropriately performed by slightly generalizing the Hardy-Ramanujan theorem, which gives the asymptotic estimate of the total number of ways to partition a large integer into smaller integers ¹⁵⁾. The generating function $G^N(t)$, where the original (one-dimensional) Hardy-Ramanujan generator is ¹⁶⁾

$$G(t) = \prod_{m=1}^{\infty} (1-t^m)^{-1}, \quad (4.1)$$

whose expansion coefficients about $t = 0$ solve the problem, has very little family resemblance to our $r(t)$, as defined in (1.3) and (2.7). However, since $G^N(t)$ has a natural boundary of singularities on the unit circle ¹⁶⁾, and therefore a finite radius of convergence, we expect a factor e^{ma} in the density of states ¹⁷⁾. In this case the constant a is an energy scale set by the universal Regge trajectory slope, and has no particular relationship with its opposite number in the statistical bootstrap. The exponent b in m^{-b} has a value which depends on the analytic properties of the generating function in question near its radius of convergence. It therefore depends on the details of the appropriate functional equation. Our equation (2.7), $H(r,t) = 0$, has no resemblance whatsoever with the corresponding equation for $G^N(t)$, and in fact the two b 's are not equal ¹⁵⁾.

D. -

In the statistical bootstrap picture, each hadron is taken to be an infinitely composite object, and in an average sense a rough estimate of the absolute magnitude of the $1-n$ particle coupling is given. [If we identify our χ with the weight function of the mass operator, these are the d_n in (1.1).] In principle therefore it must be possible to give an estimate of the q^2 behaviour, in this picture, of the elastic matrix element of a (weakly coupled) scalar current.

E. -

As hinted above, Frautschi's bootstrap constraint may be an approximate generalization of earlier bootstrap approaches which yielded linear relations between coupling constant bilinears¹⁸⁾. If such a non-linear relation between couplings properly dependent on internal symmetry indices could be found, one would have every expectation that the non-linearity would fix the identity of the associated internal symmetry group, including the number of conserved operators. To get a rough idea of how such a scheme would work it will probably be necessary to find a Lagrangian which is formally equivalent to the statistical approach.

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APPENDIX

In the form of a series of lemmas, we exhibit and prove here the relevant mathematical properties of the generating function defined in (1.3) and (2.7).

Consider $r(t)$ as defined by

$$H(r(t), t) = e^{r(t)} - 2r(t) - 1 + t = 0. \quad (\text{A.1})$$

Lemma 1

$r(t)$ possesses a convergent power series about $t = 0$, with radius of convergence $R > 0$.

Proof

The proof follows immediately from the Implicit Function Theorem ¹⁹⁾ ;
 IMPLICIT FUNCTION THEOREM : Let $F(z, w)$ be analytic in some finite domain containing $z = w = 0$, and defined by $\text{mod}(z) < a$, $\text{mod}(w) < b$. Further suppose $F(0, 0) = 0$ and $F_z(0, 0) \neq 0$. Then there are positive numbers c and d such that for each w in $\text{mod}(w) < c$, the equation $F(z, w) = 0$ has just one solution z in the domain $\text{mod}(z) < d$, and this solution can be represented as

$$z = \sum_{k=0}^{\infty} C_k w^k \quad (\text{A.2})$$

Lemma 2 ²⁰⁾

The power series

$$r(t) = \sum_{k=1}^{\infty} C_k t^k \quad (\text{A.3})$$

has radius of convergence

$$R = \lim_{k \rightarrow \infty} |C_k|^{-1/k} \quad (\text{A.4})$$

Lemma 3

$c_K > 0$, provided $c_1 > 0$.

Proof

The proof follows immediately from inserting (A.3) into (A.1).

Lemma 4

The only singularity of $r(t)$ on the finite real t axis is a square root branch point at $t = t_0 = \log 4 - 1$. At $t = t_0$, $dt/dr = 0$.

