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NON-LOCAL GAUGE INVARIANT REGULARIZATION AND WARD IDENTITY ANOMALIES

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A B S T R A C T

A gauge invariant regularization of a massive spinor field interacting with external scalar, pseudoscalar, vector and axial fields is found. It is based on ϵ separating the vertices in the total Lagrangian. Rules for its use are deduced and Ward identities are written. Anomalies, present only in the divergence of the axial current, are interpreted as the high energy contribution of the integrals associated with some loops. They can be computed in a simple way, confirming Bardeen's result.

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1. - INTRODUCTION

It is by now a well-known fact ^{1),2)} that, in some field theoretical models, anomalous terms can appear in the Ward identities. They are present, for instance, in the divergence of the axial current of a fermion field coupled to an external electromagnetic field.

This new phenomenon depends on the singular structure of the local product of fields, which forbids the use of the naïve field equations, and therefore the writing of the naïve Ward identities. If the theory is made finite by regulating the Lagrangian in a gauge invariant way, the anomalies show up in the axial current Ward identity as surviving terms after the regulators have been switched off.

Another, perhaps more transparent, way of looking at the same problem is to define the current operators as the limits of non-local product of fields, preserving gauge invariance by the insertion of appropriate field dependent phase factors. The anomalies are found by computing, via the field equations, the divergence of the axial current and then performing the necessary limiting operations ³⁾.

The currents are functional derivatives of the Lagrangian with respect to the external fields. Therefore, once a particular form for them is assumed, the Lagrangian, if it exists, and consequently the equations of motion should be completely determined. Serious objections to this approach were raised by Hagen ⁴⁾. For instance he insisted on the impossibility of constructing a reasonable theory in presence of an external axial field.

However, these difficulties can be overcome by attacking the problem in a more direct way. Instead of defining the currents, we propose to start from a Lagrangian regularized by splitting the points, and deduce from it both the currents and the field equations. The Lorentz invariance, lost in splitting the points, is formally recovered by an appropriate average over the space-time directions.

Here, along these lines, we apply a gauge invariant non-local regularization method to a massive spinor field with arbitrary internal degrees of freedom, interacting with external vector, axial vector, scalar and pseudoscalar fields. The corresponding Lagrangian leads to finite

diagrams only if an unsplit mass term is present ^{*}). This term breaks the possible chiral invariance of the theory even in the presence of pseudoscalar fields (since the pseudoscalar current contrarily to the mass term has to be split) and is responsible for the anomalies. The anomalies have been first calculated by Bardeen ⁵⁾ using a non-gauge invariant regularization procedure and later by Brown, Shih and Young ⁶⁾ employing the Pauli-Villars regularization. The result has been confirmed, using group theoretical methods by Wess and Zumino ⁷⁾. With our technique, once the rules for the use of the regularization are established, Bardeen's result can be easily obtained. The anomalies are integrated and calculated as the high energy contribution to some diagrams containing the γ_5 unregularized vertex coming from the mass term. In particular the absence of anomalies dependent on the scalar and pseudoscalar fields can be proved in a quite direct way if the regularization of the corresponding vertices is suitably chosen.

The paper is so organized. In Section 2 the regularization method is introduced for fermions interacting with vector and axial fields. In Section 3 the Ward identities and the mechanism at the origin of the anomalies for the model of Section 2 are discussed. Section 4 contains an explicit calculation of the anomalies. In Section 5, the scalar and pseudoscalar fields are introduced. Estimates of diagrams are collected in Appendices A, B and C.

^{*}) The addition of a suitably split mass term allows the treatment of a larger class of models.

2. - REGULARIZATION OF THE THEORY WITH ONLY VECTOR AND AXIAL FIELDS

Let us consider a quantized fermion field Ψ transforming like a representation of $SU(n)$ with generator λ_i ($i = 1, 2, \dots, n^2-1$), interacting with the external vector fields V_i^μ and with the external axial fields A_i^μ . The Lagrangian density of the system is taken to be ⁸⁾ :

$$L = i \bar{\Psi} \gamma_\mu \partial_\mu \Psi + \bar{\Psi} \gamma_\mu (V_\mu + \gamma_5 A_\mu) \Psi - m \bar{\Psi} \Psi \quad (2.1)$$

with

$$V^\mu = V_i^\mu \lambda_i \quad \text{and} \quad A^\mu = A_i^\mu \lambda_i$$

The sum of its first two terms is invariant under the following transformations of the spinor, vector and axial fields :

$$\delta \Psi = i (\alpha + \gamma_5 \beta) \Psi \quad (2.2)$$

$$\delta V_\mu = \partial_\mu \alpha + i [\alpha, V_\mu] + i [\beta, A_\mu] \quad (2.3)$$

$$\delta A_\mu = \partial_\mu \beta + i [\beta, V_\mu] + i [\alpha, A_\mu]$$

where $\alpha_i(x)$ and $\beta_i(x)$ are the infinitesimal parameters of the transformations of $SU(n) \otimes SU(n)$ (and $\alpha = \alpha_i \lambda_i$, $\beta = \beta_i \lambda_i$). The total Lagrangian density is invariant under the gauge group $SU(n)$ expressed by the transformations (2.2) and (2.3) with $\beta = 0$. However, these invariances are purely formal because the theory contains divergences, whose origin can be traced back to the operation of multiplying the field operators at the same point of space-time. Therefore we propose to regularize the theory by separating the points in the Lagrangian density (2.1) and trying at the same time to maintain the invariance or partial invariance properties of the model.

For this purpose let us consider the following expression :

$$T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x + \varepsilon t) dt} \right) \quad (2.4)$$

where $U = V + \gamma_5 A$ and T denotes ordering of the integral with t decreasing as one reads from left to right. Since

$$\delta \left\{ T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x+\varepsilon t) dt} \right) \right\} = i \left\{ (\alpha(x+\varepsilon) + \gamma_5 \beta(x+\varepsilon)) T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x+\varepsilon t) dt} \right) + \right. \\ \left. - T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x+\varepsilon t) dt} \right) (\alpha(x-\varepsilon) + \gamma_5 \beta(x-\varepsilon)) \right\} \quad (2.5)$$

it follows that

$$\bar{\Psi}(x+\varepsilon) \gamma^\mu T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x+\varepsilon t) dt} \right) \Psi(x-\varepsilon)$$

is $SU(n) \otimes SU(n)$ invariant. We then define a new Lagrangian density ⁹⁾

$$\overline{L(x)} = -2i \left\{ \bar{\Psi}(x+\varepsilon) \frac{\gamma \cdot \varepsilon}{\varepsilon^2} T \left(e^{i \int_{-1}^1 \varepsilon \cdot U(x+\varepsilon t) dt} \right) \Psi(x-\varepsilon) \right\} - m \bar{\Psi}(x) \Psi(x) \quad (2.6)$$

with the same invariance properties of Eq. (2.1). It is invariant under the gauge group $SU(n)$ and the only term which can break $SU(n) \otimes SU(n)$ is the mass term. The bar over the first term of Eq. (2.6) means that a suitable average over the directions of ε must be performed. If this averages washes out the terms with odd powers of ε and if $\overline{\varepsilon_\mu \varepsilon_\nu} = \frac{1}{4} g_{\mu\nu} \varepsilon^2$, it is easy to see that the expression (2.6) reproduces as $\varepsilon \rightarrow 0$ the expression (2.1). This behaviour suggests to us to consider the Lagrangian density (2.6) as a regularization of the Lagrangian density (2.1). The corresponding action integral $A = \int L(x) dx$ can be rewritten, after having performed the Fourier transforms with the following conventions

$$f(x) = \int dk e^{-ik \cdot x} f(k), \quad \Psi(x) = \frac{1}{(2\pi)^2} \int dp e^{-ip \cdot x} \Psi(p) \quad (2.7)$$

as

$$A = \int dp \bar{\Psi}(p) S^{-1}(p) \Psi(p) + \int dp dq \bar{\Psi}(p+\frac{q}{2}) \Gamma(p, q) \Psi(p-\frac{q}{2}), \quad (2.8)$$

where

$$\tilde{S}^{-1}(p) = -\gamma_p \frac{e^{-2i p \cdot \epsilon}}{\epsilon^2} - m, \quad (2.9)$$

$$\Gamma(p, q) = \sum_n \Gamma^{(n)}(p, q) \quad (2.10)$$

with

$$\Gamma^{(n)}(p, q) = \frac{1}{n!} \int \prod_i^m d\kappa_i \delta(q - \sum_i \kappa_i) T \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_i^m dt_i \prod_i^m (\hat{U}'(\kappa_i) \cdot \partial_p) \tilde{S}^{-1}(p - \sum_i \kappa_i; t_i) \quad (2.11)$$

and $U' = V - \gamma_5 A$. A particular prescription for the average corresponds to specify the value of $(e^{-2i \epsilon \cdot p} / \epsilon^2)$. By Eqs. (2.9)-(2.11), this is equivalent to fixing the form of the function $S^{-1}(p)$. If we perform the average in a Lorentz invariant way, we can rewrite $S(p)$ in the form

$$S(p) = f(\mu, p^2) \frac{\not{p} + m f(\mu, p^2)}{p^2 - m^2 f(\mu, p^2)^2} \quad (2.13)$$

The choice of the function f will be discussed later. For the moment we only note that it has to satisfy the obvious requirement of reproducing the free propagator when $\mu = 0$. This means that $\lim_{\mu \rightarrow 0} f(\mu, p^2) = 1$ must hold.