Proof

Expand $t(r)$ about its maximum at $r_0 = \log 2$ and then invert. Near t_0 we then have

$$r(t) - r_0 \cong \pm \sqrt{2(t_0 - t)} \quad (\text{A.5})$$

From (A.1) it is obvious that this branch point is the real singularity nearest $t = 0$.

Lemma 5

The circle of convergence of (A.3) passes through the singularity (real or complex) nearest $t = 0$.

Proof

The proof follows immediately from the Cauchy-Taylor theorem ²¹⁾ which states that an analytic function possesses a Taylor expansion in its domain of analyticity, and that if the expansion is taken about $z = a$, the series converges within any circle which lies entirely in the analytic domain and has its centre at a .

Lemma 6

The point $t = R$ is a singular point of (A.3).

Proof

The proof follows from the positivity of the c_K ²²⁾. (Some of the c_K could in fact vanish without affecting this particular result.)

From Lemmas 5 and 6, we have :

Lemma 7

In (A.3), $R = t_0$.

Lemma 8

The partial sum of the coefficients in (A.3) satisfies

$$\sum_{k=1}^m K C_K R^K \underset{m \rightarrow \infty}{\sim} \sqrt{2m/R\pi} \quad (\text{A.6})$$

Proof

The proof follows easily from a Tauberian theorem ²³⁾ which states that if

$$\sum_{k=0}^{\infty} a_k z^k \underset{z \rightarrow 1}{\sim} (1-z)^{-\alpha} ; \alpha > 0 \quad (\text{A.7})$$

and (this condition is trivially satisfied by our positive coefficients)

$$a_k > -C(k+1)^{\alpha-1} ; C \geq 0 ; k=0,1,2... \quad (\text{A.8})$$

then

$$\sum_{k=0}^m a_k \underset{m \rightarrow \infty}{\sim} m^{\alpha} \Gamma^{-1}(\alpha+1) \quad (\text{A.9})$$

Using (A.5) we apply this theorem to the derivative of $r(t)$, giving (A.6).

Making K a continuous variable we write (A.6) in the form

$$\int_1^m dK K C(K) R^K \underset{m \rightarrow \infty}{\sim} \sqrt{2m/R\pi} \quad (\text{A.10})$$

and differentiate to obtain

$$C(K) \underset{K \rightarrow \infty}{\sim} \exp(-K \log R) K^{-3/2} (8R\pi)^{-1/2} \quad (\text{A.11})$$

which completes our discussion.

REFERENCES AND FOOTNOTES

- 1) S.C. Frautschi - Phys.Rev. D3, 2821 (1971).
- 2) W. Nahm - to be published.
- 3) See, e.g., P. Kaus and F. Zachariasen - Phys.Rev. 171, 1597 (1968).
- 4) Likely candidates would be the class of field theories considered by A. Jaffe - Ann.Phys.(N.Y.) 32, 127 (1965).
- 5) The knowledgeable reader will note that while the spirit of the present approach generally matches that of Ref. 1), the details differ. We use a covariant density in mass squared, and begin with an inhomogeneous equation. [With respect to the latter, see C.J. Hamer and S.C. Frautschi - Phys.Rev. D4, 2125 (1971).] Several previous authors have noted that forcing covariance into the system has little substantive effect.
- 6) See, e.g., M.K. Volkov - Ann.Phys.(N.Y.) 49, 202 (1968) ; G. Efimov - Soviet Phys. JETP 17, 1417 (1963).
The exercise which follows in the text can also easily be done in momentum space. Note that our D^C differ from the standard ones by normalization factors of $(2\pi)^3$.
- 7) Such an identification cannot be taken seriously since the imaginary part of the renormalized propagator involves the propagator itself, in addition to the proper vertex and the kinematic density of states. The analogy with field theory works best with the mass operator Y of (1.4).
- 8) In this connection, see the hint of a connection between non-polynomial Lagrangians and vanishing renormalization constants in the work of M.K. Volkov - Soviet J.Nuclear Phys. 6, 799 (1968).
- 9) We are being utterly cavalier in view of the well-known ambiguity in defining the real parts of such products. For a discussion of this point see H. Lehmann and K. Pohlmeier - Commun.Math.Phys. 20, 101 (1971).

- 10) V.A. Kolkunov - Soviet Phys.JETP 16, 1025 (1963). The result (3.9) is actually somewhat stronger than is necessary here.

- 11) To check this expression, compare for example R.H. Milburn - Revs.Modern Phys. 27, 1 (1955).

- 12) R. Carlitz - Phys.Rev., to be published.

- 13) It is easy to check that most familiar field theoretic and statistical systems have state densities which grow more slowly than e^{aE} , as expected from local interactions. For example, replacing d_n in (1.1) by $1/n!$, allowing M_0 to become zero, and replacing $\rho^{(n)}$ with non-covariant phase space, leads to the photon gas result $\log \mathcal{Z}(=S) = CE^{3/4}$, as shown explicitly in Ref. 1). Using covariant phase space in this calculation yields the well-known non-polynomial Lagrangian result that $\mathcal{Z}$ is proportional to $\exp(M^{2/3})M^{-2/3}$. Again, for a degenerate Fermi gas, $\log i$ is proportional to $(E-E_0)^{1/2}$ where E_0 is the ground state energy. The general rule is that for a system with many degrees of freedom, say N , with enough energy so that its spectrum can be considered continuous, the density of states is proportional to E^n , where $n = O(N)$. As an example one can check that a system of $N \gg 1$ weakly coupled oscillators of total energy E and unit oscillator frequency has a total number of states $E^N/N!$ so that the corresponding infinitely composite system has a state density of e^E . For discussions of the general question of the rates of growth of densities of state, see :
A. Bialas and V.F. Weisskopf - Nuovo Cimento 35, 1211 (1965) ;
G. Auberson and B. Escoubes - Nuovo Cimento 36, 628 (1965).

- 14) See the review article of D. Sivers and J. Yellin - Revs.Modern Phys. 43, 125 (1971), Section VII.

- 15) K. Huang and S. Weinberg - Phys.Rev.Letters 25, 895 (1970).

- 16) G. Hardy and S. Ramanujan - Proc.London Math.Soc. 17, 75 (1917).

- 17) The presence of the exponential factor in the narrow resonance model density of states immediately leads one to suspect that a non-locality similar to the one we have discussed here is present. The existence of just such a non-locality has recently been suggested by E. Cremmer and J. Scherk - CERN Preprint TH. 1481 (1971), to be published. We thank J. Scherk for bringing his work to our attention.
- 18) For recent discussions of this subject, see :
 R.H. Capps - Phys.Rev. D5, 1018 (1972) ;
 C. Andersen and J. Yellin - Phys.Rev. D3, 846 (1971) ; D4, 2476 (1971).
- 19) N.G. De Bruijn - "Asymptotic Methods in Analysis", North Holland (Amsterdam, 1970), p. 23.
- 20) E.C. Titchmarsh - "The Theory of Functions", Oxford (Oxford, 2nd Edition, 1939), p. 213.
- 21) Ibid., Ref. 14), p. 83.
- 22) Ibid., Ref. 14, p. 214.
- 23) Ibid., Ref. 14, p. 242 ; Ref. 13), pp. 147 ff.
 There is a related theorem on Laplace integrals due to H. Delange - Ann.Sci.Ecole Normale Supérieure 71 (3), 213 (1954), as follows :
 Let $f(0) \geq 0$, with $f(t)$ non-decreasing for $t \geq 0$. Further suppose

$$F(p) = \int_0^{\infty} dt f(t) e^{-pt}$$

converges in $\text{Re } p > a$, $a > 0$. Then if

$$F(p) = (p-a)^{-b} g(p) + h(p)$$

where g and h are holomorphic in $\text{Re } p \geq a$ and b is real and $\neq 0, -1, -2 \dots$, it follows that

$$f(t) \underset{t \rightarrow \infty}{\sim} \Gamma^{-1}(b) e^{at} t^{b-1}$$