The point of view we are going to take is to consider the action as given by Eq. (2.6) with the expressions (2.7), (2.8) and (2.9) for the vertex and for the propagator as definitions to be used in constructing a regularized perturbation theory. Since the only diagrams which can have divergences are the fermion loops, our position will be justified if we can show that the action Eq. (2.6), with a suitable function f , generates finite fermion loops and that each fermion loop, already unambiguously finite with regularization, is the limit of the corresponding regularized diagram as $\mu \rightarrow 0$. These diagrams can be described in a compact way by writing the vacuum functional W , which is the generator of the connected fermion loops :

$$iW = \ln \left\langle T \left(e^{i \int d^4p d^4q \bar{\Psi}(p+q) \Gamma(p, q) \Psi(p-\frac{q}{2})} \right) \right\rangle \quad (2.14)$$

A straightforward computation leads to the following expression for W :

$$W = \sum_N W^{(N)} \quad (2.15)$$

where

$$W^{(N)} = i \frac{(-1)^N}{N} \int \prod_{i=1}^N dq_i \delta(\sum_{i=1}^N q_i) \int dp \, T_n \left\{ \Gamma(p + \frac{q_n}{2}, q_n) S(p) \Gamma(p - \frac{q_n}{2}, q_n) \right. \\ \left. S(p - q_n) \dots S(p - \sum_{j=1}^{i-1} q_j) \Gamma(p - \sum_{j=1}^{i-1} q_j - \frac{q_i}{2}, q_i) S(p - \sum_{j=1}^{i-1} q_j) \dots S(p - \sum_{j=1}^{N-1} q_j) \right\}. \quad (2.16)$$

We now turn our attention to the function $f(\mu, z)$. It is not at all obvious that the procedure we have followed until now yields a finite theory. For large p $\Gamma^{(n)}(p, q)$ behaves like $(S^{-1}(p)/p^n)$. Therefore, if $m = 0$, by Eq. (2.13) $\Gamma^{(1)}(p, q)S(p)$ behaves like $1/p$, as in the unregularized theory. This means that, in the mass zero case, no f can cure the divergences. On the other hand, if $m \neq 0$, for any f such that $(f(\mu, z))^2/z \rightarrow \infty$ as $z \rightarrow \infty$ the asymptotic form of $S(p)$ is :

$$S(p) \sim - \frac{1}{m f} \quad (2.17)$$

Since

$$\Gamma^{(n)}(p, q) \sim \frac{1}{f p^{n-1}} \quad (2.18)$$

it follows, by looking at Eq. (2.16), that the convergence of all loop integral is ensured, provided f has a high enough divergence as $z \rightarrow \infty$. For simplicity we choose f to be a polynomial and in order to be able to perform a Wick rotation in the energy plane, we choose it to have all the roots of the equation $f(\mu, z) = 0$ and $z - m^2 f(\mu, z)^2 = 0$ to be real and positive for all μ positive small enough. This requirement gives a prescription to handle the singularities of the denominators appearing in the vertices and in the propagator. The meaning of $1/f(\mu, p^2)$ will be $\lim_{\eta \rightarrow 0+} (1/f(\mu, p^2 + i\eta))$ and that of $1/p^2 - m^2 f(\mu, p^2)^2$ will be $\lim_{\eta \rightarrow 0+} (1/p^2 + i\eta - m^2 f(\mu, p^2 + i\eta)^2)$. These conventions are nothing else but the Feynman boundary conditions and the usual computational rules can be applied to each diagram.

We choose for $f(\mu, z)$ the following function :

$$f(\mu, z) = 1 - \mu z + \mu^4 z^2.$$

By simple power counting all loop integrals turn out to be finite, and the above requirements about the zeros of the denominators are easily seen to be satisfied at least for μ small enough. It remains to prove that the loop integrals which are absolutely convergent without regularization are the $\lim_{\mu \rightarrow 0}$ of the corresponding regularized loops. The absolute convergence takes place when the total number of vector and axial lines γ is > 4 . When $\gamma \leq 4$ the loop can still be absolutely convergent provided the number of derivatives acting on the external momenta are $> 4 - \gamma$. Due to gauge invariance as to the form of the vertex $\Gamma(p, q)$ [see Eq. (2.10)], any derivative on the external momentum of a vertex is equivalent to the addition of a new external line to that vertex. This implies that we can limit ourselves to analyzing only the cases with $\gamma > 4$. For the sake of simplicity in the proof, the external momenta appearing in S and S^{-1} will be taken equal to zero. In this approximation the expression for $\Gamma^{(n)}(p, q)$ becomes :

$$\Gamma^{(n)}(p, q) = \frac{1}{n!} \int \prod_i^m d k_i \delta(q - \sum_i k_i) \prod_i^m (\hat{U}(k_i) \cdot \partial_p) S^{-1}(p) \quad (2.20)$$

and the contribution to W coming from N vertices with m_i ($i = 1, 2, \dots, N$) external lines leaving each vertex is :

$$i \frac{(-1)^N}{N} \int \prod_i^N \left(\prod_j^{m_i} d k_j^{(i)} \right) \delta \left(\sum_i \sum_j k_j^{(i)} \right) \int d p T_2 \left\{ \prod_i^N \left[S(p) \prod_j^{m_i} (\hat{U}(k_j^{(i)}) \cdot \partial_p) S^{-1}(p) \right] \right\} \quad (2.21)$$

For all diagrams for which $\gamma = \sum_i^N m_i$ is > 4 , we must show that, if $\gamma > N$

$$\lim_{\mu \rightarrow 0} \int d p T_2 \left\{ \prod_i^N \left[S(p) \prod_j^{m_i} (\hat{U}(k_j^{(i)}) \cdot \partial_p) S^{-1}(p) \right] \right\} = 0, \quad (2.22)$$

$$\left\{ i k_\mu \frac{\delta}{\delta \hat{A}_{\mu^2}(k)} + \int \frac{du}{2\pi i} \left(\hat{A}_{\mu^2}(u) \frac{\delta}{\delta \hat{V}_{\mu^2}(u+k)} + \hat{V}_{\mu^2}(u) \frac{\delta}{\delta \hat{A}_{\mu^2}(u+k)} \right) \right\} W = 2m \sum_0^\infty (-1)^n \times$$

$$\times \int \prod_1^m dq_i \delta(\sum_1^m q_i + k) \times$$

$$\times \int dp T_e \left\{ \gamma_5 \lambda S(p) \Gamma(p - \frac{q_1}{2}, q_1) \dots S(p - \sum_{j=1}^{i-1} q_j) \Gamma(p - \sum_{j=1}^{i-1} q_j - \frac{q_i}{2}, q_i) S(p - \sum_{j=1}^i q_j) \dots S(p - \sum_{j=1}^{N-1} q_j) \right\}. \quad (3.6)$$

The axial current is not conserved because the mass term breaks chiral invariance. In the proof of both identities (3.5) and (3.6) no ambiguous operations involving infinities have been performed because the theory is regularized and therefore finite (at least at the perturbative level).

The right-hand side of Eq. (3.6) can also be considered to be, apart from the factor $2m$, the current associated with the external pion field π at zero value of the field, since the unregularized coupling of the pion to the ψ field should be of the form $i\pi \ell \bar{\psi} \gamma_5 \lambda \psi$. To carry over this interpretation for the limiting theory, we must check that, for all finite loops in the expression on the right-hand side of Eq. (3.6), the $\lim_{\mu \rightarrow 0}$ is interchangeable with the integration over the loop variables. By the same arguments of Section 2 we look at the right-hand side of Eq. (3.6) when the total number of vector and axial lines γ is > 3 and in the approximation for which in S and S^{-1} the external momenta are set equal to zero. The term with N vertices and with m_i ($i = 1, 2, \dots, N-1$) vector and axial lines leaving each of $N-1$ vertices is equal to :

$$2m (-1)^{N-1} \int \prod_1^{N-1} \left(\prod_j^{m_i} dk_j^{(i)} \right) \delta(\sum_1^{N-1} \sum_j^{m_i} k_j^{(i)} + k) \int dp T_e \left\{ S(p) \gamma_5 \lambda \prod_1^{N-1} \left[S(p) \prod_j^{m_i} (\hat{V}(k_j^{(i)}) \cdot \partial_p) \bar{S}(p) \right] \right\} \quad (3.7)$$

and therefore the integral to be investigated is :

$$\int dp T_e \left\{ S(p) \gamma_5 \lambda \prod_1^{N-1} \left[\prod_j^{m_i} (\hat{V}(k_j^{(i)}) \cdot \partial_p) \right] \bar{S}(p) \right\}. \quad (3.8)$$

For this case it is no longer true that, in the evaluation of the integral (3.8), the only region giving a non-zero contribution to the limit is the low energy one. For some diagrams there is a contribution also from the high energy region and destroys the possibility of interpreting the right-hand side of Eq. (3.6) as the source of the pion field. The new additional term appearing in the limit constitutes the so-called anomaly.

Before turning to the general analysis of the behaviour of the integral (3.8), we feel it is worth while to give an explicit example of a diagram for which the $\lim_{\mu \rightarrow 0}$ is not interchangeable with the integral over the loop momenta. Consider the diagram

at zero external momenta. It is represented by the analytic expression

$$\int d^4p \operatorname{Tr} \left\{ S(p) \gamma_5 \lambda \left[S(p) (\hat{V} \cdot \hat{\alpha}_p) S^{-1}(p) \right]^4 \right\} \quad (3.9)$$

Since

$$S(p) \gamma_5 S(p) = -f^2 \frac{1}{p^2 - m^2 f^2} \gamma_5 \quad (3.10)$$

we can rewrite the expression (3.9) as :

$$\operatorname{Tr} (\lambda_\alpha \hat{V}_\mu \hat{V}_\nu \hat{V}_\rho \hat{V}_\sigma) G^{\mu\nu\rho\sigma}$$

where

$$G^{\mu\nu\rho\sigma} = - \int d^4p \frac{f(\mu, p)}{p^2 - m^2 f^2} \operatorname{Tr} \left\{ \gamma_5 \Gamma_\mu^{\lambda} S(p) \Gamma_\nu^{\lambda} S(p) \Gamma_\rho^{\lambda} S(p) \Gamma_\sigma^{\lambda} \right\}. \quad (3.11)$$

We have set

$$\Gamma_\mu^{\lambda} = \Gamma_\mu^{\lambda, \alpha} \gamma_\alpha \quad (3.12)$$

and

$$\Gamma^{\mu,\alpha} = \frac{1}{f} (g^{\mu\alpha} - 2p^\mu p^\alpha \frac{f'}{f}) \quad (3.13)$$

As a consequence equality (3.11) becomes

$$G^{\mu\nu\sigma} = -m \int dp \frac{f^6}{(p^2 - m^2 f^2)^4} \Gamma^{\mu,\alpha} \Gamma^{\nu,\beta} \Gamma^{\sigma,\gamma} \Gamma^{\sigma,\delta} T_2 \left\{ \gamma_\delta \left[m^2 f^2 \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta + \right. \right. \\ \left. \left. + (\gamma_\alpha \not{p} \gamma_\beta \not{p} \gamma_\gamma \gamma_\delta + \gamma_\alpha \not{p} \gamma_\beta \gamma_\gamma \not{p} \gamma_\delta + \gamma_\alpha \gamma_\beta \not{p} \gamma_\gamma \not{p} \gamma_\delta) \right] \right\}$$

This expression can be simplified to :

$$G^{\mu\nu\sigma} = 4im \varepsilon_{\alpha\beta\gamma\delta} \int dp \frac{f^6}{(p^2 - m^2 f^2)^3} \Gamma^{\mu,\alpha} \Gamma^{\nu,\beta} \Gamma^{\sigma,\gamma} \Gamma^{\sigma,\delta} = \\ = 4im \varepsilon^{\mu\nu\sigma} \int dp \frac{f}{(p^2 - m^2 f^2)^3} (f - 2p^2 f')$$

and on passing to the Euclidean region it becomes :

$$G^{\mu\nu\sigma} = 4m\pi^2 \varepsilon^{\mu\nu\sigma} \int_0^\infty dx \frac{x g(\mu, x)}{[x + m^2 g(\mu, x)^2]^3} (g(\mu, x) - 2x g'(\mu, x)) \quad (3.14)$$

where $g(\mu, x) = f(\mu, -x)$. The last integral in expression (3.14) can be integrated by parts for any function g . The result is :

$$G^{\mu\nu\sigma} = \frac{2\pi^2}{m} \varepsilon^{\mu\nu\sigma} \frac{x^2}{[x + m^2 g(\mu, x)^2]^2} \Big|_0^\infty \quad (3.15)$$

If $\mu \neq 0$ $G^{\mu\nu\sigma} = 0$, but if $\mu = 0$ $g(\mu, x) = 1$ and consequently $G^{\mu\nu\sigma} = (2\pi^2/m) \varepsilon^{\mu\nu\sigma}$. This implies that the regularized loop contains contributions both from the low energy region and from the other regions. Because it is identically zero, the high and very high energy contributions

can be immediately computed since they are equal and opposite to the low energy contribution. This remark, which is the basis of the low energy theorems ¹⁾, shows also that in this example the anomaly, which is given by the integral over high and very high energy regions, does not depend at all on the choice of the regularizing function.

To study in general the $\lim_{\mu \rightarrow 0}$ of the integral (3.8) we need some estimates which are proved in Appendix B. The procedure and notation are the same as used at the end of Section 2 and in Appendix A. After a Wick rotation we majorize the integrand in a different way for the three domains : the low energy region, the high energy region, and the very high energy region. The majorizing functions obtained are all integrable (we are interested only in the $\gamma > 3$ cases). Their integral extended to region 1 does not go to zero as $\mu \rightarrow 0$ only for the case $N-1 = \gamma (= \sum_{i=1}^{N-1} m_i)$. But, unlike the preceding analysis of Section 2, the integral over the high energy region remains different from zero for $\gamma = 4$, while it goes to zero for $\gamma > 4$. The integral over region 2, expressed in the variables $y = \mu^2 x$, ranges from μ to μ^{-1} . In this region the leading term of f is $\mu x = y/\mu$. In Appendix B it is shown that, performing this change of variable and using this leading behaviour, if $\gamma = 4$ the variable μ appears only in the limits of integration, and therefore the $\lim_{\mu \rightarrow 0}$ is trivial. Consequently, to compute the high energy contribution of a diagram with $\gamma = 4$, we can integrate over the whole four-dimensional momentum space, putting into f $\mu = 1$, and keeping for $f(1, p^2)$ only the term proportional to p^2 . Finally the very high energy region does not show any strange behaviour. In the limit it gives no contribution at all.

In conclusion, the investigation of the integral (3.8) on $\mu \rightarrow 0$ shows the possibility of the presence of anomalies. We shall compute them explicitly in the next section.

4. - COMPUTATION OF THE ANOMALIES

The results of Section 3 have shown that in the limit $\mu \rightarrow 0$ the right-hand side of Eq. (3.6) can receive a contribution from the high energy region. This additional term, the anomaly, can appear only in those diagrams for which γ , that is the sum of the number of vector and axial lines and the number of derivatives with respect to the external momenta, is equal to 4. However, in the conjugation which follows, we are going to analyze only the diagrams without derivatives on the external legs. This simplifies considerably the work, the missing terms will be recovered at the end by gauge invariance.

We have to find the high energy contribution coming from the expression (3.7) for all possible distributions of the m_i subject to the condition $\sum_{i=1}^{N-1} m_i = 4$. As before we can use expressions for which the external momenta inside $S(p)$ and $S^{-1}(p)$ are put equal to zero, because their presence does not change the value of the anomaly. In fact the difference between Eq. (3.8) and the same expression with some of the p momenta shifted contains at least one extra derivative and therefore has $\gamma \geq 5$. Finally by the remark at the end of Section 2, the high energy contribution of the integral Eq. (3.8) can be found by computing the same integral over the whole four-dimensional momentum space with $\mu = 1$ and keeping in $f(1, p^2)$ only the term linear in p^2 . For convenience in the following, instead of $f(1, p^2)$, we shall use $h(p^2) = (1/mf(1, p^2))$. With these rules we now turn to the details of the calculation. The diagrams to be considered are :

Diagram B is trivially zero by trace arguments. The others [see Eq. (3.8)] have the following analytic expression :

$$C \equiv C_1 + C_2 + C_3 = \int dp T_2 \left\{ \lambda_1 S(p) \gamma_5 S(p) \left[\frac{\hat{U}_\mu' \hat{U}_\nu' \hat{U}_\rho' \Gamma^{\mu\nu\rho} S(p) \hat{U}_\sigma' \Gamma^\sigma}{6} \right. \right. \\ \left. \left. + \frac{\hat{U}_\mu' \hat{U}_\nu' \Gamma^{\mu\nu} S(p) \hat{U}_\rho' \hat{U}_\sigma' \Gamma^{\rho\sigma}}{4} + \frac{\hat{U}_\mu' \Gamma^\mu S(p) \hat{U}_\nu' \hat{U}_\rho' \hat{U}_\sigma' \Gamma^{\nu\rho\sigma}}{6} \right] \right\}, \quad (4.1)$$

$$D \equiv D_1 + D_2 + D_3 = - \int dp T_2 \left\{ \lambda_1 S(p) \gamma_5 S(p) \left[\frac{\hat{U}_\mu' \hat{U}_\nu' \Gamma^{\mu\nu} S(p) \hat{U}_\rho' \Gamma^\rho S(p) \hat{U}_\sigma' \Gamma^\sigma}{2} \right. \right. \\ \left. \left. + \frac{\hat{U}_\mu' \Gamma^\mu S(p) \hat{U}_\nu' \Gamma^\nu S(p) \hat{U}_\rho' \hat{U}_\sigma' \Gamma^{\rho\sigma}}{2} + \frac{\hat{U}_\mu' \Gamma^\mu S(p) \hat{U}_\nu' \hat{U}_\rho' \Gamma^{\nu\rho} S(p) \hat{U}_\sigma' \Gamma^\sigma}{2} \right] \right\} \quad (4.2)$$

$$E = \int dp T_2 \left\{ \lambda_1 S(p) \gamma_5 S(p) \hat{U}_\mu' \Gamma^\mu S(p) \hat{U}_\nu' \Gamma^\nu S(p) \hat{U}_\rho' \Gamma^\rho S(p) \hat{U}_\sigma' \Gamma^\sigma \right\} \quad (4.3)$$

where

$$\begin{aligned}\Gamma^\mu &= \Gamma^{\mu,\alpha} \gamma_\alpha \\ \Gamma^{\mu\nu} &= \Gamma^{\mu\nu,\alpha} \gamma_\alpha \\ \Gamma^{\mu\nu\sigma} &= \Gamma^{\mu\nu\sigma,\alpha} \gamma_\alpha\end{aligned}\quad (4.4)$$

and

$$\begin{aligned}\Gamma^{\mu,\alpha} &= m \partial^\mu (\beta^\alpha h) \\ \Gamma^{\mu\nu,\alpha} &= m \partial^\mu \partial^\nu (\beta^\alpha h) \\ \Gamma^{\mu\nu\sigma,\alpha} &= m \partial^\mu \partial^\nu \partial^\sigma (\beta^\alpha h)\end{aligned}\quad (4.5)$$

To simplify the writing we have omitted the K variables in the $\hat{U}$'s. They will be reinserted at the end when needed. Written in terms of $h(p^2)$ the propagator $S(p)$ has the form

$$S(p) = \frac{1}{m} \frac{\not{p}h + 1}{p^2 h^2 - 1} \quad (4.6)$$

and therefore

$$S(p) \gamma_5 S(p) = \frac{1}{m^2} \frac{1}{1 - h^2 p^2} \quad (4.7)$$

The three expressions (4.1), (4.2) and (4.3) become :

$$C = -\frac{1}{m^3} \int d^4p \frac{1}{(1 - h^2 p^2)^2} T_2 \left\{ \lambda \gamma_5 \left(\frac{\hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{6} \Gamma^{\mu\nu\sigma} \Gamma^\sigma + \frac{\hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{4} \Gamma^{\mu\nu\sigma} \Gamma^\sigma + \frac{\hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{6} \Gamma^{\mu\nu\sigma} \Gamma^\sigma \right) \right\} \quad (4.8)$$

$$\begin{aligned}D = & -\frac{1}{2m^4} \int d^4p \frac{h}{(1 - h^2 p^2)^3} T_2 \left\{ \lambda \gamma_5 \left(\hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma + \hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma + \right. \right. \\ & \left. \left. + \hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma + \hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma + \hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma + \hat{U}_\mu^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Gamma^{\mu\nu\sigma} \not{p} \Gamma^\sigma \right) \right\} \quad (4.9)\end{aligned}$$

$$\begin{aligned} E = & -\frac{1}{\omega^2} \int d^4p \frac{1}{(1-k^2 p^2)} T_2 \left\{ \lambda \gamma_5 \left[\hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} \Gamma_r^{\mu\nu} \Gamma_j^{\rho\sigma} + k^2 (\hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} \Gamma_r^{\mu\nu} \Gamma_j^{\rho\sigma} \right. \right. \\ & \left. \left. + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} \Gamma_r^{\mu\nu} \Gamma_j^{\rho\sigma} + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} \Gamma_r^{\mu\nu} \Gamma_j^{\rho\sigma} \right] \right\} \end{aligned} \quad (4.10)$$

The only term of $f(1, p^2)$ to be retained is $-p^2$. This implies that for $h(p^2)$ we must substitute $-(1/mp^2)$. The expressions (4.5) become

$$\Gamma_r^{\mu\alpha} = -(g^{\mu\alpha} p^2 - 2 p^\mu p^\alpha) \frac{1}{(p^2)^2} \quad (4.11)$$

$$\Gamma_r^{\mu\nu\alpha} = \frac{1}{(p^2)^3} \left\{ 2(g^{\mu\alpha} p^\nu + g^{\nu\alpha} p^\mu + g^{\mu\nu} p^\alpha) p^2 - 8 p^\mu p^\nu p^\alpha \right\} \quad (4.12)$$

$$\begin{aligned} \Gamma_r^{\mu\nu\rho\alpha} = & \frac{1}{(p^2)^4} \left\{ 2(g^{\mu\alpha} g^{\nu\rho} + g^{\mu\nu} g^{\rho\alpha} + g^{\mu\rho} g^{\nu\alpha}) (p^2)^2 + 48 p^\mu p^\nu p^\rho p^\alpha + \right. \\ & \left. - 8(g^{\mu\alpha} p^\nu p^\rho + g^{\nu\alpha} p^\mu p^\rho + g^{\rho\alpha} p^\mu p^\nu + g^{\mu\nu} p^\rho p^\alpha + g^{\mu\rho} p^\nu p^\alpha + g^{\nu\rho} p^\mu p^\alpha) \right\} \end{aligned} \quad (4.13)$$

We first compute the abnormal parity anomaly. By trace the C term does not give any contribution. Using Eqs. (4.11) and (4.12) the abnormal parity D term can be rewritten as :

$$\begin{aligned} D = & -\frac{i}{4\omega^4} \int d^4p \frac{h}{(1-k^2 p^2)^3} \frac{1}{(p^2)^4} \varepsilon_{\alpha\beta\gamma\delta} T_2 \left\{ \lambda \left[\hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} (g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu) p^\sigma g^{\rho\sigma} + \right. \right. \\ & + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} (g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu) g^{\rho\sigma} p^\sigma + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} g^{\mu\rho} p^\nu (g^{\rho\sigma} p^\sigma + g^{\rho\sigma} p^\nu) + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} g^{\mu\nu} p^\rho (g^{\rho\sigma} p^\sigma + g^{\rho\sigma} p^\nu) + \\ & \left. \left. + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} g^{\mu\rho} p^\nu (g^{\rho\sigma} p^\sigma + g^{\rho\sigma} p^\nu) + \hat{U}_r^{\alpha'} \hat{U}_\nu^{\alpha'} \hat{U}_j^{\alpha'} \hat{U}_\sigma^{\alpha'} g^{\mu\nu} p^\rho (g^{\rho\sigma} p^\sigma + g^{\rho\sigma} p^\nu) \right] \right\} \end{aligned} \quad (4.14)$$

Also the diagrams D give zero since the average of the momenta over the directions leads to a symmetric tensor which is contracted with the ϵ tensor. The abnormal parity part of E becomes after taking the trace and averaging over the directions of the momenta

$$E = \frac{i}{2\omega^5} \epsilon^{\mu\nu\rho\sigma} \int d^4p \frac{1}{(1-h^2 p^2)^4 (p^2)^4} T_2 \left\{ \lambda \left[2 \hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\rho} \hat{U}_{\rho}^{\sigma} - p^2 h^2 (\hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\rho} \hat{U}_{\rho}^{\sigma} + \hat{U}_{\mu}^{\rho} \hat{U}_{\rho}^{\nu} \hat{U}_{\nu}^{\sigma}) \right] \right\} \quad (4.15)$$

Here we have used the relation

$$\overline{T_2(\gamma_5 \Gamma_{\mu}^{\nu} \Gamma_{\nu}^{\rho} \Gamma_{\rho}^{\sigma})} = \frac{i}{(p^2)^3} \epsilon^{\mu\nu\rho\sigma} \quad (4.16)$$

where the bar indicates in this section the average over the directions of the momenta. If we substitute for $h(p^2) = 1/mp^2$, the integral can be performed by going to the Euclidean region. The final result is :

$$E = -\frac{\pi^2}{6\omega} \epsilon^{\mu\nu\rho\sigma} T_2 \left\{ \lambda \left[\hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\rho} \hat{U}_{\rho}^{\sigma} + \hat{U}_{\mu}^{\rho} \hat{U}_{\rho}^{\nu} \hat{U}_{\nu}^{\sigma} + \hat{U}_{\mu}^{\sigma} \hat{U}_{\sigma}^{\nu} \hat{U}_{\nu}^{\rho} \right] \right\} \quad (4.17)$$

Therefore the anomaly $\tilde{G}_\ell(k)$ in momentum space is

$$\begin{aligned} \tilde{G}_\ell(k) = & -\frac{\pi^2}{3} \epsilon^{\mu\nu\rho\sigma} \int d^4k_1 d^4k_2 d^4k_3 d^4k_4 \delta(\sum_i k_i + k) T_2 \left\{ \lambda \left[\hat{U}_{\mu}^{\nu}(k_1) \hat{U}_{\nu}^{\rho}(k_2) \hat{U}_{\rho}^{\sigma}(k_3) \hat{U}_{\sigma}^{\mu}(k_4) \right. \right. \\ & \left. \left. + \hat{U}_{\mu}^{\rho}(k_1) \hat{U}_{\rho}^{\nu}(k_2) \hat{U}_{\nu}^{\sigma}(k_3) \hat{U}_{\sigma}^{\mu}(k_4) + \hat{U}_{\mu}^{\sigma}(k_1) \hat{U}_{\sigma}^{\nu}(k_2) \hat{U}_{\nu}^{\rho}(k_3) \hat{U}_{\rho}^{\mu}(k_4) \right] \right\} \quad (4.18) \end{aligned}$$

where the $\sim$ over G_ℓ means anti-Fourier transforms. In configuration space Eq. (4.18) becomes

$$G_\ell(x) = -\frac{\epsilon^{\mu\nu\rho\sigma}}{48\pi^2} T_2 \left\{ \lambda \left[\hat{U}_{\mu}^{\nu}(x) \hat{U}_{\nu}^{\rho}(x) \hat{U}_{\rho}^{\sigma}(x) \hat{U}_{\sigma}^{\mu}(x) + \hat{U}_{\mu}^{\rho}(x) \hat{U}_{\rho}^{\nu}(x) \hat{U}_{\nu}^{\sigma}(x) \hat{U}_{\sigma}^{\mu}(x) + \hat{U}_{\mu}^{\sigma}(x) \hat{U}_{\sigma}^{\nu}(x) \hat{U}_{\nu}^{\rho}(x) \hat{U}_{\rho}^{\mu}(x) \right] \right\} \quad (4.19)$$

and written in terms of the V's and the A's it has the form :

$$G_2 = -\frac{1}{12\pi^2} \varepsilon^{\mu\nu\sigma} T_2 \left\{ \lambda \left[3 V_\mu V_\nu V_\sigma + (V_\mu A_\nu V_\sigma + A_\mu V_\nu A_\sigma + V_\mu A_\nu A_\sigma) + \right. \right. \\ \left. \left. - 3 A_\mu V_\nu V_\sigma - (A_\mu A_\nu V_\sigma + V_\mu V_\nu A_\sigma + A_\mu A_\nu A_\sigma) \right] \right\} \quad (4.20)$$

This expression must be completed in such a way as to obtain a gauge invariant result. This means that the final anomaly must be built out of A_μ , of

$$G_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i [V_\mu, V_\nu] \quad (4.21)$$

and of

$$D_\mu A_\nu = \partial_\mu A_\nu - i [V_\mu, A_\nu] \quad (4.22)$$

The terms to be taken into account are :

$$\varepsilon^{\mu\nu\sigma} T_2 (\lambda G_{\mu\nu} G_\sigma), \quad \varepsilon^{\mu\nu\sigma} T_2 (\lambda A_\mu A_\nu A_\sigma), \quad \varepsilon^{\mu\nu\sigma} T_2 (\lambda D_\mu A_\nu D_\sigma A_\sigma) \\ \varepsilon^{\mu\nu\sigma} T_2 (\lambda A_\mu G_{\nu\sigma} A_\sigma), \quad \varepsilon^{\mu\nu\sigma} \left[T_2 (\lambda A_\mu A_\nu G_\sigma) + T_2 (\lambda G_{\mu\nu} A_\sigma A_\sigma) \right]$$

Equation (4.20) yields in a unique way the final result :

$$G_2 = \frac{\varepsilon^{\mu\nu\sigma}}{16\pi^2} T_2 \left\{ \lambda \left[G_{\mu\nu} G_\sigma + \frac{4}{3} D_\mu A_\nu D_\sigma A_\sigma + \frac{8}{3} i A_\mu G_{\nu\sigma} A_\sigma + \frac{2}{3} i (A_\mu A_\nu G_\sigma + G_{\mu\nu} A_\sigma A_\sigma) + \frac{4}{3} A_\mu A_\nu A_\sigma \right] \right\} \quad (4.23)$$

Expressed in the variables

$$V_{\mu\nu} = G_{\mu\nu} - i [A_\mu, A_\nu] \quad A_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$$

Equation (4.23) reproduces Bardeen's result ⁵⁾

$$G_2 = \frac{\epsilon^{\mu\nu\sigma}}{4\pi^2} T_2 \left\{ \lambda \left[\frac{1}{4} V_{\mu\nu} V_{\sigma} + \frac{1}{12} A_{\mu} A_{\nu} A_{\sigma} + \frac{2}{3} i (A_{\mu} V_{\nu} A_{\sigma} + V_{\mu} A_{\nu} A_{\sigma} + A_{\mu} A_{\nu} V_{\sigma}) - \frac{8}{3} A_{\mu} A_{\nu} A_{\sigma} \right] \right\} \quad (4.24)$$

It is well known and easy to prove that such an anomaly cannot be compensated by introducing some additional gauge invariant vacuum functionals.

The determination of the normal parity anomaly is slightly longer because all diagrams C, D and E can contribute. Because of the relations

$$\overline{T_2(\Gamma^{\mu\nu} \Gamma^{\sigma})} = -\frac{4}{(p^2)^3} (g^{\mu\nu} g^{\sigma} + g^{\mu\sigma} g^{\nu} + g^{\nu\sigma} g^{\mu}) \quad (4.25)$$

$$\overline{T_2(\Gamma^{\mu\nu} \Gamma^{\sigma\tau})} = \frac{8}{(p^2)^3} (g^{\mu\sigma} g^{\nu\tau} + g^{\mu\tau} g^{\nu\sigma}) \quad (4.26)$$

where for the Γ 's the expressions (4.11), (4.12), (4.13) have been used, the formula (4.8) for C can be transformed into :

$$C = \frac{1}{2\omega^3} \int d^4p \frac{1}{(4-h^2)^2} T_2 \left\{ \lambda \left[\frac{g^{\mu\nu\sigma\tau} g^{\rho\sigma} g^{\sigma\tau}}{3} (\hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} + \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} \hat{U}_{\tau}^{\mu}) - (g^{\mu\nu} g^{\sigma\tau} + g^{\mu\sigma} g^{\nu\tau}) \hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} \right] \right\} \quad (4.27)$$

Upon integration over the momenta in the Euclidean region we reach the final result for C :

$$C = -i \frac{\pi^2}{2\omega} T_2 \left\{ \lambda \left[\frac{g^{\mu\nu\sigma\tau} g^{\rho\sigma} g^{\sigma\tau}}{3} (\hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} + \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} \hat{U}_{\tau}^{\mu}) - (g^{\mu\nu} g^{\sigma\tau} + g^{\mu\sigma} g^{\nu\tau}) \hat{U}_{\mu}^{\nu} \hat{U}_{\nu}^{\sigma} \hat{U}_{\sigma}^{\tau} \right] \right\} \quad (4.28)$$

The relations

$$\overline{T_2(\Gamma^{\mu\nu} \Gamma^{\sigma})} = \frac{4g^{\mu\nu} g^{\sigma}}{(p^2)^3} \quad (4.29)$$

$$\overline{T_2(\Gamma^\mu \Gamma^\nu \not{\Gamma}^\sigma)} = -\frac{4}{(p^2)^3} (g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}) \quad (4.30)$$

imply that the formula (4.9) for D can be written as :

$$D = -\frac{1}{2\omega^4} \int d^4p \frac{h}{(1-h^2)^3 (p^2)^3} T_2 \left\{ \lambda \chi_5 \left[(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}) (\hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} + \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4}) - 2(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}) \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} \right] \right\} \quad (4.31)$$

The integration over the momenta yields the following final expression for D :

$$D = -i \frac{\pi^2}{2\omega^4} T_2 \left\{ \lambda \chi_5 \left[(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}) \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} - \frac{g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}}{2} (\hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} + \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4}) \right] \right\} \quad (4.32)$$

In the same way the final formula for the normal parity part of E is :

$$E = -\frac{i\pi^2}{2\omega^4} T_2 \left\{ \lambda \chi_5 \left[\frac{g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho}}{3} (\hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} + \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} + \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4}) - \frac{2}{3} g^{\mu\nu} g^{\sigma\rho} \hat{U}_\mu^{\lambda_1} \hat{U}_\nu^{\lambda_2} \hat{U}_\sigma^{\lambda_3} \hat{U}_\rho^{\lambda_4} \right] \right\} \quad (4.33)$$

To deduce Eq. (4.33) from Eq. (4.10) the following identities have been used :

$$\overline{T_2(\Gamma^\mu \Gamma^\nu \Gamma^\sigma \Gamma^\rho)} = \frac{4}{(p^2)^4} (g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\sigma}) \quad (4.34)$$

$$\overline{T_2(\Gamma^\mu \not{\Gamma}^\nu \Gamma^\sigma \not{\Gamma}^\rho)} = -\frac{2}{(p^2)^3} (g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\sigma}) \quad (4.35)$$

$$\overline{T_2(\Gamma^\mu \not{\Gamma}^\nu \Gamma^\sigma \not{\Gamma}^\rho)} = \frac{4}{(p^2)^3} g^{\mu\sigma} g^{\nu\rho} \quad (4.36)$$

The results, Eqs. (4.28), (4.32) and (4.33), can be summed up and yield for the normal parity anomaly $G'_\ell(x)$ the following expression :

$$G'_2(x) = -\frac{i}{48\pi^2} T_2 \left\{ \lambda \gamma_5 \left[\frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}}{2} U'_\mu(x) U'_\nu(x) U'_\rho(x) U'_\sigma(x) + \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}}{2} U'_\mu(x) U'_\nu(x) U'_\rho(x) U'_\sigma(x) + \right. \right. \\ \left. \left. + (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) U'_\mu(x) U'_\nu(x) U'_\rho(x) U'_\sigma(x) - 2 g^{\mu\rho} g^{\nu\sigma} U'_\mu(x) U'_\nu(x) U'_\rho(x) U'_\sigma(x) \right] \right\} \quad (4.37)$$

Rewritten in terms of the V's and the A's Eq. (4.37) becomes :

$$G'_2 = -\frac{i}{12\pi^2} T_2 \left\{ \lambda \gamma_5 \left[(A_\mu V_\nu V_\rho V_\sigma - V_\mu V_\nu V_\rho A_\sigma) (-2 g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) + (V_\mu A_\nu V_\rho V_\sigma - V_\mu V_\nu A_\rho V_\sigma) \times \right. \right. \\ \times (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + 3 g^{\mu\sigma} g^{\nu\rho}) + (V_\mu A_\nu A_\rho A_\sigma - A_\mu A_\nu A_\rho V_\sigma) (2 g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) + \\ \left. \left. + (A_\mu V_\nu A_\rho A_\sigma - A_\mu A_\nu V_\rho A_\sigma) (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + 3 g^{\mu\sigma} g^{\nu\rho}) \right] \right\} \quad (4.38)$$

This anomaly can be eliminated. If we add to the vacuum functional W a new vacuum functional W' with

$$W' = -\frac{1}{24\pi^2} \int dx T_2 \left\{ -2i G^{\mu\nu} A_\mu A_\nu + (D_\mu A^\mu)^2 + 2 (A_\mu A^\mu)^2 \right\} \quad (4.39)$$

where $G_{\mu\nu}$ is given by Eq. (4.21) and $D_\mu A_\nu$ by Eq. (4.22), the vector current remains divergenceless, while in the divergence of the axial current the normal parity anomaly disappears. This happens because W' generates a vector current which is conserved, and an axial current whose divergence has the terms without derivatives exactly equal and opposite to the expression (4.38).

5. - INTRODUCTION OF THE SCALAR AND PSEUDOSCALAR FIELDS

We are now going to introduce the additional interaction of the fermion field with external scalar fields σ_i and pseudoscalar fields π_i

$$L_1 = \bar{\Psi} (\sigma + i\gamma_5 \pi) \Psi \quad (5.1)$$

where $\sigma = \sigma_i \lambda_i$ and $\pi = \pi_i \lambda_i$. More briefly with the notation $\tau = \sigma + i\gamma_5 \pi$, Eq. (5.1) can be rewritten as

$$L_1 = \bar{\Psi} \tau \Psi \quad (5.2)$$

If the transformation of τ with respect to the gauge group is

$$\delta \tau = i[\alpha, \tau] \quad (5.3)$$

L_1 is gauge invariant. Much in the same way as in Section 1 the Lagrangian density L_1 can be regularized, by ϵ separating the points, in a gauge invariant way. By Eqs. (2.2) and (2.5) it is easy to see that the Lagrangian density

$$\bar{L}_1(x) = \bar{\Psi}(x+\epsilon) T \left(e^{i \int_0^1 \epsilon \cdot U(x+\epsilon t) dt} \right) \tau(x) T \left(e^{-i \int_{-1}^0 \epsilon \cdot U(x+\epsilon t) dt} \right) \Psi(x-\epsilon), \quad (5.4)$$

where the bar means that a suitable average over the ϵ directions is performed, is gauge invariant. The corresponding action integral $A_1 = \int \bar{L}_1(x) dx$ can be written as :

$$A_1 = \int dp dq \bar{\Psi}(p+\frac{q}{2}) \Gamma_4(p, q) \Psi(p-\frac{q}{2}) \quad (5.5)$$

where

$$\Gamma_4(p, q) = \sum_{n, m}^{\infty} \Gamma_4^{(n, m)} \quad (5.6)$$

$$\Gamma_1^{(n,m)}(p,q) = \frac{1}{n!m!} \int \prod_{j=1}^m dk'_j \prod_{i=1}^m dk_i \delta(q - \sum_{j=1}^m k'_j + \sum_{i=1}^m k_i) \prod_{j=1}^m (\hat{U}(k'_j) \cdot \gamma_p) \hat{U}(k_0) \prod_{i=1}^m (\hat{U}(k_i) \cdot \gamma_p) \times$$

$$\times T \int_0^{\frac{1}{2}} \prod_{j=1}^m dt'_j \int_{-\frac{1}{2}}^0 \prod_{i=1}^m dt_i \phi(\varepsilon^2, (p - \sum_{j=1}^m k'_j t'_j - \sum_{i=1}^m k_i t_i)^2) \quad (5.7)$$

and

$$\phi(\varepsilon^2, p^2) = e^{\frac{2ip \cdot \varepsilon}{\varepsilon^2}} \quad (5.8)$$

We have a large freedom in the choice of the function ϕ , provided it satisfies the obvious requirements of making the loops finite and of being equal to 1 for $\varepsilon^2 = 0$. We take it to be

$$\phi(\mu, z) = \left(\frac{1}{1 - \mu z} \right)^3 \quad (5.9)$$

where $\mu = \varepsilon^2 > 0$ is the same μ of the preceding sections. The usual Feynman boundary conditions are understood to give a meaning to the denominators of Eq. (5.9) ($z \rightarrow z + i\eta$).

The theory defined by the action

$$A + A_1 = \int dp \bar{\Psi}(p) S^{-1}(p) \Psi(p) + \int dp dq \bar{\Psi}(p + \frac{q}{2}) \Gamma'(p, q) \Psi(p - \frac{q}{2}) \quad (5.10)$$

where

$$\Gamma'(p, q) = \Gamma(p, q) + \Gamma_1(p, q) \quad (5.11)$$

is gauge invariant. The vector current is conserved in the sense of the co-variant derivatives which have now to include also the σ and the π fields. Concerning the axial current in the right-hand side of Eq. (3.6) $\Gamma(p,q)$ must be replaced by the new vertex $\Gamma'(p,q)$, it being understood that the left-hand side undergoes modifications similar to the ones for the vector current. What we want to show is that the true, abnormal parity, anomaly already computed [Eq. (4.18)-Eq. (4.24)] is not modified by the introduction of the scalar and pseudoscalar fields. This amounts to prove that, in an unambiguously finite loop with at least one Γ_1 vertex, there is no contribution as $\mu \rightarrow 0$ from the domain of the high loop momenta. The estimates necessary to draw this conclusion are reported in Appendix C.

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APPENDIX A

We have to majorize the integral

$$\int d p T_2 \left\{ \prod_i^N (S(p) \prod_j^{m_i} (\hat{U}(k_j) \cdot \gamma_p) S^{-1}(p)) \right\} \quad (A.1)$$

where N is the number of vertices in the diagram and m_i is the number of vector and axial lines coming out from each vertex. The position of the zeros of $f(\mu, z)$ and of $z - m^2 f(\mu, z)^2$ allows us to Wick rotate the p^0 line of integration by performing the substitution $p^0 \rightarrow i p^0$. Then in the variable $x = p^0{}^2 + \vec{p}^2$ $f(\mu, p^2)$ becomes $1 + \mu x + \mu^4 x^2$ and $p^2 - m^2 f(\mu, p^2)^2$ becomes $-x - m^2(1 + \mu x + \mu^4 x^2)^2$. It is convenient to separate the region of the four-dimensional integration into three domains : region 1 or low energy region with $0 \leq x \leq \mu^{-1}$, region 2 or high energy region with $\mu^{-1} \leq x \leq \mu^{-3}$, region 3 or very high energy region with $\mu^{-3} \leq x$. We will now estimate the terms contained in the integrand of Eq. (A.1). In all the inequalities we are going to write, it is understood that a constant independent of μ and x can always multiply each side.

In region 1, the following majorizations are easily seen to hold :

$$\prod_j^{m_i} \| (\hat{U}(k_j) \cdot \gamma_p) S^{-1}(p) \| \leq \begin{cases} \mu^{\frac{m-1}{2}} & \text{if } m \text{ is odd} \\ \mu^{\frac{m}{2}} x^{\frac{1}{2}} & \text{if } m \text{ is even} \end{cases} \quad (A.2)$$

$$\| S(p) \|^N \leq \left[\frac{x^{\frac{1}{2}} + m}{x + m^2} \right]^N \leq \sum_0^N \alpha \frac{x^{\frac{\alpha}{2}}}{(x + m^2)^N} \quad (A.3)$$

where the symbol $\| \|$ denotes the norm in the Dirac matrix space. Therefore the integrand $I_{N, \gamma, \delta}$ of integral (A.1) can be estimated as :

$$I_{N, \gamma, \delta} \leq \sum_0^N \alpha \frac{x^{\frac{\alpha+\delta}{2}} \mu^{\frac{\gamma-N+\delta}{2}}}{(x + m^2)^N} \quad (A.4)$$

where $\gamma = \sum_{i=1}^N m_i$ and δ the number of vertices with an even number of vector and axial lines coming out (obviously $\gamma \geq N$ and $\delta \leq N$). Consequently the contribution of region 1 to the integral (A.1) is smaller for each α than

$$\mu^{\frac{\gamma-N+\delta}{2}} \int_0^{\mu^{-1}} dx \frac{x^{\frac{\alpha+\delta}{2}+1}}{(x+m^2)^N} \quad (A.5)$$

If $\gamma > N$ the expression (A.5) can be majorized by

$$\mu^{\frac{\gamma-N+\delta}{2}} \left(\int_0^1 dx \frac{x^{\frac{\alpha+\delta}{2}+1}}{(x+m^2)^N} + \int_1^{\mu^{-1}} dx x^{\frac{\alpha+\delta-2N+2}{2}} \right) \quad (A.6)$$

which tends to zero as $\mu \rightarrow 0$ if $\gamma > 4$. If $\gamma = N$ and then $\delta = 0$ the integral is majorized by

$$\int_0^{\mu^{-1}} dx \frac{x^{\frac{\alpha}{2}+1}}{(x+m^2)^N} \quad (A.7)$$

which is obviously convergent as $\mu \rightarrow 0$ for $\gamma > 4$ since α can be at most N .

In region 2, the corresponding estimates are :

$$\prod_{j=1}^m \left\| (\hat{U}(k_j) \cdot \hat{a}_p) S^{-1}(p) \right\| \leq \mu^{-1} x^{-\frac{\delta+1}{2}} \quad (A.8)$$

$$\|S(p)\| \leq \sum_{\alpha}^N \frac{\mu^{2N-\alpha} x^{N-\frac{\alpha}{2}}}{(m^2 \mu^2 x + 1)^N} \quad (A.9)$$

$$I_{N,\gamma,\delta} \leq \sum_{\alpha}^N \frac{\mu^{N-\alpha} x^{\frac{N-\gamma-\alpha}{2}}}{(m^2 \mu^2 x + 1)^N} \quad (A.10)$$

The contribution of region 2 to the integral (A.1) can, for each α , be majorized by

$$\mu^{N-\alpha} \int_{\mu^{-1}}^{\mu^{-3}} dx \frac{x^{\frac{N-\alpha-\delta}{2}+1}}{(m^2 \mu^2 x + 1)^N} \quad (A.11)$$

This expression is smaller than

$$\mu^{\gamma-4} \left\{ \int_{\mu}^1 dx x^{\frac{N-\gamma-d+2}{2}} + \int_1^{\mu^{-1}} dx x^{-\frac{N+\gamma+d-2}{2}} \right\} \quad (\text{A.12})$$

which tends to zero as $\mu \rightarrow 0$ for $\gamma > 4$.

Finally in region 3 the estimates are :

$$\prod_{j=1}^m \left\| \left(\hat{U}(k_j) \cdot \lambda_p \right) \bar{S}(p) \right\| \leq \mu^{-4} x^{-\frac{\gamma+3}{2}} \quad (\text{A.13})$$

$$\| S(p) \|^N \leq \sum_{\alpha=0}^N \mu^{-4\alpha} x^{-\frac{3}{2}\alpha} \quad (\text{A.14})$$

$$I_{N,\gamma,\delta} \leq \sum_{\alpha=0}^N \mu^{-4(N+\alpha)} x^{-\frac{\gamma+3N+3\alpha}{2}} \quad (\text{A.15})$$

The contribution of region 3 to the integrand of Eq. (A.1) is, for each α , smaller than

$$\mu^{-4(N+\alpha)} \int_{\mu^{-3}}^0 dx x^{-\frac{\gamma+3(N+\alpha)-2}{2}} \quad (\text{A.16})$$

which can be seen by explicit computation to tend to zero as $\mu \rightarrow 0$ if $\gamma > 4$.

APPENDIX B

We have to majorize the integral

$$\int dp T_2 \left\{ S(p) \gamma_5 \prod_{i=1}^{N-1} \left[S(p_i) \prod_{j=1}^{m_i} (\hat{U}(k_j^{(i)}) \cdot \gamma_p) S^{-1}(p_i) \right] \right\} \quad (B.1)$$

which represents a diagram with N vertices, one being a γ_5 vertex and the remaining $N-1$ having vector and axial lines coming out. The convention and notation are the same as in Appendix A. In this case the variable index α of Eq. (A.3) cannot reach N because $\alpha = N$ means that in each of the N propagators we are taking the term with γ^μ . Since there are $N-1$ vertices with γ^μ and one vertex with γ_5 , the term with $\alpha = N$ drops by trace since the total number of γ 's is $2N-1+4$. Besides $\alpha \leq N-1$, the other restrictions are obviously $\delta \leq N-1$ and $\gamma \geq N-1$.

By inequalities (A.2) and (A.3) the integrand $I_{N,\gamma,\delta}$ of the integral (B.1) can be estimated in region 1 as :

$$I_{N,\gamma,\delta} \leq \sum_{\alpha=0}^{N-1} x^{\frac{\alpha+\delta}{2}} \mu^{\frac{\gamma-N+\delta+1}{2}} \frac{1}{(m^2+x)^N} \quad (B.2)$$

Consequently the contribution of region 1 to the integral (B.1) is, for each α , smaller than

$$\mu^{\frac{\gamma-N+\delta+1}{2}} \int_0^{\mu^{-1}} dx \frac{x^{\frac{\alpha+\delta+2}{2}}}{(m^2+x)^N} \quad (B.3)$$

If $\gamma > N-1$, the integral (B.3) can be majorized by :

$$\mu^{\frac{\gamma-N+\delta+1}{2}} \left\{ \int_0^1 dx x^{\frac{\alpha+\delta+2}{2}} + \int_1^{\mu^{-1}} dx x^{\frac{\alpha+\delta+2-2N}{2}} \right\} \quad (B.4)$$

which tends to zero as $\mu \rightarrow 0$ if $\gamma > 2$. If $\gamma = N-1$ and consequently $\delta = 0$, the integral is majorized by

$$\int_0^{\mu^{-1}} dx \frac{x^{\frac{\alpha+2}{2}}}{(m^2+x)^N} \quad (B.5)$$

which is convergent for $\gamma > 2$.

In region 2 the corresponding estimate is :

$$I_{N,\gamma,\delta} \leq \sum_{\alpha=0}^{N-1} \mu^{N-\alpha+1} \frac{x^{\frac{N-\gamma-\alpha+1}{2}}}{(m^2 \mu^2 x + 1)^N} \quad (B.6)$$

Therefore the contribution of region 2 to the integral (B.1) is, for each α , smaller than :

$$\int_{\mu^{-1}}^{\mu^{-3}} dx \mu^{N-\alpha+1} \frac{x^{\frac{N-\alpha+1-\gamma}{2}}}{(m^2 \mu^2 x + 1)^N} = \mu^{\gamma-4} \int_{\mu}^{\mu^{-1}} dy \frac{y^{\frac{N-\alpha+3-\gamma}{2}}}{(1+m^2 y)^N} \quad (B.7)$$

where the change of variable $y = \mu^2 x$ has been performed. If $\gamma = 4$ the majorant (B.7) is convergent and different from zero as $\mu \rightarrow 0$. If $\gamma > 4$ the expression (B.7) is smaller than :

$$\mu^{\gamma-4} \left\{ \int_{\mu}^1 dy y^{\frac{N-\alpha+3-\gamma}{2}} + \int_1^{\mu^{-1}} dy y^{-\frac{N+\alpha+\gamma-3}{2}} \right\} \quad (B.8)$$

which tends to zero as $\mu \rightarrow 0$.

In region 3 the estimate is :

$$I_{N,\gamma,\delta} \leq \sum_{\alpha=0}^{N-1} \mu^{-4(N+\alpha-1)} x^{-\frac{\gamma+3(N+\alpha-1)}{2}} \quad (B.9)$$

The corresponding integral is majorized by :

$$\mu^{-4(N+\alpha-1)} \int_{\mu^{-3}}^{\infty} dx x^{-\frac{\gamma+3(N+\alpha-1)}{2}+1}$$

which tends to zero as $\mu \rightarrow 0$ for $\gamma > 3$.

APPENDIX C

We are interested in estimating a loop integral with N vertices in two cases. In the first the N vertices are all Γ and Γ_1 vertices, while in the second one of them is a γ_5 vertex of the type coming from Eq. (3.6). The conventions used are: γ is the total number of vector and axial lines coming out of the Γ vertices, similarly γ_1 is the same number for the Γ_1 vertices, δ is the number of Γ vertices emitting an even number of vector and axial lines, η is the number of Γ_1 vertices emitting an odd number of vector and axial lines, and p is the number of Γ_1 vertices. We will also use the symbol $P = \gamma + \gamma_1$. Here, in the same way as in Section 2, it is understood that any derivative on the external momenta counts as an outgoing vector or axial line.

By the form (5.9) chosen for the function ϕ , a Γ_1 vertex with r vector and axial lines has the following majorization in the three regions:

$$\text{low energy region } (0 \leq x \leq \mu^{-1}) \quad \begin{cases} \mu^{\frac{n}{2}} & n \text{ even} \\ \mu^{\frac{n+1}{2}} x^{\frac{1}{2}} & n \text{ odd} \end{cases} \quad (C.1)$$

$$\text{high energy region } (\mu^{-1} \leq x \leq \mu^{-3}) \quad \mu^{-3} x^{-\frac{2+6}{2}} \quad (C.2)$$

$$\text{very high energy region } (\mu^{-3} \leq x) \quad \mu^{-3} x^{-\frac{2+6}{2}}. \quad (C.3)$$

In region 1, using Eq. (A.4), one sees that the integrand with only Γ and Γ_1 vertices is majorized by:

$$\sum_{\alpha=0}^N \alpha x^{\frac{\alpha+\delta+\eta}{2}} \frac{\mu^{\frac{P+P+\delta+\eta-N}{2}}}{(m^2+x)^N} \quad (C.4)$$

while, by Eq. (B.2), the integrand with one γ_5 can be majorized by:

$$\sum_{\alpha=0}^N \alpha x^{\frac{\alpha+\delta+\eta}{2}} \frac{\mu^{\frac{P+P+\delta+\eta+1-N}{2}}}{(m^2+x)^N} \quad (C.5)$$

In the same way as in Appendix A [Eqs. (A.5)-(A.7)], the integral of Eq. (C.4) tends to zero as $\mu \rightarrow 0$ if $p + p > N$ and $p + p > 4$. If $p + p = N$ and $p + p > 4$, the majorization of the integral is finite. The same result holds for the integral of (C.5) if we replace the condition $p + p > 4$ by the condition $p + p > 3$.

In region 2, using Eq. (A.10), the integrand with only Γ and Γ_1 vertices is majorized by :

$$\sum_{\alpha}^N \mu^{N-2p-\alpha} \frac{x^{\frac{N-\alpha-p-5p}{2}}}{(m^2 \mu^2 x + 1)^N} \quad (C.6)$$

while, using Eq. (B.6), the integrand with one γ_5 is majorized by :

$$\sum_{\alpha}^N \mu^{N-2p-\alpha+1} \frac{x^{\frac{N-\alpha-p-5p+1}{2}}}{(m^2 \mu^2 x + 1)^N} \quad (C.7)$$

The integral of (C.6) is, for each α , smaller than

$$\int_{\mu}^{\mu^{-1}} dy \mu^{p+3p-4} \frac{y^{\frac{N-\alpha-p-5p}{2}+1}}{(m^2 y + 1)^N} \quad (C.8)$$

which tends to zero as $\mu \rightarrow 0$ if $p + p > 4$. The same result holds for (C.7) with the replacement of the condition $p + p > 4$ by the conditions $p + p > 3$ and $p > 0$.

Finally in region 3, using Eq. (A.15) the integrand with only Γ and Γ_1 vertices is smaller than :

$$\sum_{\alpha}^N \mu^{p-4(N+\alpha)} \frac{x^{-\frac{p+3(N+\alpha+p)}{2}}}{x} \quad (C.9)$$

and the integrand with γ_5 is, by Eq. (B.9), smaller than

$$\sum_{\alpha}^N \mu^{p-4(N+\alpha-1)} \frac{x^{-\frac{p+3(N+\alpha+p-1)}{2}}}{x} \quad (C.10)$$

The integral of Eq. (C.9) tends to zero if $\rho + p > 4$ and the integral of Eq. (C.10) tends to zero if both conditions $\rho + p > 3$ and $p > 0$ are satisfied.

A loop with only Γ and Γ_1 vertices is unambiguously convergent if $\rho + p > 4$, and a loop with a γ_5 vertex is unambiguously convergent if $\rho + p > 3$. The above estimates imply that for all finite loops when $p \neq 0$ only the low energy contribution can appear as $\mu \rightarrow 0$.

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- 8) Our conventions are :

$$g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

and $\epsilon_{0123} = 1$

- 9) Expressions of this type have been first used by :
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